Computation of Belyi maps with prescribed ramification and applications in Galois theory

Dissertation zur Erlangung des naturwissenschaftlichen Doktorgrades der Julius-Maximilians-Universität Würzburg

vorgelegt von

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CHAPTER 1

Introduction

This dissertation deals with the explicit computation of high degree genus-0 three-branch-point covers of $\mathbb{P}^1\mathbb{C}$, also called Belyi maps, with prescribed monodromy as well as the corresponding verification process.

The calculation of ramified covers of $\mathbb{P}^1\mathbb{C}$ with prescribed monodromy groups is of great importance for the inverse Galois theory problem as monodromy groups occur as Galois groups over the rational function field $\mathbb{C}(t)$. In combination with well known theoretical descent criteria this also translates to function fields over small number fields. We therefore obtain explicit defining equations for function field extensions with prescribed Galois groups. Under certain conditions this yields explicit polynomials defining regular Galois extensions of $\mathbb{Q}(t)$ with prescribed Galois groups as well as extensions of $\mathbb{Q}$ thanks to Hilbert's irreducibility theorem. Under certain conditions upon suitable specialization in t this also allows the regular realization of index-2 subgroups of the original Galois group.

The determination of large Belyi maps with prescribed ramification is considered to be a challenging problem, see [**41**]. Several powerful computational approaches for computing Belyi maps beyond the standard techniques are introduced by e.g. Klug, Musty, Schiavone, Voight [**23**], Roberts [**39**] and Monien [**35**], [**36**].

We will introduce another technique for computing complex approximations of Belyi maps with prescribed ramification and present examples of degree up to 280 using Magma [**13**] and Matlab [**32**]. As a consequence we are able to determine polynomials having almost simple Galois groups over $\mathbb{Q}(t)$, see Theorem 5.1. Furthermore, we give an explicit version of a theorem of Magaard, see Theorem 5.3, which theoretically characterizes all sporadic groups occurring as a composition factor of the monodromy group of a rational function, namely the five Mathieu groups, the Higman-Sims group HS, the Janko groups J_1 and J_2 as well as the Conway group Co_3.

In particular, we present the first (to the best of our knowledge) explicit realizations of $\mathrm{PSp}(4,4)$, $\mathrm{PSp}(4,4){:}2$, HS, Aut(HS), $\mathrm{O}^+(8,2)$ as Galois groups over $\mathbb{Q}(t)$ and J_1 as a Galois group over $K(t)$ with K being a degree-7 number field.

This dissertation is structured as follows: The theoretical background will be introduced in chapter 2. Well established techniques for computing Belyi maps with prescribed ramification are discussed in chapter 3. In chapter 4 we explain our newly developed computation method and present a detailed documentation of the explicit realization of a degree-100 Belyi map having monodromy group Aut(HS). In chapter 5 we present several more examples of Belyi maps of large degree with rational coefficients as well as Belyi maps with monodromy groups having sporadic composition factors. The final chapter 6 documents the code used for computations in the previous chapters.

All of the results were achieved in cooperation with Dominik Barth. The entire collaboration has reached the following milestones:

- developing a new method for computing high degree Belyi maps with prescribed ramification, explicit realization of Belyi maps over $\mathbb{Q}$ and small number fields up to degree 280, see [**10**] (originally presented in [**5**], [**8**], [**6**] and [**7**])
- calculation of multi-branch-point covers with prescribed ramification (collaboration with J. König), Galois group verification for 2-transitive groups, explicit realization of $\mathrm{PSp}_6(2)$ of degree 28 and 36, $\mathrm{PSp}_4(3)$ of degree 27 and $\mathrm{PSL}_6(2)$ of degree 63, see [**4**], [**9**], [**11**].

In his upcoming dissertation Dominik Barth will give a thorough survey on computing multi-branch-points covers.

Notation.

$\mathbb{N}$	set of natural numbers 1, 2, 3, ...
$\mathbb{N}_0$	set of natural numbers including 0
$\mathbb{Z}$	ring of integers
$\mathbb{Q}$	field of rational numbers
$\mathbb{R}$	field of real numbers
$\mathbb{C}$	field of complex numbers
$\lvert M \rvert$	cardinality of a finite set M
$f^{-1}(M)$	pre-image of a set or element M by a map f
$[z]_\sim$	equivalence class containing z in the equivalence relation $\sim$
$\overline{K}$	algebraic closure of a field K
$\mathbb{P}^1 K$	projective line over a field K
$\mathbb{H}$	upper complex half-plane
$\mathbb{D}$	complex unit disc
$\mathrm{Aut}(U)$	automorphism group of an open subset $U \subseteq \mathbb{C}$
$\langle M \rangle$	group generated by a set or element M
x^G	orbit of x of a group G acting on a set containing x
$\mathrm{Sym}(M)$	symmetry group of a set M
S_n	symmetric group of $\{1, \ldots, n\}$
A_n	alternating group of $\{1, \ldots, n\}$
$\mathrm{Aut}(G)$	automorphism group of a group G
$N_G(S)$	normalizer of a subset S in a group G
L/K	field extension $L \supseteq K$
$[L : K]$	degree of the field extension L/K
$\mathrm{Aut}(L/K)$	automorphism group of the field extension L/K
$\mathrm{Gal}(L/K)$	Galois group of a Galois extension L/K
$\mathrm{Gal}(f/K)$	Galois group of a polynomial f over K
$R[X]$	ring extension of a ring R by an element X
$K(X)$	field extension of a field K by an element X
$\deg(f)$	degree of a univariate polynomial or rational function

$\overline{z}$	complex conjugate of a complex number z
$\mathrm{Real}(z)$	real part of a complex number z
$\mathrm{Imag}(z)$	imaginary part of a complex number z
$\|z\|$	value of a complex number z
∂M	border of a real or complex subset M
M°	interior of a real or complex subset M
$\overline{M}$	closure of a subset of a real or complex subset M
$[a, b]$	closed real interval from a to b
(a, b)	open real interval from a to b

CHAPTER 2

Theoretical Background

This chapter is in parts taken over from [**10**, Chapter 2].

Let $f \in \mathbb{C}(X)$ be a rational function of degree n. An element $a \in \mathbb{P}^1\mathbb{C}$ is called a *critical value* of f if $|f^{-1}(a)| < n$ holds. The *ramification locus* of f is defined to be an ordered set of all critical values of f.

2.1. Monodromy and ramification tuples

Let f be a rational function of degree n with ramification locus

$$\mathcal{R} := (r_1, \ldots, r_m) \subseteq \mathbb{P}^1\mathbb{C}$$

for some $m \in \mathbb{N}$. For a fixed $p_0 \in \mathbb{P}^1\mathbb{C} \setminus \mathcal{R}$ let $\pi_1(\mathbb{P}^1\mathbb{C} \setminus \mathcal{R}, p_0)$ be the *topological fundamental group* of $\mathbb{P}^1\mathbb{C} \setminus \mathcal{R}$ with base point p_0. For the sake of simplicity we will denote a path and its homotopy class by the same symbol. It is well known that $\pi_1(\mathbb{P}^1\mathbb{C} \setminus \mathcal{R}, p_0)$ is generated by closed paths γ_k for $k = 1, \ldots, m$ starting in p_0 and turning counter-clockwise only around r_k, respectively, such that $\prod_{k=1}^m \gamma_k = 1$.

For any closed path γ in $\mathbb{P}^1\mathbb{C} \setminus \mathcal{R}$ starting in p_0 let $\overline{\gamma}^q$ be the lifted path of γ under f starting at some point $q \in f^{-1}(p_0)$, i.e. $\overline{\gamma}^q$ is a path in $\mathbb{P}^1\mathbb{C} \setminus f^{-1}(\mathcal{R})$ such that $f(\overline{\gamma}^q) = \gamma$. In a natural way the following homomorphism arises:

$$\mathrm{mon} : \begin{cases} \pi_1(\mathbb{P}^1\mathbb{C} \setminus \mathcal{R}, p_0) \to \mathrm{Sym}(f^{-1}(p_0)) \cong S_n \\ \gamma \mapsto (q \mapsto \text{end point of } \overline{\gamma}^q) \end{cases}$$

An embedding of the image of mon in S_n, denoted by $\mathrm{mon}(f)$, is called the monodromy group of f and the tuple

$$(\sigma_1, \ldots, \sigma_m) := (\mathrm{mon}(\gamma_1), \ldots, \mathrm{mon}(\gamma_m)) \in (S_n \setminus \{1\})^m$$

up to simultaneous conjugation will be referred to as the *ramification tuple* of f (with respect to $\mathcal{R}$). In this context two tuples $(s_1, \ldots, s_m)$, $(\tilde{s}_1, \ldots, \tilde{s}_m) \in {S_n}^m$

are called *simultaneous conjugate* if there exists an element $g \in S_n$ such that

$$(s_1, \dots, s_m) = (g^{-1}\tilde{s}_1 g, \dots, g^{-1}\tilde{s}_m g).$$

Important properties are collected in the following lemma, see also [**26**, Chapter 1.2].

Lemma 2.1. *Let $f \in \mathbb{C}(X)$ be a rational function with ramification tuple $(\sigma_1, \dots, \sigma_m) \in (S_n \setminus \{1\})^m$. Then the following conditions hold:*
(a) $\prod_{k=1}^m \sigma_k = 1$.
(b) $\mathrm{mon}(f) = \langle \sigma_1, \dots, \sigma_m \rangle$.
(c) $\mathrm{mon}(f)$ *is a transitive subgroup of* S_n.
(d) Riemann-Hurwitz genus formula: $2(n-1) = \sum_{k=1}^m \mathrm{ind}(\sigma_k)$. *Here,* $\mathrm{ind}(\sigma)$ *for any $\sigma \in S_n$ is defined to be n minus the number of cycles of σ.*

Motivated by the previous lemma we will call $(\sigma_1, \dots, \sigma_m) \in (S_n \setminus \{1\})^m$ a *genus-0 tuple for a transitive subgroup G* of S_n if the following conditions are satisfied:

- $\prod_{k=1}^m \sigma_k = 1$.
- $G = \langle \sigma_k : k \in \{1, \dots, m\} \rangle$.
- $2(n-1) = \sum_{k=1}^m \mathrm{ind}(\sigma_k)$.

Riemann's famous existence theorem, see e.g. [**26**, Theorem 1.8.14], guarantees the existence of rational functions with prescribed ramification:

Theorem 2.2. *For every genus-0 tuple $(\sigma_1, \dots, \sigma_m)$ for a transitive subgroup G of S_n and every ordered m-element subset $\mathcal{R}$ of $\mathbb{P}^1\mathbb{C}$ there exists a unique rational function $f \in \mathbb{C}(X)$ up to inner $\mathbb{C}$-Möbius transformation with ramification locus $\mathcal{R}$ and ramification tuple $(\sigma_1, \dots, \sigma_m)$.*

Remark 2.3. Recall that *K-Möbius transformations* for any field K are given by degree-1 rational functions in $K(X)$. They are of type

$$\frac{AX + B}{CX + D}$$

for some $A, B, C, D \in K$ fulfilling $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} \neq 0$. It is well known that these maps act sharply 3-transitive on $\mathbb{P}^1 K$.

2.2. Function field setting

Let K be a subfield of $\mathbb{C}$ and

$$f := \frac{p}{q} \in K(X)$$

with coprime polynomials $p, q \in K[X]$ a rational function of degree n with ramification locus $\mathcal{R}$ and ramification tuple $(\sigma_1, \ldots, \sigma_m)$ for some $m \in \mathbb{N}$.

Furthermore, let L be the splitting field of $p(X) - tq(X)$ over $K(t)$ with t being a transcendental, $\hat{K}$ the algebraic closure of K in L, and x a root of the irreducible polynomial $p(X) - tq(X)$ in L. In the function field extensions $L/K(t)$ and $K(x)/K(t)$ the set of ramified places corresponds to $\mathcal{R}$.

The groups $A := \mathrm{Gal}(L/K(t))$ and $G := \mathrm{Gal}(L/\hat{K}(t))$ are called the *arithmetic monodromy group* and the *geometric monodromy group* of f. Both groups are considered as transitive permutation groups acting on the n roots of $p(X) - tq(X)$ contained in L. It is well known that G is normal in A. If $K = \hat{K}$, or equivalently $G = A$, then $L/K(t)$ is called a *regular Galois extension*.

Decomposition and inertia subgroups of A carry valuable information, see [**24**, Lemma 3.2, 3.6 and 3.7]:

Lemma 2.4. *For a place in L extending a place P in $K(t)$ let G_Z be the corresponding decomposition group and G_T the inertia group. Then the following holds:*

(a) G_T is cyclic and normal in G_Z.
(b) If a generator σ of G_T has cycle lengths $m_1, \ldots, m_k$, then the specialization of $p(X) - tq(X)$ at the place P has roots of multiplicity $m_1, \ldots, m_k$.
(c) Assume P is of degree 1. If G_Z has orbits $O_1, \ldots, O_r$ where each O_i is a union of k_i orbits of G_T then the specialization of $p(X) - tq(X)$ at P has degree-k_i factors in $K[X]$ with multiplicity $\frac{|O_i|}{k_i}$.

Remark. If a specialization reduces the degree of the polynomial by k we will consider ∞ to be a root of this polynomial with multiplicity k.

Note, that G and $\mathrm{mon}(f)$ are isomorphic in such a way that the conjugacy classes of $(\sigma_1, \ldots, \sigma_m)$ can be identified with the unique conjugacy classes of the respective inertia group generators of any extension of ramified places in $K(t)$ to L.

2.3. Belyi maps

A rational function f will be called a *Belyi map* if its ramification locus is equal to $(0, 1, \infty)$. The elements of $f^{-1}(0)$, $f^{-1}(1)$ and $f^{-1}(\infty)$ are called *zeros*, *ones* and *poles* of f, respectively.

If $(\sigma_0, \sigma_1, \sigma_\infty)$ denotes the ramification triple of f then each cycle of σ_0, σ_1 or σ_∞ corresponds to a zero, one or pole of f of multiplicity given by its respective cycle length.

2.3.1. Dessin d'enfant.

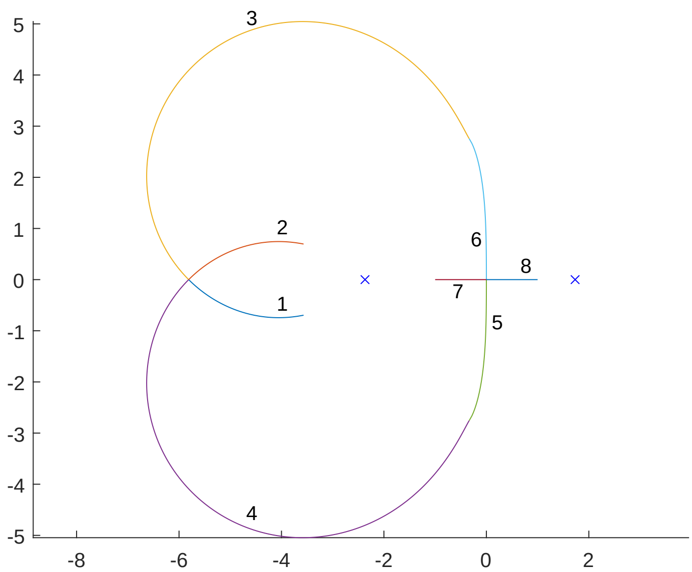
The *dessin d'enfant* or *dessin* of a degree-n Belyi map $f \in \mathbb{C}(X)$ is defined to be set $f^{-1}([0, 1])$.

In a natural way the dessin of f can be considered as a bipartite graph with n edges embedded in $\mathbb{P}^1\mathbb{C}$ where the vertices are given by the zeros and ones of f and the edges are given by the connected components of $f^{-1}(]0, 1[)$. Note that dessins are connected and all the edges meeting in a vertex locally form a regular star configuration. We will label the edges from 1 to n in an oriented counter clock-wise fashion such that the labelling corresponds to the ramification triple of f, see also [**26**, Section 1.3.3].

Example 2.5. A dessin with ramification triple $(\sigma_0, \sigma_1, \sigma_\infty)$ where

$$\sigma_0 = (1, 2, 3, 4)(5, 8, 6, 7), \quad \sigma_1 = (3, 6)(4, 5), \quad \sigma_\infty = (1, 4, 7, 6, 2)(3, 8, 5)$$

including the poles (marked with blue crosses) is shown in the following figure:



For the sake of clarity we occasionally omit drawing the labels to our dessins since we will be dealing with high degree Belyi maps.

Remark 2.6. The complex conjugate $\bar{f}$ of a degree-n Belyi map f with ramification triple $(\sigma_0, \sigma_1, \sigma_\infty)$ is also a degree-n Belyi map. If its ramification triple is denoted by $(\overline{\sigma_0}, \overline{\sigma_1}, \overline{\sigma_\infty})$ it is easy to deduce that

$$(\overline{\sigma_0}, \overline{\sigma_1}, \overline{\sigma_\infty}) = \left({\sigma_0}^{-1}, {\sigma_1}^{-1}, ({\sigma_\infty}^{-1})^{{\sigma_1}^{-1}}\right).$$

2.3.2. Real Belyi maps.

According to [**24**, Section 4.1] the ramification triples of real Belyi maps can be characterized as follows:

Lemma 2.7. *Let $(\sigma_0, \sigma_1, \sigma_\infty) \in {S_n}^3$ be a genus-0 tuple for a transitive subgroup of S_n. Then there exists a real Belyi map $f \in \mathbb{R}(X)$ with ramification triple $(\sigma_0, \sigma_1, \sigma_\infty)$ if and only if there exists an involution $\rho \in N_{S_n}(\langle \sigma_0, \sigma_1 \rangle)$ satisfying the following conditions:*
(a) $({\sigma_0}^\rho, {\sigma_1}^\rho, {\sigma_\infty}^\rho) = ({\sigma_0}^{-1}, {\sigma_1}^{-1}, ({\sigma_\infty}^{-1})^{{\sigma_1}^{-1}})$.
(b) There exists at least one fixed point in ρ, $\rho\sigma_1$ or $\rho\sigma_1\sigma_\infty$.

We will take advantage of the following obvious geometric property of real Belyi maps: The dessin of a real degree-n Belyi map f lies axially symmetric to the real line and the element ρ from the previous lemma describes the symmetry of the dessin of f: the fixed points of ρ reveal the real edges and the cycles of length 2 of ρ tell us the complex conjugate pairs of edges.

Furthermore, the real zeros, ones and poles of f can be characterized in the following way:

- A cycle O of σ_0 or σ_1 belongs to a real zero or one of f if and only if
$$\{o^\rho : o \in O\} = \{o : o \in O\}.$$
- A cycle O of σ_∞ belongs to a real pole of f if and only if
$$\{o^{\sigma_0\rho} : o \in O\} = \{o : o \in O\}.$$

Consequently, we will call a cycle of σ_0, σ_1, σ_∞ a *real cycle* if it belongs to a real root, one or pole of f. Furthermore, two cycles of σ_0, σ_1, σ_∞ are called *complex conjugate cycles* if they belong to complex conjugate pairs of zeros, ones or poles of f.

2.3.3. Belyi maps defined over $\mathbb{Q}$.

In the following we will apply the famous rational rigidity criterion, see [**30**, Theorem I.4.8], for genus-0 triples which guarantees the existence of regular Galois extensions of $\mathbb{Q}(t)$.

For elements a, b in a subgroup G of S_n we write $a \sim_G b$ if a is conjugate to b in G. For $(\sigma_0, \sigma_1, \sigma_\infty) \in G^3$ let $\ell(\sigma_0, \sigma_1, \sigma_\infty)$ be the number of elements in the set

$$\{(s_0, s_1, s_\infty) \in G^3 : G = \langle s_0, s_1 \rangle, s_0 s_1 s_\infty = 1, s_i \sim_G \sigma_i\}$$

up to simultaneous conjugation in G.

The triple $(\sigma_0, \sigma_1, \sigma_\infty)$ is said to be *rigid* if $\ell(\sigma_0, \sigma_1, \sigma_\infty) = 1$. Furthermore, a conjugacy class C of G is called *rational* if $C^k = C$ for all integers k prime to $|G|$.

Theorem 2.8. *Let G be a transitive subgroup of S_n with trivial center and $(\sigma_0, \sigma_1, \sigma_\infty)$ a genus-0 triple for G with the following properties:*
(a) $(\sigma_0, \sigma_1, \sigma_\infty)$ is rigid.
(b) σ_0, σ_1 and σ_∞ are contained in rational conjugacy classes of G.
(c) At least one exponent in the cycle structure description of $\sigma_0, \sigma_1, \sigma_\infty$ is odd.
Then there exists a Belyi map

$$f = \frac{p}{q} \in \mathbb{Q}(X)$$

with ramification triple $(\sigma_0, \sigma_1, \sigma_\infty)$ such that

$$p(X) - tq(X) \in \mathbb{Q}(t)[X]$$

defines a regular Galois extension of $\mathbb{Q}(t)$ with Galois group G.

Proof. Since $(\sigma_0, \sigma_1, \sigma_\infty)$ is a genus-0 triple with properties (a) and (b) the rational rigidity criterion [**30**, Theorem I.4.8] guarantees the existence of a regular Galois extension $L/\mathbb{Q}(t)$ ramified only at (t), $(t-1)$ and $(\frac{1}{t})$ with

$$G \cong \mathrm{Gal}(L/\mathbb{Q}(t))$$

and the inertia groups over ramified places correspond to the conjugacy classes of σ_0, σ_1 and σ_∞ in G.

Let K be the fixed field of a point stabilizer of G in L. Then, according to [**30**, Theorem I.9.1] and the Riemann-Hurwitz genus formula for function field extensions [**42**, Theorem 3.4.13] the genus of K turns out to be 0. Furthermore, from (c) and again [**30**, Theorem I.9.1] we see that there exists a place of odd degree in K. According to [**43**, Section 9.6.1] K must be a rational function field, i.e. $K = \mathbb{Q}(x)$ for some $x \in K$. Since $[\mathbb{Q}(x) : \mathbb{Q}(t)] = n$ we find a degree-n rational function

$$f = \frac{p}{q} \in \mathbb{Q}(X)$$

with coprime polynomials $p, q \in \mathbb{Q}[X]$ such that $t = f(x)$. In particular,

$$p(X) - tq(X) \in \mathbb{Q}(t)[X]$$

is a minimal polynomial of x over $\mathbb{Q}(t)$ defining the regular Galois extension $L/\mathbb{Q}(t)$ with Galois group G. As $L/\mathbb{Q}(t)$ is only ramified at the places (t), $(t-1)$ and $(\frac{1}{t})$ the rational function f turns out to be a Belyi map with ramification triple contained in the same conjugacy classes as $(\sigma_0, \sigma_1, \sigma_\infty)$. In combination with (a) the ramification triple of f has to coincide with $(\sigma_0, \sigma_1, \sigma_\infty)$. $\square$

2.3.4. Index-2 subgroups.

The existence of Belyi maps defining regular extensions of $\mathbb{Q}(t)$ implies that index-2 subgroups of the corresponding monodromy groups also occur regularly over $\mathbb{Q}(t)$, see [**40**, Lemma 4.5.1]:

Lemma 2.9. *Let G be the Galois group of a regular extension $L/\mathbb{Q}(t)$ ramified at three places which are rational over $\mathbb{Q}$, and let H be a subgroup of G of index 2. Then the fixed field of H is rational. In particular, H occurs regularly as a Galois group over $\mathbb{Q}$.*

We will now turn the theoretical result of the previous lemma into an explicit one: Again, let

$$f = \frac{p}{q} \in \mathbb{Q}(X)$$

be a Belyi map with coprime polynomials $p, q \in \mathbb{Q}[X]$ and ramification triple $(\sigma_0, \sigma_1, \sigma_\infty)$ such that

$$p(X) - tq(X) \in \mathbb{Q}(t)[X]$$

defines a regular Galois extension $L/\mathbb{Q}(t)$ with Galois group G. Let H be an index-2 subgroup of G with corresponding fixed field denoted by K.

We will now determine a polynomial with Galois group H by finding a suitable specialization of $p(X) - tq(X)$ in the variable t. Note, that

$$K = \mathbb{Q}(t)(s) \qquad \text{with} \qquad s^2 = P(t)$$

for some unique square-free polynomial $P \in \mathbb{Q}[t]$ with leading coefficient $\frac{1}{c}$ where c is a square-free integer.

Since exactly one element of the ramification triple of f is contained in H we are in one of the following cases:

case 1: If $\sigma_0 \in H$ then only the places $(t-1)$ and $(\frac{1}{t})$ are ramified in $K/\mathbb{Q}(t)$, therefore

$$P(t) = \frac{1}{c}(t-1).$$

Since $t = cs^2 + 1$ we see $\mathbb{Q}(t)(s) = \mathbb{Q}(s)$ and

$$\begin{aligned} H &= \mathrm{Gal}(p(X) - tq(X) \mid K) \\ &= \mathrm{Gal}\left(p(X) - (cs^2+1)q(X) \mid \mathbb{Q}(s)\right). \end{aligned}$$

case 2: If $\sigma_1 \in H$ we find in a similar fashion

$$P(t) = \frac{1}{c}t.$$

Obviously, $\mathbb{Q}(t)(s) = \mathbb{Q}(s)$ and

$$\begin{aligned} H &= \mathrm{Gal}(p(X) - tq(X) \mid K) \\ &= \mathrm{Gal}\left(p(X) - cs^2 q(X) \mid \mathbb{Q}(s)\right). \end{aligned}$$

case 3: If $\sigma_\infty \in H$ then

$$P(t) = \frac{1}{c}t(t-1).$$

With $r := \frac{s}{t}$ we find

$$r^2 = \frac{s^2}{t^2} = \frac{1}{c}\left(1 - \frac{1}{t}\right),$$

hence

$$t = \frac{1}{1 - cr^2}.$$

This implies $t \in \mathbb{Q}(r)$ and $s = rt \in \mathbb{Q}(r)$, therefore $\mathbb{Q}(t)(s) = \mathbb{Q}(r)$ and

$$\begin{aligned} H &= \mathrm{Gal}(p(X) - tq(X) \mid K) \\ &= \mathrm{Gal}\left(p(X) - \frac{1}{1-cr^2}q(X) \mid \mathbb{Q}(r)\right). \end{aligned}$$

In the following subsections we describe three different ways to explicitly calculate c.

2.3.4.1. *Non-rational conjugacy classes.*
We will assume the following:

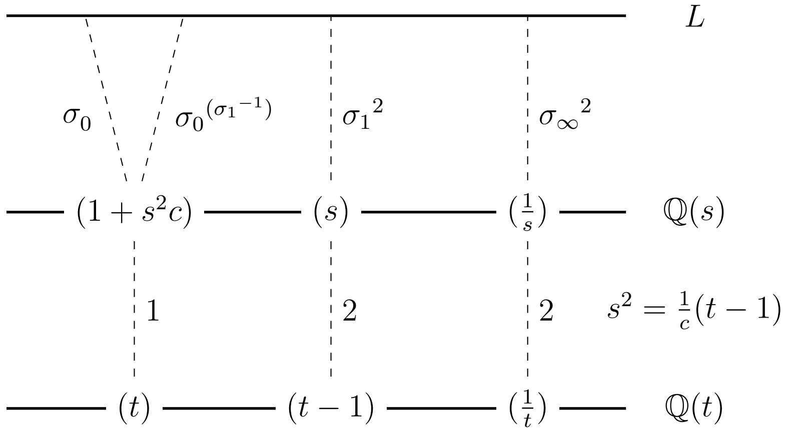
- $\sigma_0 \in H$ (this will be the case for most of our examples).
- σ_0 is contained in a non-rational conjugacy class in H.

Since we deal with case 1 from the previous section we have

$$K = \mathbb{Q}(s) \quad \text{with} \quad s^2 = \frac{1}{c}(t-1)$$

for some square-free integer $c \in \mathbb{Z}$.

The ramification of $\mathbb{Q}(s)/\mathbb{Q}(t)$ and $L/\mathbb{Q}(s)$ which can be computed via [**30**, Theorem I.6.3] is illustrated the following figure:



Let $\gamma \in \mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{-c}))$ and ζ be a primitive $|H|$-th root of unity. If $m \in \mathbb{N}$ is picked in such a way that

$$\gamma^{-1}(\zeta) = \zeta^m$$

then as the ramification locus of $L\overline{\mathbb{Q}}/\overline{\mathbb{Q}}(s)$ is pointwise fixed under γ the permutation σ_0 must be conjugate to $\sigma_0{}^m$ in H according to Fried's branch cycle lemma [**44**, Lemma 2.8]. In combination with the well known formula in [**22**, Satz V.13.1] we find

$$\gamma(\chi(\sigma_0)) = \chi(\sigma_0{}^m) = \chi(\sigma_0)$$

for any character χ of H. This implies that the character values belonging to σ_0 are contained in $\mathbb{Q}(\sqrt{-c})$. Since the class of σ_0 in H is non-rational we can determine c from the corresponding character values.

2.3.4.2. *Discriminant computation.*
In the special case for H being an even and G an odd group we can obtain c via a discriminant consideration:

Let δ be the discriminant of $p(X) - tq(X) \in \mathbb{Q}(t)[X]$, then $K = \mathbb{Q}(t, \sqrt{\delta})$ since K is the fixed field of $H = G \cap A_n$. In combination with $K = \mathbb{Q}(t, \sqrt{P(t)})$ we find that the square-free parts of δ and $P(t)$ coincide allowing us to compute c by an explicit computation of δ.

2.3.4.3. *Magma computation.*
The value for c can also be determined via a Galois group computation in Magma: Pick $t_0 \in \mathbb{Q}$ such that

$$G = \mathrm{Gal}(p(X) - t_0 q(X) \mid \mathbb{Q})$$

which can be checked with the Magma command `GaloisGroup`. Then apply `GaloisSubgroup` to find the degree-2 number field corresponding to the index-2 subgroup H suggesting a square-free integer value for c denoted by c^*. As both commands `GaloisGroup` and `GaloisSubgroup` do not return proven results we still have to rigorously prove $c = c^*$ which will be accomplished in the following way:

Let P^* be the polynomial P where c is replaced by c^*. Set

$$t_\square := \begin{cases} c^*\lambda^2 + 1 & \text{in case 1} \\ c^*\lambda^2 & \text{in case 2} \\ \frac{1}{1-c^*\lambda^2} & \text{in case 3} \end{cases}$$

for any $\lambda \in \mathbb{Q}$ such that $t_\square \notin \{0, 1, \infty\}$ then $P^*(t_\square)$ is a square in $\mathbb{Q}$. If we somehow confirm that $\mathrm{Gal}(p(X) - t_\square q(X) \mid \mathbb{Q})$ is a subgroup of H we find that $P(t_\square)$ must be a rational square. Therefore,

$$\frac{P(t_\square)}{P^*(t_\square)} = \frac{c^*}{c}$$

is also a rational square. As both c and c^* are square-free integers the latter yields $\frac{c^*}{c} = 1$, hence $c = c^*$.

CHAPTER 3

Known methods for Belyi map computation

In their survey [**41**] Sijsling and Voight discuss various techniques for computing Belyi maps, including a Gröbner basis approach, complex analytic techniques, modular form calculations and p-adic methods.

This chapter depicts some of the well established techniques for computing Belyi maps with prescribed ramification that helped us developing the new computation method presented in chapter 4.

3.1. Gröbner basis method

The standard method for explicitly computing Belyi maps with prescribed ramification consist of Gröbner basis calculations:

Let $f \in \mathbb{C}(X)$ be a Belyi map of degree n with ramification triple

$$(\sigma_0, \sigma_1, \sigma_\infty) \in {S_n}^3.$$

After applying a suitable inner Möbius transformation we may assume that $f(\infty) \in \mathbb{C} \setminus \{0, 1\}$. Therefore, f is of type

$$f = c \cdot \frac{p}{q} = 1 + (c-1) \cdot \frac{r}{q} \tag{1}$$

for some scalar c and polynomials p, q and r satisfying the following factorization conditions:

- If σ_0 has cycle structure $(\alpha_1^{A_1}, \ldots, \alpha_u^{A_u})$ then there exist monic polynomials $p_k \in \mathbb{C}[X]$ with $\deg(p_k) = A_k$ for $k = 1, \ldots, u$ such that
$$p = {p_1}^{\alpha_1} \cdot \cdots \cdot {p_u}^{\alpha_u}.$$
- If σ_1 is of cycle structure $(\beta_1^{B_1}, \ldots, \beta_v^{B_v})$ then there exist monic polynomials $r_k \in \mathbb{C}[X]$ with $\deg(r_k) = B_k$ for $k = 1, \ldots, v$ and
$$r = {r_1}^{\beta_1} \cdot \cdots \cdot {r_v}^{\beta_v}.$$

- If σ_∞ is of cycle structure description $(\gamma_1^{C_1}, \ldots, \gamma_w^{C_w})$ then there are monic polynomials $q_k \in \mathbb{C}[X]$ with $\deg(q_k) = C_k$ for $k = 1, \ldots, w$ and
$$q = {q_1}^{\gamma_1} \cdot \cdots \cdot {q_w}^{\gamma_w}.$$

Plugging the factorization of p, r and q into (1) yields

$$c \cdot {p_1}^{\alpha_1} \cdot \cdots \cdot {p_u}^{\alpha_u} - (c-1) \cdot {r_1}^{\beta_1} \cdot \cdots \cdot {r_v}^{\beta_v} - {q_1}^{\gamma_1} \cdot \cdots \cdot {q_w}^{\gamma_w} = 0. \tag{2}$$

If we compare the coefficients of (2) and consider c as well as the coefficients of the monic polynomials $p_1, \ldots, p_u$, $r_1, \ldots r_v$, $q_1, \ldots, q_w$ as unknowns we obtain a system of non-linear polynomial equations consisting of n equations and $n+3$ unknowns.

From a theoretical perspective the solutions of this system are obtainable by a Gröbner base computation using Buchberger's algorithm allowing us to calculate Belyi maps with prescribed ramification.

However, the Gröbner base approach comes with various problems:

- The explicit Gröbner base calculation quickly becomes rather expensive as the permutation degree rises: computer experiments have shown that the implementation feasibility reaches its limit at the permutation degree somewhere around 20 using a clever differentiation trick by Atkin and Swinnerton-Dyer, see [**41**, Chapter 2].
- The solutions of (2) include all Belyi maps with ramification triple that only fit the cycle structure description of $(\sigma_0, \sigma_1, \sigma_\infty)$ but not necessarily have ramification triple $(\sigma_0, \sigma_1, \sigma_\infty)$. In particular we may obtain Belyi maps with different monodromy groups. From all of these solutions one has to find the desired one.
- The polynomial system (2) also contains *parasitic solutions*, i.e. solutions in which least two of the polynomials $p_1, \ldots, p_u$, $r_1, \ldots r_v$, $q_1, \ldots, q_w$ have common complex roots. Parasitic solutions obviously do not correspond to Belyi maps with prescribed ramification triple $(\sigma_0, \sigma_1, \sigma_\infty)$.

3.2. Computing Shabat polynomials

Let $(\sigma_0, \sigma_1, \sigma_\infty)$ be the ramification triple of a degree-n Belyi map $f \in \mathbb{C}[X]$, also called *Shabat polynomial*. Then σ_∞ must be equal to an n-cycle and the dessin of f turns out to be a tree in $\mathbb{P}^1\mathbb{C}$.

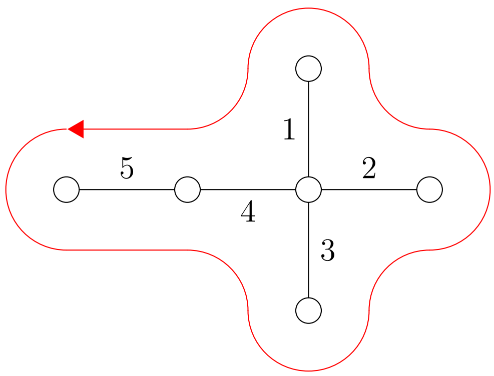
In 2014 Bishop [**12**] showed that this tree is balanced with respect to harmonic measure. This result can be interpreted in the following way: a particle starting at ∞ travelling along a random path towards the tree is equally likely to hit any edge of the tree on either side. In particular, as the complement of such a tree in $\mathbb{P}^1\mathbb{C}$ is simply connected we can conformally map it onto $\mathbb{D}$ to find out that the vertices of the tree are mapped uniformly distributed to $\partial\mathbb{D}$. This fact can be used to explicitly compute the dessins of Shabat polynomials with prescribed ramification triple.

With the help of conformal mappings Marshall and Rohde, see [**31**], were able to explicitly compute the dessins of Shabat polynomials with thousands of edges using Marshall's *Zipper* algorithm. A Matlab implementation by Barnes can be found in [**3**].

Example 3.1. Following Bishop's approach given in [**12**, Chapter 1] we construct an approximate dessin of a Shabat polynomial with ramification triple $(\sigma_0, \sigma_1, \sigma_\infty)$ where

$$\begin{aligned}\sigma_0 &= (1,4,3,2),\\ \sigma_1 &= (4,5),\\ \sigma_\infty &= (1,2,3,4,5).\end{aligned}$$

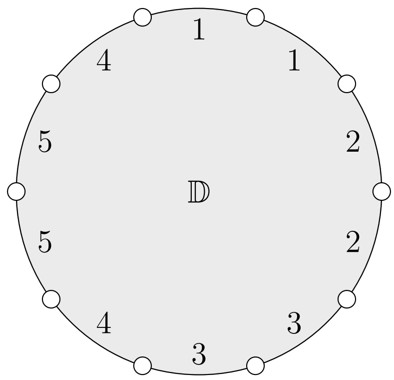
Our first step consists of sketching the expected dessin:



Now, if we walk counter-clockwise around the dessin and write down the edges we pass by along the way we obtain

$$(1, 4, 5, 5, 4, 3, 3, 2, 2, 1).$$

For the next step we draw the unit disc $\mathbb{D}$, divide it into 10 (= length of the circular walk) equal segments and label them in such a way that they fit the above circular walk:

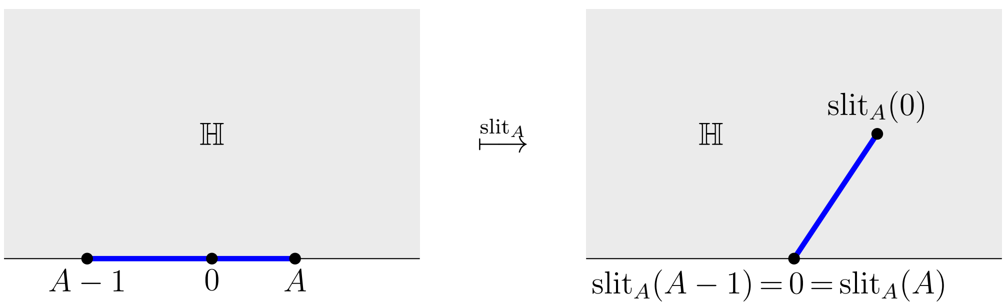


In the following step we conformally map the above labelled unit disc onto $\mathbb{H}$ and weld adjacent real line segments having the same number using slit maps of the following type:

$$\mathrm{slit}_A : \begin{cases} \mathbb{H} \to \mathbb{H} \\ z \mapsto (z - A)^A (z + 1 - A)^{A-1} \end{cases}$$

where $0 < A < 1$ is a real number.

It is easy to see that slit_A welds the line segments $[A - 1, 0]$ and $[0, A]$ into $\overline{\mathbb{H}}$ which is visualized in the figure below:



The algorithmic approach for welding all the line segments having the same number can be described in the following way:

(i) Pick two adjacent real line segments having the same number.

(ii) Map these line segments onto $[A-1,0]$ and $[0,A]$ using an automorphism of $\mathbb{H}$ and then apply slit_A.

Note that A is picked in such a way that the corresponding edges of the resulting dessin form a star configuration.

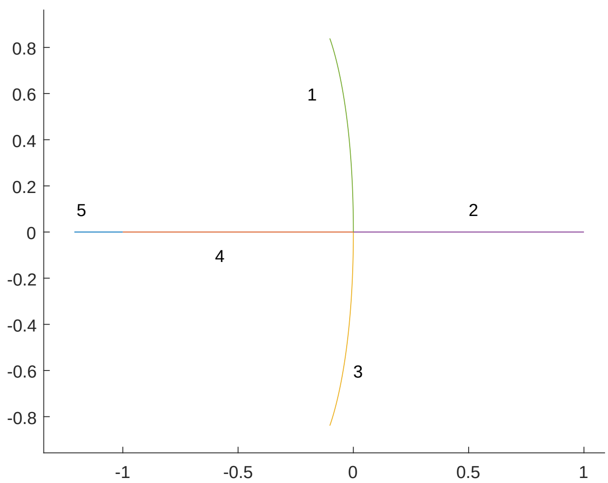
(iii) If there are at least 2 pairs of edges left that needed to be welded jump to step (i), otherwise continue.

(iv) Map $\mathbb{H}$ conformally onto $\mathbb{D}$ such that the upper semicircle of $\partial\mathbb{D}$ has to be glued to the lower semicircle. Then then last welding step can be achieved by applying the conformal map

$$\omega : \begin{cases} \mathbb{D} \to \mathbb{P}^1\mathbb{C} \\ z \mapsto z + \dfrac{1}{z} \end{cases}$$

with the following obvious properties:
- $\omega(\mathbb{D}) = (\mathbb{C} \cup \{\infty\}) \setminus [-2,2]$.
- $\omega(z) = 2 \cdot \text{Real}(z)$ for $z \in \partial\mathbb{D}$.

The resulting welded line segments become the edges of an approximate dessin contained in $\mathbb{P}^1\mathbb{C}$ corresponding to $(\sigma_0, \sigma_1, \sigma_\infty)$. After applying a Möbius transformation we obtain the following dessin:



Using the coordinates of the zeros and ones, which can be read off from the constructed dessin, one then can compute the corresponding Shabat polynomial using Newton's method for the system (2) from section 3.1.

3.3. Computing Belyi maps using modular forms

In [**23**] Klug, Musty, Schiavone and Voight describe a method using modular forms for computing Belyi maps with *hyperbolic* ramification triples up to degree 50. Additional computation results are also presented in [**38**].

Using a similar approach Monien was also able to explicitly realize the sporadic Janko group J_2 of degree 100 and the sporadic Conway group Co_3 of degree 276, see [**36**] and [**37**].

In this section we give a brief description on how modular forms can be used to explicitly compute Belyi maps with prescribed ramification, see also [**23**] for a more detailed explanation.

3.3.1. Belyi maps between orbit spaces.

Let $(\sigma_0, \sigma_1, \sigma_\infty)$ be a genus-0 triple for a transitive subgroup of S_n and

$$a := \mathrm{ord}(\sigma_0), \qquad b := \mathrm{ord}(\sigma_1), \qquad c := \mathrm{ord}(\sigma_\infty).$$

Furthermore assume that $(\sigma_0, \sigma_1, \sigma_\infty)$ is *hyperbolic*, i.e.

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1.$$

The corresponding triangle group

$$\Delta := \left\langle \delta_a, \delta_b, \delta_c \,|\, \delta_a^a = \delta_b^b = \delta_c^c = \delta_c\delta_b\delta_a = 1 \right\rangle$$

comes with the embedding

$$e_\Delta : \Delta \to \mathrm{SL}_2(\mathbb{R})$$

such that

$$\delta_a \mapsto \begin{pmatrix} \cos\left(\frac{\pi}{a}\right) & \sin\left(\frac{\pi}{a}\right) \\ -\sin\left(\frac{\pi}{a}\right) & \cos\left(\frac{\pi}{a}\right) \end{pmatrix} \qquad \text{and} \qquad \delta_b \mapsto \begin{pmatrix} \cos\left(\frac{\pi}{b}\right) & \mu\sin\left(\frac{\pi}{b}\right) \\ -\frac{1}{\mu}\sin\left(\frac{\pi}{b}\right) & \cos\left(\frac{\pi}{b}\right) \end{pmatrix}$$

where

$$\lambda := \frac{\cos\left(\frac{\pi}{a}\right)\cos\left(\frac{\pi}{b}\right) + \cos\left(\frac{\pi}{c}\right)}{2\sin\left(\frac{\pi}{a}\right)\sin\left(\frac{\pi}{b}\right)} \qquad \text{and} \qquad \mu := \lambda + \sqrt{\lambda^2 - 1}.$$

Since we have a natural action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ given by

$$\theta : \begin{cases} \mathrm{SL}_2(\mathbb{R}) \to \mathrm{Aut}(\mathbb{H}) \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \left(z \mapsto \dfrac{Az+B}{Cz+D} \right) \end{cases}$$

we may also let Δ act on $\mathbb{H}$ via:

$$\begin{cases} \Delta \to \mathrm{Aut}(\mathbb{H}) \\ \delta \mapsto (\theta \circ e_\Delta)(\delta) \end{cases}$$

Using the natural epimorphism

$$\varphi : \begin{cases} \Delta \to G := \langle \sigma_0, \sigma_1 \rangle \\ \text{where } \delta_a \mapsto \sigma_0 \text{ and } \delta_b \mapsto \sigma_1 \end{cases}$$

let Γ the pre-image of a point stabilizer of G under φ.

The central object of investigation is then given by the following mapping:

$$\Phi : \begin{cases} \mathbb{H}/\Gamma \to \mathbb{H}/\Delta \\ [z]_\Gamma \mapsto [z]_\Delta \end{cases}$$

Here $\mathbb{H}/\Gamma$ and $\mathbb{H}/\Delta$ denote the orbit spaces of $\mathbb{H}$ modulo Γ and Δ, respectively. By construction the genera of both $\mathbb{H}/\Gamma$ and $\mathbb{H}/\Delta$ are equal to 0.

The map Φ describes a degree-n cover ramified only over

$$[i]_\Delta, \quad [\mu i]_\Delta \quad \text{and} \quad [\gamma]_\Delta$$

with monodromy group $G = \langle \sigma_0, \sigma_1 \rangle$ where

$$\begin{aligned} \gamma := & \frac{\mu^2 - 1}{2(\cot\left(\frac{\pi}{a}\right) + \mu \cot\left(\frac{\pi}{b}\right))} \\ & + i \sqrt{\csc^2\left(\frac{\pi}{a}\right) - \left(\frac{\mu^2 - 1}{2(\cot\left(\frac{\pi}{a}\right) + \mu \cot\left(\frac{\pi}{b}\right))} - \cot\left(\frac{\pi}{a}\right) \right)^2}. \end{aligned}$$

The functional inverse of Φ can be locally expressed as the quotient of two holomorphic functions that form a basis for the vector space of solutions to a second order linear differential equation with well studied solutions. As a consequence, a power series expression of Φ can be obtained.

3.3.2. Modular forms.

Recall that a *modular form* for Γ of *weight* $k \in 2\mathbb{Z}$ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ such that

$$f(\gamma z) = j(\gamma, z)^k f(z) \qquad \text{for all } z \in \mathbb{H} \text{ and all } \gamma \in \Gamma \tag{3}$$

where

$$j(\gamma, z) := cz + d \qquad \text{for} \qquad \gamma = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \in \mathrm{PSL}_2(\mathbb{R}).$$

In a natural way the set of all modular forms for Γ of weight k, denoted by $S_k(\gamma)$, forms a $\mathbb{C}$-vector space. The dimension of $S_k(\Gamma)$ can be computed explicitly using the famous Riemann-Roch theorem.

In order to compute any element in $S_k(\Gamma)$ write it in its Taylor series expression and compare the coefficients of both sides in (3) after plugging in a sufficient amount of $z \in \mathbb{H}$ and $\gamma \in \Gamma$. A truncation of the desired power series equation is therefore obtainable by solving a system of linear equations.

For the explicit computation of Φ as a rational function pick $k \in \mathbb{N}$ such that $S_k(\Gamma)$ is at least 2-dimensional. Then two linearly independent elements of $S_k(\Gamma)$ can be used to describe an isomorphism

$$\iota : \mathbb{H}/\Gamma \to \mathbb{P}^1\mathbb{C}.$$

The power series expression of Φ in combination with ι then allows the computation of the desired Belyi map as a rational function.

CHAPTER 4

A new method for computing Belyi maps

In this chapter we explain our computation method that allows the explicit realization of Belyi maps with prescribed ramification triples. Using this particular method we are able to calculate Belyi maps of degree up to 280. The main idea combines elements from the methods described in the sections 3.2 and 3.3.

Let $(\sigma_0, \sigma_1, \sigma_\infty)$ be a prescribed hyperbolic genus-0 triple for a transitive group. Our main goal is to explicitly compute a Belyi map

$$f = \frac{p}{q} = 1 + \frac{r}{q}$$

with ramification triple $(\sigma_0, \sigma_1, \sigma_\infty)$.

The basic algorithm can be divided into the following four steps:

(i) Construct Φ from section 3.3.

(ii) Transform the dessin of Φ to an approximated dessin of f contained in $\mathbb{P}^1\mathbb{C}$.

(iii) Compute the coefficients of f by using the coordinates of the zeros, ones and poles of the constructed approximate dessin.

(iv) Verify that the computed equation for f indeed describes a Belyi map with ramification triple $(\sigma_0, \sigma_1, \sigma_\infty)$.

In order to demonstrate our algorithm we give a detailed documentation of the explicit realization of the sporadic group Aut(HS) of degree 100, see the gray boxes in the upcoming pages. Note that segments of this documentation are taken over from [**10**, Chapter 3].

Aut(HS)-realization, Part 1.
A genus-0 triple $(\sigma_0, \sigma_1, \sigma_\infty)$ for Aut(HS) of degree 100 is given by

$$\begin{aligned}
\sigma_0 \ &= (1,23,53,86)(2,36,29,43)(3,15,46,6)(4,80,71,81)(5,75,16,47)\\
&\quad (7,32,60,8)(9,76,100,51)(10,50,49,34)(11,28,74,84)\\
&\quad (12,72,37,52)(13,21,96,88)(14,41,40,87)(17,42,45,79)\\
&\quad (18,63,19,20)(22,99,39,89)(24,59,77,38)(25,68,26,35)\\
&\quad (27,69,73,48)(30,92,33,82)(31,56,93,58)(44,98,67,64)\\
&\quad (54,95,85,62)(55,65,94,61)(57,78,83,97)(66,90,70,91),\\
\sigma_1 \ &= (1,3)(2,99)(4,24)(6,18)(7,53)(8,42)(11,30)(12,96)(13,31)(15,80)\\
&\quad (17,38)(19,83)(20,100)(21,69)(22,70)(25,67)(27,28)(29,52)(32,93)\\
&\quad (33,76)(34,65)(35,92)(36,94)(37,89)(40,82)(41,46)(44,84)(45,90)\\
&\quad (47,71)(51,97)(58,68)(61,88)(72,79)(75,86)(91,95),\\
\sigma_\infty &= (1,6,20,76,92,26,68,93,7,23)(2,22,90,42,60,32,56,31,88,94)\\
&\quad (3,86,5,47,80)(4,38,79,12,21,27,11,82,41,15)(19,78,57,97,100)\\
&\quad (8,17,77,59,24,81,71,16,75,53)(9,51,83,63,18,46,14,87,40,33)\\
&\quad (10,34,55,61,96,52,36,65,49,50)(13,58,25,98,44,74,28,48,73,69)\\
&\quad (29,37,39,99,43)(30,84,64,67,35)(45,66,91,54,62,85,95,70,89,72).
\end{aligned}$$

This triple has the following cycle structure description:

	σ_0	σ_1	σ_∞
cycle structure	4^{25}	$2^{35}.1^{30}$	$10^8.5^4$

Note that $(\sigma_0, \sigma_1, \sigma_\infty)$ is rigid and each permutation is contained in a rational conjugacy class of Aut(HS). According to Theorem 2.8 there exists a Belyi map

$$f = \frac{p}{q} \in \mathbb{Q}(X)$$

with ramification triple $(\sigma_0, \sigma_1, \sigma_\infty)$ such that

$$p(X) - tq(X) \in \mathbb{Q}(t)[X]$$

defines a regular extension of $\mathbb{Q}(t)$ with Galois group Aut(HS).

4.1. Preparations

We stick to the notations from section 3.3 with the only exception of replacing $\mathbb{H}$ with $\mathbb{D}$ using the following conformal map:

$$w : \begin{cases} \mathbb{H} \to \mathbb{D} \\ z \mapsto \dfrac{z-i}{z+i} \end{cases}$$

This enables us to consider Δ as a subgroup of $\operatorname{Aut}(\mathbb{D})$. The elements δ_a, δ_b and δ_c each describe hyperbolic rotations in $\mathbb{D}$ with the below properties:

hyberbolic rotation	center	angle
δ_a	$m_a := w(i) = 0$	$\frac{2\pi}{a}$
δ_b	$m_b := w(\mu i) = \frac{\mu-1}{\mu+1} \in \mathbb{R}$	$\frac{2\pi}{b}$
δ_c	$m_c := w(\gamma)$	$\frac{\pi}{c}$

Furthermore, instead of Φ from 3.3.1 we will work with:

$$\Phi_{\mathbb{D}} : \begin{cases} \mathbb{D}/\Gamma \to \mathbb{D}/\Delta \\ [z]_\Gamma \mapsto [z]_\Delta \end{cases}$$

Note that the orbit spaces $\mathbb{D}/\Gamma$ and $\mathbb{D}/\Delta$ are both genus-0 Riemann surfaces and the map $\Phi_{\mathbb{D}}$ shares similar properties as Φ: It is a degree-n cover ramified over $[m_a]_\Delta, [m_b]_\Delta$ and $[m_c]_\Delta$ with ramification triple $(\sigma_0, \sigma_1, \sigma_\infty)$.

We will define the *dessin* of $\Phi_{\mathbb{D}}$ to be the set

$$\left\{ \Phi_{\mathbb{D}}^{-1}([z]_\Delta) : z \in [m_a, m_b] \right\}.$$

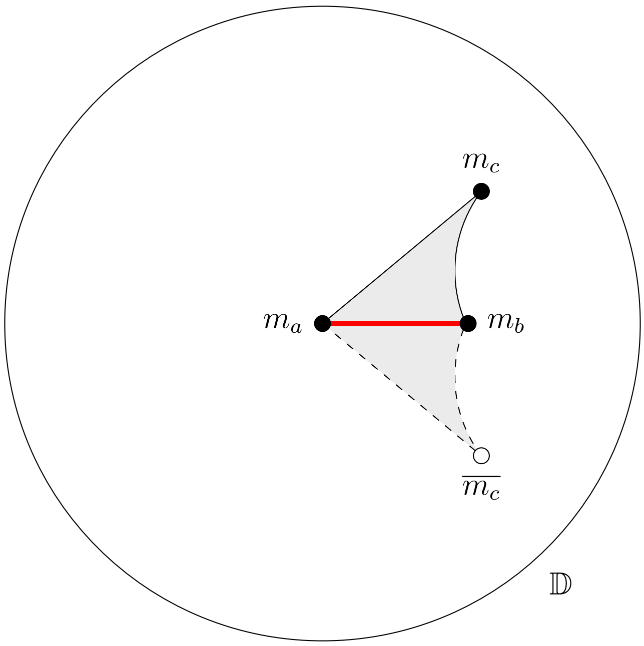
Remark 4.1. The latter definition coincides with the traditional definition of dessins to be the pre-image of $[0,1]$. As $\mathbb{D}/\Delta$ is of genus 0 there exists a homeomorphism $\mathbb{D}/\Delta \to \mathbb{P}^1\mathbb{C}$ with $[m_a]_\Delta \mapsto 0$, $[m_b]_\Delta \mapsto 1$ and $[m_c]_\Delta \to \infty$. Using this identification the notion of being the pre-image of $[0,1]$ then translates to being the pre-image of $[m_a, m_b]$.

4.2. Fundamental domains

A crucial part for the upcoming dessin construction process heavily relies on the choice of sufficiently nice fundamental domains for the orbit spaces $\mathbb{D}/\Delta$ and $\mathbb{D}/\Gamma$.

4.2.1. Fundamental domain for $\mathbb{D}/\Delta$.

We will consider the hyperbolic kite $\diamondsuit$ in $\mathbb{D}$ with vertices m_a, $\overline{m_c}$, m_b, m_c containing the interior as well as the closed edges connecting m_a with m_c and m_c with m_b:



According to the geometric properties of δ_a, δ_b and δ_c from the previous section the inside angles of $\diamondsuit$ turn out to be $\frac{2\pi}{a}$, $\frac{2\pi}{b}$ and $\frac{\pi}{c}$.

It is easy to show that for every $z \in \mathbb{D}$ there exists an automorphism $\delta \in \Delta$ and a unique $z^* \in \diamondsuit$ such that

$$z^\delta = z^*.$$

Therefore, $\diamondsuit$ is a fundamental domain for the orbit space $\mathbb{D}/\Delta$.

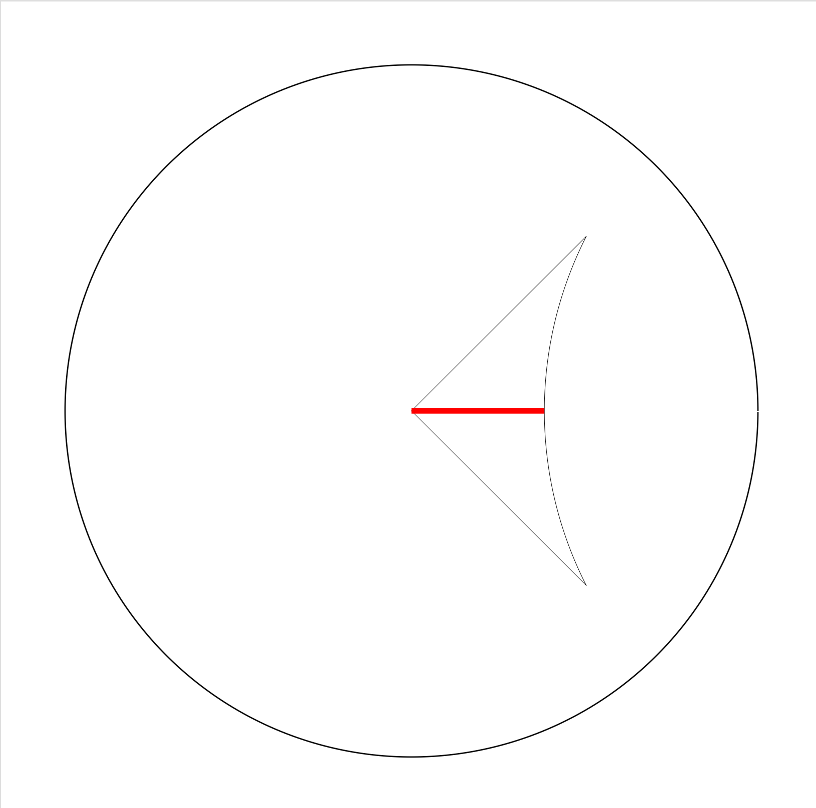
Additionally note that the pre-image of the red line in the above figure belongs to the dessin of $\Phi_{\mathbb{D}}$.

Aut(HS)-realization, Part 2.

For the triple $(\sigma_0, \sigma_1, \sigma_\infty)$ from Part 1 we have

$$(a, b, c) = (4, 2, 10).$$

Then a fundamental domain $\diamondsuit$ for $\mathbb{D}/\Delta$ is visualized in the following figure:



Note that in the case $b = 2$ the hyperbolic kite $\diamondsuit$ turns out to be a hyperbolic triangle.

4.2.2. Fundamental domain for $\mathbb{D}/\Gamma$.

We will enumerate the images of $\diamondsuit$ under elements of Δ in the following way:

$$\text{label}(\diamondsuit^{\gamma}) := 1^{\varphi(\gamma)}.$$

Obviously, this kind of labelling is well defined.

As $\diamondsuit$ is a fundamental domain for $\mathbb{D}/\Delta$ we can derive the following sufficient criterion for being a fundamental domain for $\mathbb{D}/\Gamma$.

Lemma 4.2. *A subset $\mathcal{D} \subseteq \mathbb{D}$ is a fundamental domain for $\mathbb{D}/\Gamma$ if there exist $\gamma_k \in \Delta$ for $k \in \{1, \dots, n\}$ such that:*

- $\mathcal{D} = \bigcup_{k \in \{1,\dots,n\}} \diamondsuit^{\gamma_k}$.
- $\text{label}(\diamondsuit^{\gamma_k}) = k$ *for* $k \in \{1, \dots, n\}$.

For $\gamma \in \Delta$ the corresponding σ_0*-petal* and σ_1*-petal* are defined to be

$$\bigcup_{k \in \{1,\dots,a\}} \left(\diamondsuit^{(\delta_a{}^k)}\right)^{\gamma} \qquad \text{and} \qquad \bigcup_{k \in \{1,\dots,b\}} \left(\diamondsuit^{(\delta_b{}^k)}\right)^{\gamma}.$$

If the context is clear we just call them *petals*. A *picked* σ_0*-petal* and a *picked* σ_1*-petal* are defined to be of type

$$\bigcup_{k \in \{1,\dots,a\} \setminus \{s,\dots,t\}} \left(\diamondsuit^{(\delta_a{}^k)}\right)^{\gamma} \qquad \text{and} \qquad \bigcup_{k \in \{1,\dots,b\} \setminus \{s,\dots,t\}} \left(\diamondsuit^{(\delta_b{}^k)}\right)^{\gamma}$$

for some $\gamma \in \Delta$ and $s, t \in \mathbb{N}$ such that:

- $a > |\{1, \dots, a\} \setminus \{s, \dots, t\}| \geq 2$.
- $b > |\{1, \dots, b\} \setminus \{s, \dots, t\}| \geq 2$.

Again, if the context is clear we just call them *picked petals*. If a petal is not equal to a picked petal we call it a *full petal*.

For our approach we require sufficiently nice connected fundamental domains $\mathcal{D}$ for $\mathbb{D}/\Gamma$, see Remark 4.4. We will use the following:

- We say that $\mathcal{D}$ is k*-picked* or *picked* if $\mathcal{D}$ is equal to a union of full petals and exactly k picked petals where the intersection of two distinct petals is either empty or equal to $\diamondsuit^{\gamma}$ for some $\gamma \in \Delta$.
- $\mathcal{D}$ is called *nice* if $\mathcal{D}$ is 0-picked.
- An entry $w \in \{1, \dots, n\}$ of a permutation $\sigma \in S_n$ is called *bad* if

$$|w^{\langle\sigma\rangle}| \neq \text{ord}(\sigma).$$

The following lemma guarantees that for all of the triples appearing in this dissertation we find a sufficiently nice fundamental domain.

Lemma 4.3. *Let* $(\sigma_0, \sigma_1, \sigma_\infty) \in {S_n}^3$ *be a hyperbolic genus-0 triple with*

$$ord(\sigma_1) = 2.$$

Furthermore, let k *be the number of cycles of* σ_0 *that contain bad entries which are fixed by* σ_1. *Then there exists a connected fundamental domain* $\mathcal{D}$ *for* $\mathbb{D}/\Gamma$ *which is* k*-picked.*

Proof. Consider the directed Graph $\mathcal{G} = (V, E)$ where

$$V := \text{set of cycles of } \sigma_0$$

and

$$E := \{(v_1, v_2) \in V \times V : v_1 \neq v_2 \text{ and } \sigma_1(v_1) \cap v_2 \neq \emptyset\}.$$

As $\mathrm{ord}(\sigma_1) = 2$ we obviously have $(v_2, v_1) \in E$ for $(v_1, v_2) \in E$. This allows us to consider $\mathcal{G}$ as an undirected graph. As $(\sigma_0, \sigma_1, \sigma_\infty)$ is a transitive triple the graph $\mathcal{G}$ must be connected.

We now pick a spanning tree of $\mathcal{G}$ which yields a connected ℓ-picked fundamental domain, were ℓ denotes the number of cycles of σ_0 having cycle length smaller than $\mathrm{ord}(\sigma_0)$ and greater than 1.

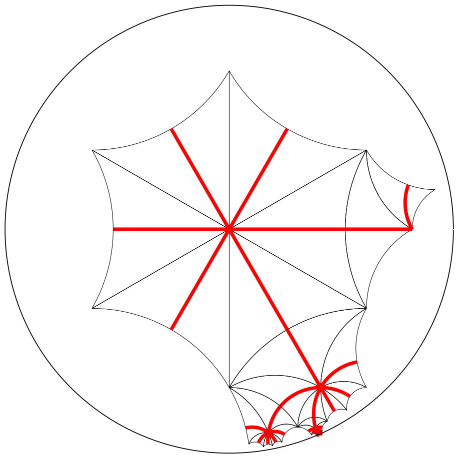
We can improve the value of ℓ arising from all picked σ_0-petals that belong to cycles that only contain bad entries which are not fixed by σ_1. The reason for that lies in the fact any picked σ_0-petal with this property can be replaced by full σ_1-petals attached somewhere else. A proper rearranging process then yields a connected k-picked fundamental domain for $\mathbb{D}/\Gamma$. $\square$

It is worth pointing out that the proof of the previous lemma gives an explicit instruction for computing k-picked or nice fundamental domains with k being small.

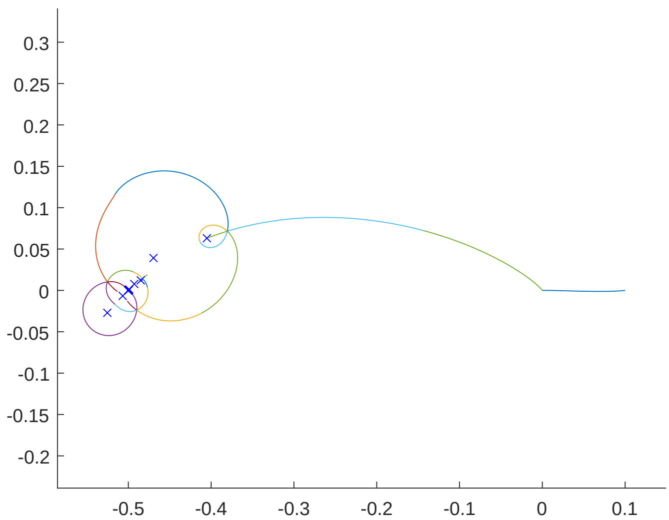
Also note that this lemma is also the reason why we permute (if possible!) all triples in such a way that the second element is always of order 2.

Remark 4.4. The significance of nice fundamental domain lies in the fact that using our computation method these will lead to approximate dessins that at least fulfil the expected geometric properties of the desired dessin. If we instead have to work with a k-picked fundamental domain the resulting approximate dessin will always have at least k wrong angles.

Assume we are given the following 1-picked fundamental domain where the picked petal can be found to the right of the center:



Then the upcoming welding process will lead to an approximate dessin where the edges at the coordinates $(0, 0)$ do not meet at an expected angle of π:



Aut(HS)-realization, Part 3.

Recall that the cycle structures of σ_0 and σ_1, see Part 1, are given by:

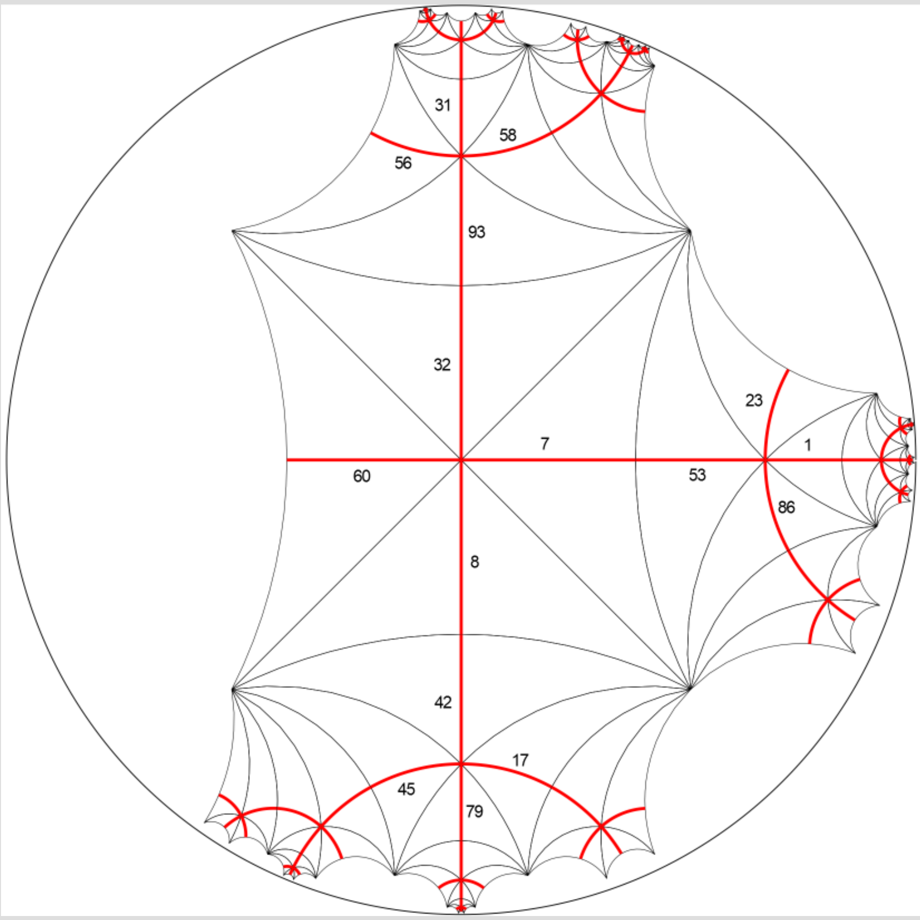
	σ_0	σ_1
cycle structure	4^{25}	$2^{35}.1^{30}$

According to Lemma 4.3 there exists a nice fundamental domain for $\mathbb{D}/\Gamma$ which we will now construct explicitly:

We begin with the arbitrarily chosen cycle $(7, 32, 60, 8)$ from σ_0 yielding the starting σ_0-petal centred at the origin in $\mathbb{D}$. Since

$$\sigma_1(7) = 53, \quad \sigma_1(32) = 93, \quad \sigma_1(60) = 60, \quad \sigma_1(8) = 42$$

its neighbours are given by $(1, 23, 53, 86)$, $(31, 56, 93, 58)$ and $(17, 42, 45, 79)$. The corresponding σ_0-petals will be attached accordingly. Repeating this procedure eventually yields a nice fundamental domain:



4.3. Obtaining an approximate dessin

Our goal is to transform the dessin of $\Phi_{\mathbb{D}}$ into a dessin of $\mathbb{P}^1\mathbb{C}$ using conformal mappings as they leave regular star configurations invariant.

4.3.1. Mapping a fundamental domain conformally onto $\mathbb{H}$.

Assume that $\mathcal{D}$ is a fundamental domain for $\mathbb{D}/\Gamma$ such that $\mathcal{D}^\circ$ is simply connected and $\mathcal{D}$ is of type

$$\mathcal{D} = \bigcup_{k\in\{1,\dots,n\}} \diamondsuit^{\gamma_k}$$

with $\diamondsuit$ being a hyperbolic kite and $\gamma_k \in \Delta$ for $k \in \{1,\dots,n\}$. Thanks to Riemann's mapping theorem we can conformally map $\mathcal{D}^\circ$ onto $\mathbb{H}$.

Among all the techniques to explicitly compute such a map we choose to work with the *Schwarz-Christoffel mapping* which conformally maps $\mathbb{H}$ onto the interior of a given polygon, see for example [**16**, Theorem 2.1].

With the help of the Schwarz-Christoffel toolbox [**15**] implemented in Matlab we are able to map $\mathcal{D}^\circ$ onto $\mathbb{H}$ in the following way:

The border $\partial\mathcal{D}$ consists of circular line segments which we individually approximate by a polygonal chain of length 2 with equidistant vertices, yielding a polygon P. Via the Schwarz-Christoffel toolbox we can compute a numerical approximation

$$\mathcal{W}_{\text{approx}} : P \to \mathbb{H}$$

of a conformal map

$$\mathcal{W} : \mathcal{D} \to \mathbb{H}$$

as well as its inverse.

We call two points $z_1, z_2 \in \overline{\mathbb{H}}$ equivalent modulo $\Gamma_{\mathcal{W}}$ if $\mathcal{W}^{-1}(z_1)$ and $\mathcal{W}^{-1}(z_2)$ are equivalent modulo Γ yielding an equivalence relation $\Gamma_{\mathcal{W}}$ on $\overline{\mathbb{H}}$.

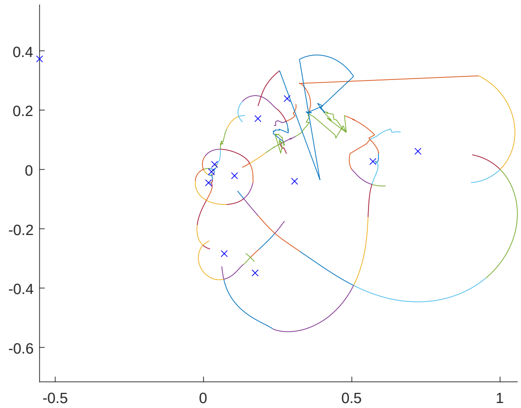
The image of $\overline{\mathcal{D}}$ under $\mathcal{W}$ respecting the quotient structure dictated by Γ can be computed as

$$\mathcal{W}(\overline{\mathcal{D}}/\Gamma) = \mathcal{W}(\overline{\mathcal{D}})/\Gamma_{\mathcal{W}} = \mathcal{W}(\mathcal{D}) \cup (\mathcal{W}(\partial\mathcal{D})/\Gamma_{\mathcal{W}}) = \mathbb{H} \cup (\partial\mathbb{H}/\Gamma_{\mathcal{W}})\,.$$

As P and $\mathcal{W}_{\text{approx}}$ approximate $\mathcal{D}$ and $\mathcal{W}$ the latter formula suggests a quotient relation $\Gamma_{\mathcal{W}_{\text{approx}}}$ on $\partial\mathbb{H}$ that approximates $\mathcal{W}(\overline{\mathcal{D}}/\Gamma)$:

$$\mathcal{W}(\overline{\mathcal{D}}/\Gamma) \approx \mathcal{W}_{\text{approx}}\left(\overline{P}/\Gamma\right) = \mathbb{H} \cup \left(\partial\mathbb{H}/\Gamma_{\mathcal{W}_{\text{approx}}}\right).$$

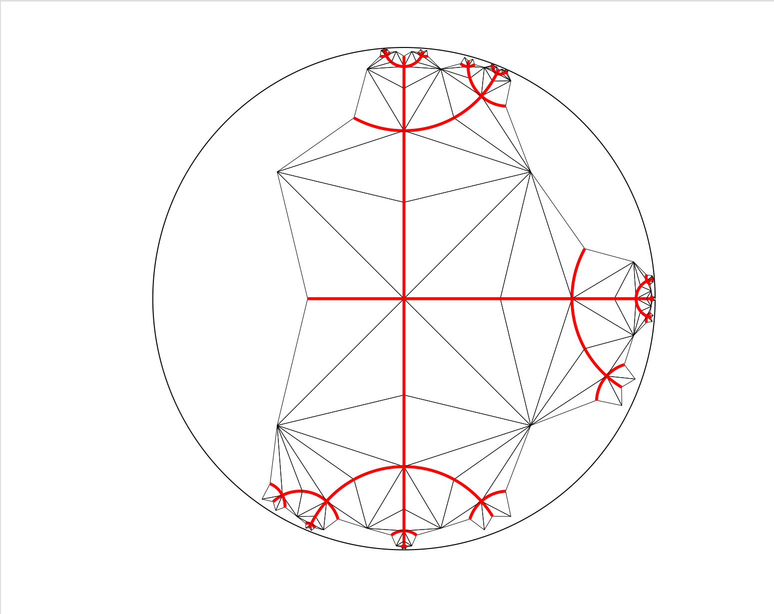
Remark 4.5. As the number of edges rises and the shapes of fundamental domains get more irregular the Schwarz-Christoffel toolbox in Matlab struggles to compute approximation of conformal maps. In a significant number of examples the implementation fails to determine the Schwarz-Christoffel mapping or is only capable of computing conformal maps of low numerical precision such that the computed results are useless for further calculations. An example of a complete dessin computation with a failed attempt at calculating the Schwarz-Christoffel map is shown in the following figure:



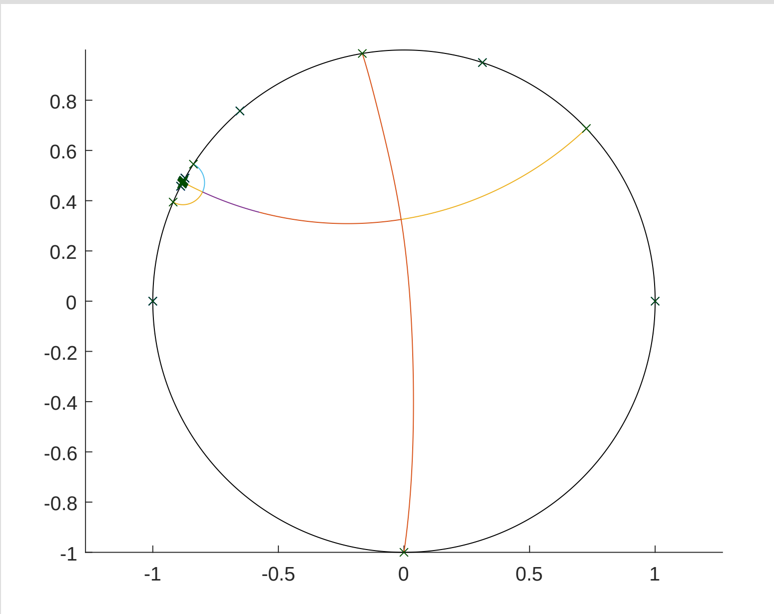
If we ever had to deal with issues like this we simply chose other fundamental domains even if they turn out to be not nice.

Alternatively one could also apply the Zipper algorithm [**31**] in order to compute a conformal map from a fundamental domain onto $\mathbb{H}$ avoiding all the issues that come with the Schwarz-Christoffel toolbox.

Aut(HS)-realization, Part 4.
Our approximated fundamental domain constructed in Part 3 is shown in the following figure:



The conformal image of the above polygon onto $\mathbb{D}$ (instead of $\mathbb{H}$) is visualized below.



4.3.2. Conformal welding.

For the sake of simplicity we will write $\Gamma_{\mathcal{W}}$ instead of $\Gamma_{\mathcal{W}_{\text{approx}}}$.

Our next goal is to find an approximate conformal image of $\mathbb{H}$ onto $\mathbb{P}^1\mathbb{C}$ minus a union of slits respecting the quotient relation dictated by $\partial\mathbb{H}/\Gamma_{\mathcal{W}}$, i.e. $\Gamma_{\mathcal{W}}$-equivalent points on $\partial\mathbb{H}$ are mapped to the same point. Recall that $\partial\mathbb{H}$ is divided into pairs of equivalent line segments modulo $\Gamma_{\mathcal{W}}$.

In the following give a detailed explanation on how to weld these pairs of equivalent line segments:

4.3.2.1. *The first part of the welding process.*

The key to successfully weld adjacent line segments on $\partial\mathbb{H}$ lies in use of slit maps from section 3.2:

$$\text{slit}_A : \begin{cases} \mathbb{H} \to \mathbb{H} \\ z \mapsto (z-A)^A(z+1-A)^{A-1} \end{cases}$$

for some $A \in (0,1)$.

The algorithmic approach for the first part of the welding process consists of the following steps:

(i) Pick two adjacent real line segments that are equivalent modulo $\Gamma_{\mathcal{W}}$.

(ii) Apply an automorphism of $\mathbb{H}$ that maps these line segments onto $[A-1,0]$ and $[0,A]$ and apply slit_A.

Note that A is picked in such a way that the corresponding edges of the resulting approximate dessin form a regular star configuration.

(iii) If there are at least 2 pairs of equivalent line segments on $\partial\mathbb{H}$ remaining jump to step (i), otherwise continue with the second part of the welding process, see section 4.3.2.2.

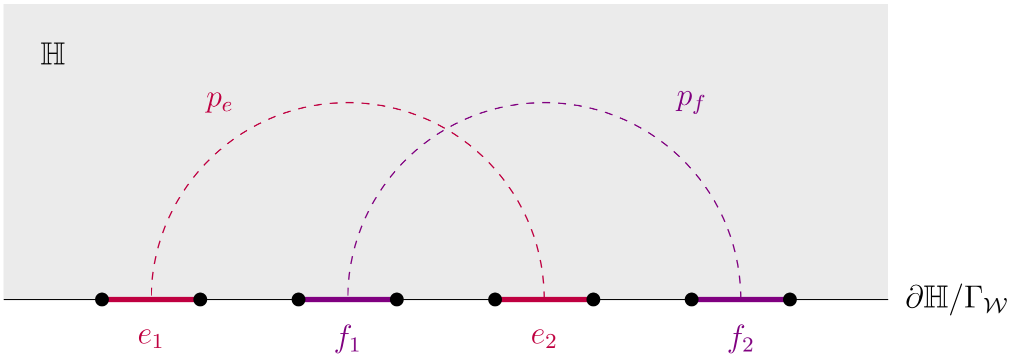
Lemma 4.6. *In step (i) we are always able to find two adjacent line segments on $\partial\mathbb{H}$ that are considered equivalent modulo $\Gamma_{\mathcal{W}}$.*

Proof. Assume this is not the case, then it is easy to see that there exist real line segments e_1, e_2, f_1, f_2 with the following properties:

(a) e_1 and e_2 are equivalent modulo $\Gamma_{\mathcal{W}}$ as well as f_1 and f_2.

(b) f_1 and f_2 are not contained in the same connected component of $\partial\mathbb{H} \setminus ({e_1}^\circ \cup {e_2}^\circ)$.

Now, let p_e be a semicircle in $\overline{\mathbb{H}}$ connecting any point of $e_1{}^\circ$ to its equivalent point in $e_2{}^\circ$. Analogously, define p_f. This configuration is sketched in the below figure:



Due to (b) the semicircles p_e and p_f intersect in exactly one point in $\mathbb{H}$. Thanks to (a) the images of p_e and p_f in $\overline{\mathbb{H}}/\Gamma_{\mathcal{W}}$ describe closed paths. These observations imply that $\overline{\mathbb{H}}/\Gamma_{\mathcal{W}}$ contains closed paths properly intersecting in exactly one point which is not possible on genus-0 Riemann surfaces. As $\overline{\mathbb{H}}/\Gamma_{\mathcal{W}}$ is of genus 0 we have a contradiction. $\square$

4.3.2.2. *The second part of the welding process.*
In order to perform the second and final part of the welding process we will apply a conformal map $\mathbb{H} \to \mathbb{D}$ in such a way that the upper semicircle of $\partial\mathbb{D}$ has to be glued to the lower semicircle. Then the last welding step can be achieved by using the conformal map

$$\omega : \begin{cases} \mathbb{D} \to \mathbb{P}^1\mathbb{C} \\ z \mapsto z + \dfrac{1}{z} \end{cases}$$

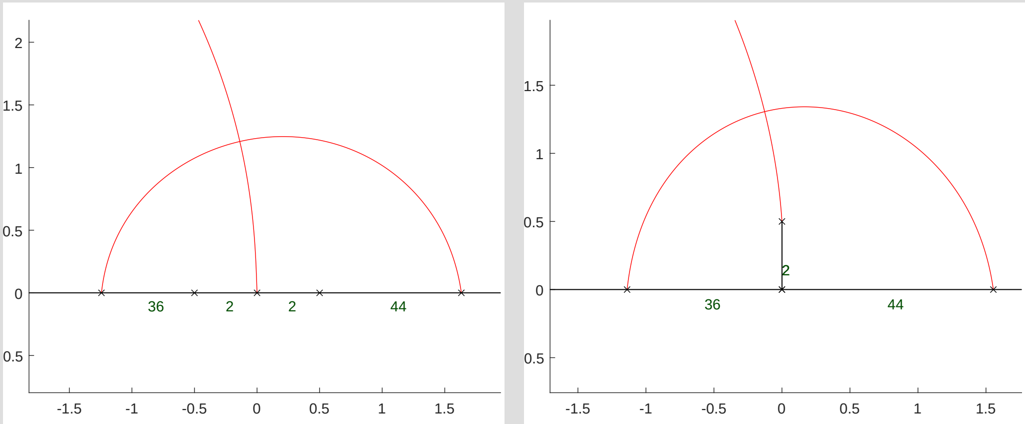
from section 3.2.

As a result we obtain the conformal image of $\mathbb{H}$ onto $\mathbb{P}^1\mathbb{C}$ minus a union of slits respecting the quotient relations of $\partial\mathbb{H}/\Gamma_{\mathcal{W}}$. In particular, this gives us the transformed dessin of $\Phi_{\mathbb{D}}$ contained in $\mathbb{P}^1\mathbb{C}$.

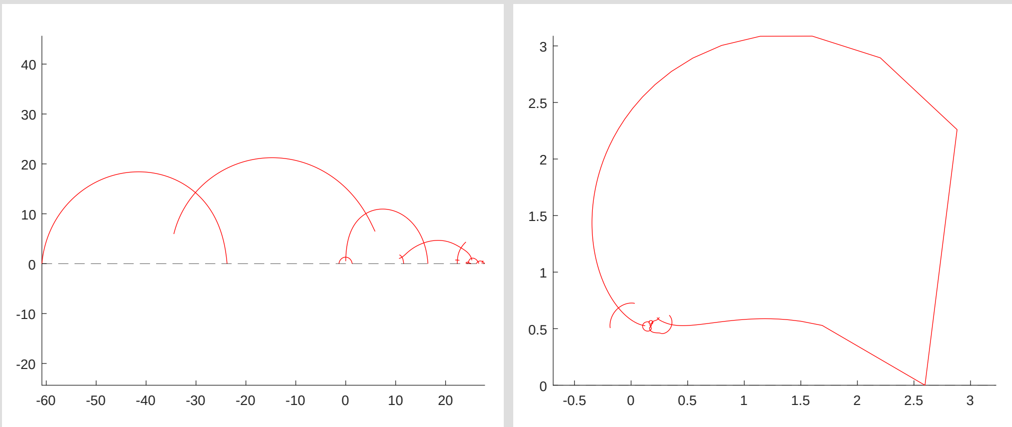
Aut(HS)-realization, Part 5.

Our goal is to glue the pairs of line segments shown in Part 4 having the same label. Note that there are a total of 106 line segments, therefore the first part of the welding process consists of 52 applications of slit_A.

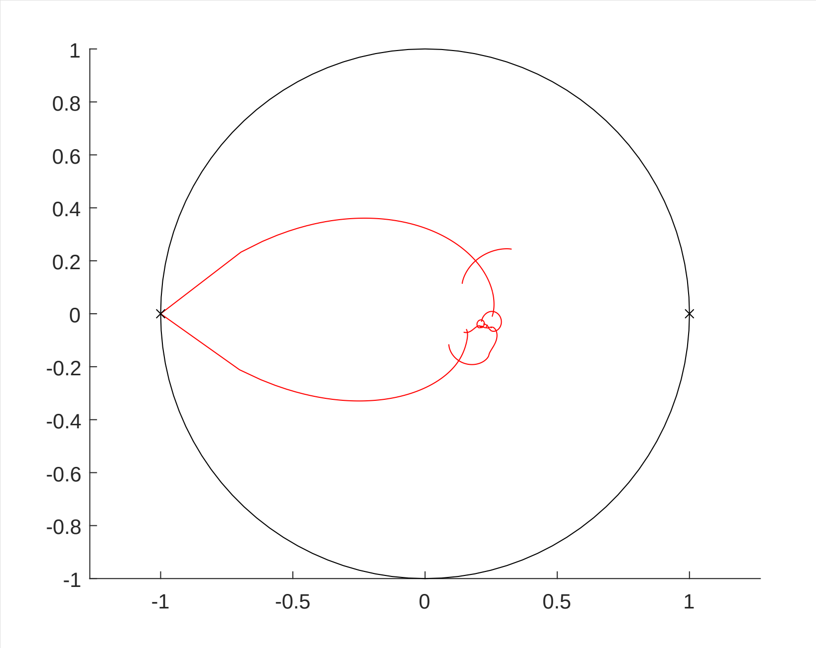
In the first welding step we glue the two adjacent real line segments labelled with 2. In order to achieve this we apply an automorphism of $\mathrm{Aut}(\mathbb{H})$ that maps both real line segments to $[-\frac{1}{2}, \frac{1}{2}]$ followed by an application of $\mathrm{slit}_{\frac{1}{2}}$. This first welding process is visualized in the following figures:



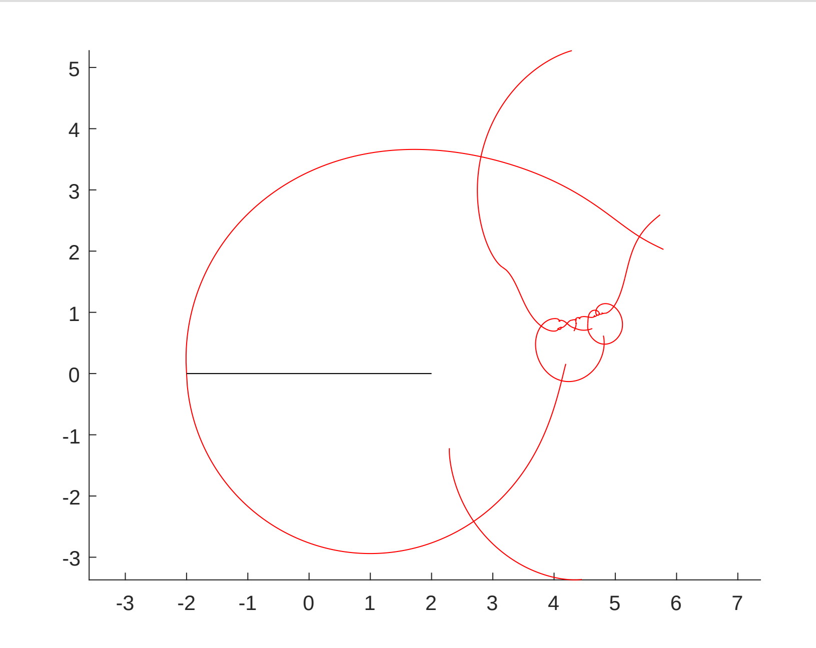
After applying 25 and 52 slit maps we end up with the following results:



In order to perform the second part of the welding process we start by conformally mapping $\mathbb{H}$ onto $\mathbb{D}$ in such a way that the upper semicircle needs to be glued to the lower semicircle:



After applying ω we finally get the desired dessin of $\Phi_{\mathbb{D}}$ contained in $\mathbb{P}^1\mathbb{C}$:



4.3.3. Smoothing Dessins.

In some cases the dessins produced by the above procedure do not suffice for a successful application of Newton's method. In this situation it is still worth trying to modify the constructed dessin in one of the following ways. This deformation process will be called *smoothing*.

4.3.3.1. *Symmetrizing real dessins.*

For a given constructed dessin of a Belyi map defined over $\mathbb{R}$ we apply a Möbius transformation such that three zeros, ones or poles belonging to real cycles are mapped to real numbers. Then the resulting dessin is expected to be axially symmetric to the real line.

By forcing supposedly real zeros, ones and poles to be on the real line and averaging out complex conjugate pairs of vertices of the dessin to be perfectly aligned to each other we obtain a proper real approximate dessin. This symmetrization process will be done in the following way:

- Let $x \in \mathbb{C}$ be any point of the constructed dessin that belongs to a real point of the dessin. Then, the symmetrized coordinates x_{sym} of x are given by
$$x_{\mathrm{sym}} := \mathrm{Real}(x).$$
- Let $x \in \mathbb{C}$ be any point of the constructed dessin that belongs to a point in $\mathbb{H}$ and x^c its complex conjugate point on the dessin. Then, the symmetrized coordinates x_{sym} and ${x^c}_{\mathrm{sym}}$ of x and x^c are obtained via
$$x_{\mathrm{sym}} := \frac{1}{2}\left(x + \overline{x^c}\right) \qquad \text{and} \qquad {x^c}_{\mathrm{sym}} := \overline{x_{\mathrm{sym}}}.$$

4.3.3.2. *Averaging out dessins.*

This method comes into play if several approximate dessins of a fixed genus-0 triple from different fundamental domains were computed. By applying suitable Möbius transformations we align all the dessins at the same three fixed points and average out the coordinates of the other remaining vertices.

Aut(HS)-realization, Part 6.

We will now establish the symmetry properties of the dessin of a real Belyi map f with ramification triple $(\sigma_0, \sigma_1, \sigma_\infty)$ from Part 1 as explained in section 2.3.2. The symmetry of the dessin of f is described by

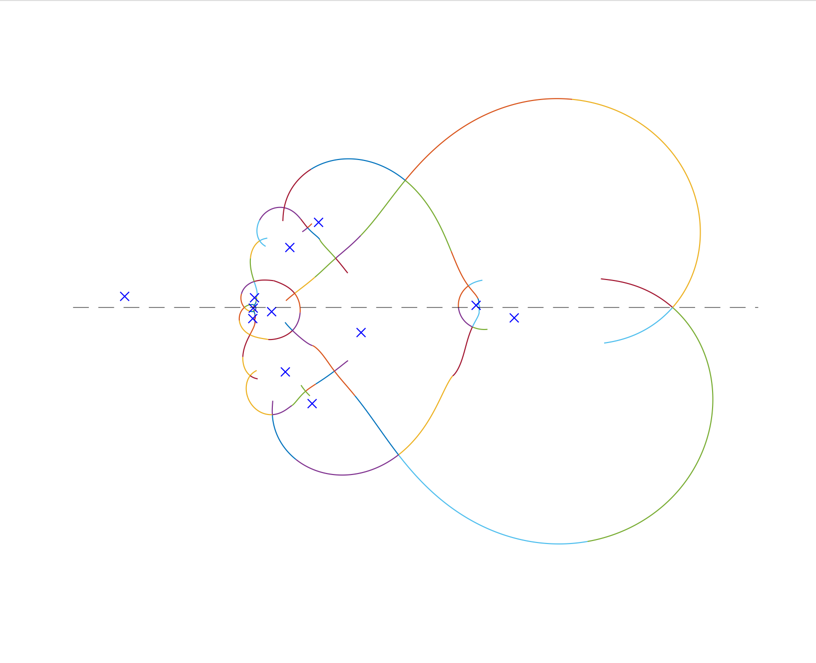
$$\begin{aligned}\rho = {} & (1,35)(2,99)(3,92)(4,28)(5,64)(6,33)(7,58)(8,31)(9,63)(10,54)(11,80)\\ & (12,72)(13,42)(14,87)(15,30)(16,98)(17,21)(18,76)(19,51)(20,100)\\ & (22,36)(23,26)(24,27)(25,86)(29,89)(32,93)(34,95)(37,52)(38,69)\\ & (39,43)(40,41)(44,47)(45,88)(46,82)(48,59)(49,85)(50,62)(53,68)\\ & (55,66)(56,60)(57,78)(61,90)(65,91)(67,75)(70,94)(71,84)(73,77)\\ & (74,81)(79,96)(83,97)\end{aligned}$$

yielding the following symmetry data:

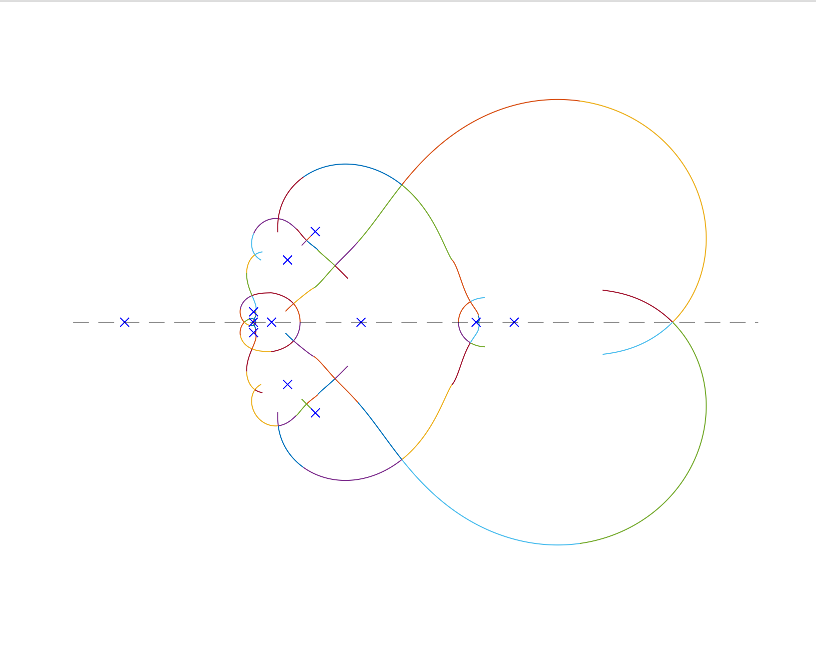
- As ρ does not fix any elements of $\{1, \ldots, 100\}$ the dessin of f does not have any edges on the real line.
- The real zeros of f corresponds to the following cycles of σ_0:
$$(12,72,37,52),\ (14,41,40,87),\ (57,78,83,97).$$
- The real ones of f corresponds to the following cycles of σ_1:
$$(2,99),\ (32,93),\ (20,100).$$
- The real poles of f corresponds to the following cycles of σ_∞:
$$\begin{aligned}&(19,78,57,97,100),\\ &(29,37,39,99,43),\\ &(1,6,20,76,92,26,68,93,7,23),\\ &(2,22,90,42,60,32,56,31,88,94),\\ &(4,38,79,12,21,27,11,82,41,15),\\ &(9,51,83,63,18,46,14,87,40,33).\end{aligned}$$

We now pick three of these real zeros, ones or poles and compute a Möbius transformation which maps them to -1, 0, 1.

If we apply this Möbius transformation to our approximate dessin we obtain an ”almost” symmetrical dessin:



The symmetrization process from section 4.3.3.1 then yields the following symmetrical dessin:



4.4. Belyi map computation

In this section we will explain in greater detail on how we can use the approximated dessin to compute the corresponding Belyi map via Newton's method. In preparation for this approach we need the following:

- For any permutation σ let $\mathcal{R}_\sigma$ be the ordered system of the smallest representatives (sorted by size) of the orbits of $\langle\sigma\rangle$. Furthermore, let $|\mathcal{R}_\sigma|$ denote the cardinality of $\mathcal{R}_\sigma$, and $\mathcal{R}_\sigma^k$ the k-th entry of $\mathcal{R}_\sigma$.
- Let $||\cdot||$ denote the Euclidean norm for polynomials with complex coefficients:

$$||\sum_{k=0}^{m} p_k X^k|| := \sqrt{\sum_{k=0}^{m} |p_k|^2}.$$

Example 4.7. If $\sigma = (1,3)(2,4,5)(6)$ then $\mathcal{R}_\sigma = (1,2,6)$, and $\mathcal{R}_\sigma^3 = 6$.

4.4.1. Assembling defining equations.

Similarly to section 3.1 we again establish defining equations for complex and real Belyi maps with prescribed ramification.

4.4.1.1. *Complex Belyi maps.*

If $f \in \mathbb{C}(X)$ is a Belyi map with ramification triple $(\sigma_0, \sigma_1, \sigma_\infty) \in {S_n}^3$, then after applying a suitable inner Möbius transformation we may assume $f(\infty) \notin \{0,1\}$. Therefore, f is given by

$$f = c \cdot \frac{p}{q} = 1 + (c-1) \cdot \frac{r}{q} \tag{4}$$

for some scalar c and monic complex polynomials p, q and r of type:

$$p = \prod_{k=1}^{|\mathcal{R}_{\sigma_0}|} (X - n_k)^{|(\mathcal{R}_{\sigma_0}^k)^{\langle\sigma_0\rangle}|},$$

$$r = \prod_{k=1}^{|\mathcal{R}_{\sigma_1}|} (X - o_k)^{|(\mathcal{R}_{\sigma_1}^k)^{\langle\sigma_1\rangle}|}$$

and

$$q = \prod_{k=1}^{|\mathcal{R}_{\sigma_\infty}|} (X - w_k)^{|(\mathcal{R}_{\sigma_\infty}^k)^{\langle\sigma_\infty\rangle}|}$$

with mutually distinct complex numbers n_k, o_k and w_k, see Lemma 2.4. Obviously, n_k are the zeros of f, o_k are the ones of f, and w_k are the poles of f.

Using the notation

$$F\left(X, c, n_1, \dots, n_{|\mathcal{R}_{\sigma_0}|}, o_1, \dots, o_{|\mathcal{R}_{\sigma_1}|}, w_1, \dots, w_{|\mathcal{R}_{\sigma_\infty}|}\right) := cp - q - (c-1)r$$

equation (4) can be rewritten as

$$F = 0.$$

If we compare coefficients with respect to X in the latter equation and consider c and all of the zeros n_k, ones o_k and poles w_k as unknowns we obtain a complex polynomial system consisting of n equations and $n+3$ unknowns.

4.4.1.2. *Real Belyi maps.*
If f turns out to be a real Belyi map we can express the above polynomials p, q and r in the following way: Using the notation

$$\mathrm{Real}(\mathcal{R}_\sigma) := \{k : \mathcal{R}_\sigma^k \text{ belongs to a real cycle of } \sigma\}$$

and

$$\mathrm{Conj}(\mathcal{R}_\sigma) := \{\min\{k, j\} : \mathcal{R}_\sigma^k \text{ and } \mathcal{R}_\sigma^j \text{ are in complex conjugate cycles of } \sigma\}$$

for $\sigma \in \{\sigma_0, \sigma_1, \sigma_\infty\}$ we have:

$$p = \prod_{\substack{k=1\\ k\in\mathrm{Conj}(\mathcal{R}_{\sigma_0})}}^{|\mathcal{R}_{\sigma_0}|} (X^2 - n_{1,k}X + n_{2,k})^{|(\mathcal{R}_{\sigma_0}^k)^{\langle\sigma_0\rangle}|} \cdot \prod_{\substack{k=1\\ k\in\mathrm{Real}(\mathcal{R}_{\sigma_0})}}^{|\mathcal{R}_{\sigma_0}|} (X - n_k)^{|(\mathcal{R}_{\sigma_0}^k)^{\langle\sigma_0\rangle}|},$$

$$r = \prod_{\substack{k=1\\ k\in\mathrm{Conj}(\mathcal{R}_{\sigma_1})}}^{|\mathcal{R}_{\sigma_1}|} (X^2 - o_{1,k}X + o_{2,k})^{|(\mathcal{R}_{\sigma_1}^k)^{\langle\sigma_1\rangle}|} \cdot \prod_{\substack{k=1\\ k\in\mathrm{Real}(\mathcal{R}_{\sigma_1})}}^{|\mathcal{R}_{\sigma_1}|} (X - o_k)^{|(\mathcal{R}_{\sigma_1}^k)^{\langle\sigma_1\rangle}|}$$

and

$$q = \prod_{\substack{k=1\\ k\in \mathrm{Conj}(\mathcal{R}_{\sigma_\infty})}}^{|\mathcal{R}_{\sigma_\infty}|} (X^2 - w_{1,k}X + w_{2,k})^{|(\mathcal{R}^k_{\sigma_\infty})^{\langle\sigma_\infty\rangle}|}$$
$$\cdot \prod_{\substack{k=1\\ k\in \mathrm{Real}(\mathcal{R}_{\sigma_\infty})}}^{|\mathcal{R}_{\sigma_\infty}|} (X - w_k)^{|(\mathcal{R}^k_{\sigma_\infty})^{\langle\sigma_\infty\rangle}|}$$

with suitable real numbers c, n_k, $n_{1,k}$, $n_{2,k}$, o_k $o_{1,k}$, $o_{2,k}$, w_k, $w_{1,k}$, $w_{2,k}$. Clearly, n_k, o_k and w_k are the real zeros, ones and poles of f, and $(n_{1,k}, n_{2,k})$, $(o_{1,k}, o_{2,k})$ and $(w_{1,k}, w_{2,k})$ are the sum and product of pairs of conjugate zeros, ones and poles of f.

In the same fashion as before we define

$$F := cp - q - (c-1)r$$

where F is similarly considered to be a multivariate polynomial in X, c, n_k, $n_{1,k}$, $n_{2,k}$, o_k $o_{1,k}$, $o_{2,k}$, w_k, $w_{1,k}$, $w_{2,k}$. Again, by comparing the coefficients with respect to X in the equation

$$F = 0$$

we obtain a real polynomial system with n equations and $n+3$ unknowns.

4.4.2. Approximate Belyi data from approximate dessins.

From an approximate dessin as constructed in the previous sections we can read off the approximate coordinates of the zeros n_k^*, ones o_k^* and poles w_k^* of the desired Belyi map with respective multiplicities.

In order find a sufficiently good value for c, which is similarly denoted by $c^* := a^* + b^* i$, we solve the following problem:

$$\mathrm{minimize}_{a^*,b^*\in\mathbb{R}} \; ||F\left(a^* + b^* i, n_1^*, \dots, o_1^*, \dots, w_1^*, \dots\right)||^2$$

By a gradient computation of F the desired value for c^* can be determined.

The collection of the coefficient c^* and the zeros n_k^*, ones o_k^* and poles w_k^*, denoted by (c^*, p^*, r^*, q^*), will be called the *approximate Belyi data* of the given dessin.

4.4.3. Newton iteration.

Using Newton's method we want to find a solution to

$$F = 0.$$

With the approximated dessin as a starting point we carry out the following procedure:

(1) Apply any Möbius transformation to the approximated dessin such that three zeros, ones or poles are mapped to $-1, 0, 1$ and calculate the corresponding approximate Belyi data (c^*, p^*, r^*, q^*).

(2) Evaluate $||F||^2$ at (c^*, p^*, r^*, q^*). If the resulting value is sufficiently small continue, otherwise go back to step (1).

(3) Pick three zeros, ones or poles of (c^*, p^*, r^*, q^*) and consider them fixed such that $F = 0$ is a system of n equations and n unknowns. Apply Newton's method to find a solution to $F = 0$ using the approximate Belyi data (c^*, p^*, r^*, q^*) as the starting point.

If we want to compute a Belyi map with real coefficients we can use the same approach with the following changes:

- In step (1) we only allow real zeros, ones or poles to be mapped to -1, 0, 1.
- In step (3) we only choose real zeros, ones or poles to be fixed in the real system $F = 0$.

If we observe quadratic convergence behaviour after several Newton iterations we keep computing an approximate solution

$$(c_{\text{Newton}}, p_{\text{Newton}}, r_{\text{Newton}}, q_{\text{Newton}})$$

of $F = 0$ having sufficiently high precision. In particular, we obtain a rational function

$$f_{\text{Newton}} = c_{\text{Newton}} \cdot \frac{p_{\text{Newton}}}{q_{\text{Newton}}} = 1 + (c_{\text{Newton}} - 1) \cdot \frac{r_{\text{Newton}}}{q_{\text{Newton}}}$$

where the ramification behaviour above 0, 1 and ∞ has the same cycle structures as σ_0, σ_1, σ_∞.

Remark 4.8. Since our approximate Belyi data rarely suffice for a successful application of Newton's method we exhaustively repeat step (3). If this fails we return to step (1) and compute new approximate Belyi data by starting from a different fundamental domain.

4.4.4. Recognizing coefficients of Belyi maps.

Assume we were able to obtain via Newton's method a complex approximation f_{Newton} of a Belyi map belonging to the ramification triple $(\sigma_0, \sigma_1, \sigma_\infty)$ additionally satisfying the conditions stated in Theorem 2.8.

Pick a zero, one or pole, denoted by x_k, of f_{Newton} with label $k \in \{1, \dots, n\}$. Further assume that f_{Newton} is modified by an inner Möbius transformation such that x_k is mapped to ∞ and f_{Newton} fulfils a normalization condition:

> A Belyi map
> $$f = \frac{p}{q} = 1 + \frac{r}{q} \in K(X)$$
> over any field K is said to fulfil a *normalization condition* if there exists a divisor
> $$d = a_n X^n + a_{n-1} X^{n-1} + \dots + a_1 X + a_0 \in K[X]$$
> of p, q or r for some $n \geq 3$ where $a_{n-1} = 0$ and $a_1 = a_0$.

Additionally assume that the orbit of k in the corresponding decomposition group G_Z is a union of ℓ orbits of the inertia group G_T. Then the coefficients of f_{Newton} are contained in a degree-ℓ number field, see Lemma 2.4 (c).

Provided we are working with sufficiently high numerical precision we can recognize the coefficients of the desired Belyi map as algebraic numbers.

Aut(HS)-realization, Part 7.

Using the approximated symmetric dessin from Part 6 we are able to find via Newton's method a complex approximation

$$f_{\text{Newton}} = \frac{p_{\text{Newton}}}{q_{\text{Newton}}} = 1 + \frac{r_{\text{Newton}}}{q_{\text{Newton}}} \in \mathbb{C}(X)$$

with ramification triple $(\sigma_0, \sigma_1, \sigma_\infty)$. With regards to Lemma 2.4 we pick a pole w of f_{Newton} belonging to a cycle of length 5 of σ_∞ and apply an inner Möbius transformation to f_{Newton} such that w is mapped to ∞ and f_{Newton} fulfils a normalization condition. This enables us to recognize the coefficients of f_{Newton} in the degree-4 number field $\mathbb{Q}(\sqrt[4]{5})$.

In order to descend further to a Belyi map

$$f = \frac{p}{q} = 1 + \frac{r}{q} \in \mathbb{Q}(X)$$

we naively guess the degree-4 divisor q_4 of q with multiplicity 5 having small coefficients. In order to test if our guess is correct we have to check whether there exists a Möbius transformation that transforms q_4 to the corresponding degree-3 factor of q_{Newton} with multiplicity 5.

Fortunately, there indeed exists such a polynomial q_4 having surprisingly small coefficients, namely

$$q_4 = X^4 - 5.$$

If we modify f_{Newton} by an inner Möbius transformation accordingly we are able to determine the desired Belyi map

$$f = \frac{p}{q} = 1 + \frac{r}{q}$$

consisting of the polynomials

$$\begin{aligned} p &= (7X^5 - 30X^4 + 30X^3 + 40X^2 - 95X + 50)^4 \\ &\quad \cdot (2X^{10} - 20X^9 + 90X^8 - 240X^7 + 435X^6 - 550X^5 \\ &\qquad + 425X^4 - 100X^3 - 175X^2 + 250X - 125)^4 \\ &\quad \cdot (2X^{10} + 5X^8 - 40X^6 + 50X^4 - 50X^2 + 125)^4, \\ q &= 2^8 \cdot (X^4 - 5)^5 \\ &\quad \cdot (X^8 - 20X^6 + 60X^5 - 70X^4 + 100X^2 - 100X + 25)^{10} \end{aligned}$$

and

$$
\begin{aligned}
r = 2^{13}5^2 3 \cdot \big(X^{30} &- \tfrac{50}{3}X^{29} + \tfrac{535}{4}X^{28} - 705X^{27} + \tfrac{33125}{12}X^{26} - \tfrac{98935}{12}X^{25} \\
&+ \tfrac{562115}{32}X^{24} - \tfrac{465025}{24}X^{23} - \tfrac{1204025}{48}X^{22} + \tfrac{4117075}{24}X^{21} \\
&- \tfrac{38437675}{96} \cdot X^{20} + \tfrac{5480375}{12} \cdot X^{19} + \tfrac{924625}{6} \cdot X^{18} \\
&- 1678750X^{17} + \tfrac{156003125}{48}X^{16} - \tfrac{32481875}{12}X^{15} - \tfrac{39794375}{24}X^{14} \\
&+ \tfrac{32584375}{4}X^{13} - \tfrac{176578125}{16}X^{12} + \tfrac{15528125}{3}X^{11} + \tfrac{22056250}{3}X^{10} \\
&- \tfrac{200265625}{12}X^{9} + \tfrac{1354578125}{96}X^{8} - \tfrac{40140625}{24}X^{7} - \tfrac{452640625}{48}X^{6} \\
&+ \tfrac{258546875}{24}X^{5} - \tfrac{437609375}{96}X^{4} - \tfrac{15546875}{12}X^{3} \\
&+ \tfrac{10703125}{4}X^{2} - \tfrac{4140625}{3}X + \tfrac{546875}{2}\big) \\
\cdot \big(X^{35} - \tfrac{81}{4}X^{34} &+ \tfrac{385}{2}X^{33} - \tfrac{4545}{4}X^{32} + 4585X^{31} - \tfrac{51795}{4}X^{30} \\
&+ \tfrac{48615}{2}X^{29} - \tfrac{84585}{4}X^{28} - \tfrac{151925}{4}X^{27} + \tfrac{3222625}{16}X^{26} - \tfrac{3661625}{8}X^{25} \\
&+ \tfrac{9866325}{16}X^{24} - \tfrac{425125}{2}X^{23} - \tfrac{10719625}{8}X^{22} + \tfrac{15940125}{4}X^{21} \\
&- \tfrac{45347625}{8}X^{20} + \tfrac{7695625}{4}X^{19} + \tfrac{168171875}{16}X^{18} - \tfrac{208346875}{8}X^{17} \\
&+ \tfrac{434509375}{16}X^{16} + 3018750X^{15} - \tfrac{114028125}{2}X^{14} + 94578125X^{13} \\
&- 71718750X^{12} - \tfrac{49921875}{4}X^{11} + \tfrac{1672234375}{16}X^{10} - \tfrac{1129859375}{8}X^{9} \\
&+ \tfrac{1678421875}{16}X^{8} - \tfrac{66015625}{2}X^{7} - \tfrac{157578125}{8}X^{6} + \tfrac{106640625}{4}X^{5} \\
&- \tfrac{29453125}{8}X^{4} - \tfrac{64453125}{4}X^{3} + \tfrac{272265625}{16}X^{2} - \tfrac{62890625}{8}X + \tfrac{23828125}{16}\big)^2.
\end{aligned}
$$

4.5. Verification

In the following we will explain our general approach on how to verify that a given rational function

$$f = \frac{p}{q} \in K(X)$$

(with K being a number field) is a Belyi map with prescribed ramification triple and compute its algebraic and geometric monodromy group A and G.

4.5.1. Belyi map property.

The first step consists of verifying that f is indeed a Belyi map. This can be done by combining the Riemann-Hurwitz genus formula from Lemma 2.1 (d) with the inseparability behaviour of the polynomials p, q, $p - q$.

4.5.2. Computing the subdegrees of A.

Let $O_1, \ldots, O_k$ for some $k \in \mathbb{N}$ be the orbits of any point stabilizer of the transitive group A. Then, A is called a rank-k group with subdegrees $|O_1|$, $\ldots$, $|O_k|$.

In order to find the subdegrees of A we factorize the polynomial

$$\triangledown_f := p(X) - f(t)q(X) \in K(t)[X].$$

Lemma 4.9. *The subdegrees of A are given by the degrees of the irreducible factors of $\triangledown_f \in K(t)[X]$.*

Proof. Let x be a root of $p(X) - tq(X) \in K(t)[X]$. Then

$$t = \frac{p(x)}{q(x)} = f(x)$$

and the corresponding point stabilizer in A leads to the fixed field $K(t, x) = K(x)$. This yields that the irreducible factors of

$$p(X) - tq(X) = p(X) - f(x)q(X)$$

in $K(x)[X]$ describe the subdegrees of A. □

4.5.3. Primitivity of A.

According to Ritt's theorem the arithmetic monodromy group A of f is primitive if and only if f does not decompose non-trivially. Thanks to [**1**, Proposition 3.1] the rational function f decomposes of type $f = g \circ h$ for some $h = \frac{r}{s}$

with coprime polynomials r, s if and only if the polynomial

$$\bigtriangledown_h := r(X) - h(t)s(X) \in K(t)[X]$$

is a divisor of $\bigtriangledown_f$ in $K(t)[X]$. As $\bigtriangledown_h$ vanishes at t, $\deg(\bigtriangledown_h) = \deg(h)$ and $\deg(\bigtriangledown_h)$ divides $\deg(f) = \deg(g) \cdot \deg(h)$ we obtain the following criterion summarizing the previous results:

Lemma 4.10. *Let S be the set of all proper divisors of $\bigtriangledown_f \in K(t)[X]$ vanishing at t of respective degree greater than 1 dividing $\deg(f)$. Then A is primitive if and only if no elements in S are of type $r(X) - \frac{r(t)}{s(t)}s(X)$ where $r, s \in K[X]$.*

Using the conditions (9) and (10) in Proposition 4.3 from [**2**] one can easily check whether any polynomial is of type $r(X) - \frac{r(t)}{s(t)}s(X)$.

As the degree of each divisor in S is equal to the sum of some subdegrees of A containing 1 we obtain the following divisibility criterion as a special case of Lemma 4.10:

Corollary 4.11. *If there is no subset of the subdegrees containing 1 adding up to a non-trivial divisor of the permutation degree then A is primitive.*

4.5.4. Narrowing down the possible candidates for A and G. By using the Magma database (and therefore the classification of finite simple groups!) we obtain a list of all finite primitive groups having the desired subdegrees. Clearly, A is equal to one of these groups.

This helps to narrow down the possibilities for G: Recall that G is normal in A and G contains generating triples having the same cycle structure as $(\sigma_0, \sigma_1, \sigma_\infty)$. For every group appearing in the above list we look for genus-0 triples of the desired cycle structures generating a normal subgroup. This yields all possible candidates for G. Fortunately, in all of our cases we remain with only one possibility for G.

Once G has been determined we check if G contains exactly one genus-0 triple $(\sigma_0, \sigma_1, \sigma_\infty)$ (up to simultaneous conjugation) having the desired cycle structure. In most of our examples this is the case and ensures that the ramification triple is indeed given by $(\sigma_0, \sigma_1, \sigma_\infty)$.

Aut(HS)-realization, Part 8.
The previously described verification process will be now applied to the computed polynomials from Part 7.

Theorem 4.12. *Let p, q and r be the polynomials from Part 7. Then the following holds:*

(a) The rational function

$$f := \frac{p}{q} = 1 + \frac{r}{q}$$

is a Belyi map with ramification triple $(\sigma_0, \sigma_1, \sigma_\infty)$*.*

(b) The polynomial

$$p(X) - tq(X) \in \mathbb{Q}(t)[X]$$

defines a regular extension of $\mathbb{Q}(t)$ *with Galois group* Aut(HS)*.*

Proof. From the Riemann-Hurwitz genus formula (Lemma 2.1 (d)) in combination with the inseparability behaviour of p, q and r we see that the ramification locus of f is given by $(0, 1, \infty)$ with ramification triple of the same cycle structure as $(\sigma_0, \sigma_1, \sigma_\infty)$.

Since the polynomial

$$p(X) - f(t)q(X) \in \mathbb{Q}(t)[X]$$

splits into irreducible factors of degree 1, 22, 77 we see using Lemma 4.9 and 4.11 that A is a primitive rank-3 group with subdegrees 1, 22, 77. According to the classification of finite primitive rank-3 groups this implies $A = \mathrm{HS}$ or $A = \mathrm{Aut(HS)}$. Due to the fact that G is normal in A we also have $G = \mathrm{HS}$ or $G = \mathrm{Aut(HS)}$. As the even group HS does not contain elements having the same cycle structure as σ_1 we end up with $G = \mathrm{Aut(HS)}$ which implies $A = \mathrm{Aut(HS)}$.

Recall that the monodromy group of f is isomorphic to $G = \mathrm{Aut(HS)}$ with ramification triple having the same cycle structure as $(\sigma_0, \sigma_1, \sigma_\infty)$. As Aut(HS) only contains exactly one triple (up to simultaneous conjugation) having the same cycle structures as $(\sigma_0, \sigma_1, \sigma_\infty)$ the ramification triple of f coincides with $(\sigma_0, \sigma_1, \sigma_\infty)$. $\square$

The proof of Theorem 4.12 relies on the classification of finite simple groups which can be avoided by using the following graph-theoretical approach:

An undirected k-regular graph $\mathcal{G}$ with n vertices is called strongly regular if there exist $\lambda, \mu \in \mathbb{N}_0$ such that adjacent vertices have exactly λ common neighbours and non-adjacent vertices have exactly μ common neighbours. We say that $\mathcal{G}$ is of type $\mathrm{srg}(n, k, \lambda, \mu)$ if $\mathcal{G}$ fulfils the latter conditions.

We firstly collect some well known properties of strongly regular graphs, see [**14**, Theorem 1.1, Theorem 3.1].

Lemma 4.13. *Let $\mathcal{G}$ be of type $\mathrm{srg}(n, k, \lambda, \mu)$, then the following conditions hold:*

(i) $k(k - \lambda - 1) = (n - k - 1)\mu$.

(ii) $\frac{1}{2}\left(n - 1 \pm \frac{(n-1)(\mu-\lambda)-2k}{\sqrt{(\mu-\lambda)^2+4(k-\mu)}}\right)$ *are non-negative integers.*

Another key observation is the fact that rank-3 groups can be considered as automorphism groups of strongly regular graphs, see [**14**, Theorem 6.1].

Lemma 4.14. *Let A be a rank-3 subgroup of S_n having subdegrees 1, k, ℓ with $1 < k < \ell$. Then there exists a strongly regular graph $\mathcal{G}$ of type $\mathrm{srg}(n, k, \lambda, \mu)$ for some $\lambda, \mu \in \mathbb{N}_0$ such that A is a subgroup of $\mathrm{Aut}(\mathcal{G})$.*

The following lemma replaces the usage of the classification of finite simple groups in the proof of Theorem 4.12.

Lemma 4.15. *Let A be a rank-3 group with subdegrees 1, 22, 77, then $A = \mathrm{HS}$ or $A = \mathrm{Aut}(\mathrm{HS})$.*

Proof. Due to Lemma 4.14 there exists a strongly regular graph $\mathcal{G}$ of type $\mathrm{srg}(100, 22, \lambda, \mu)$ for some $\lambda, \mu \in \{0, \ldots, 100\}$ such that A is a subgroup of $\mathrm{Aut}(\mathcal{G})$. According to Lemma 4.13 (i) the parameters λ and μ have to satisfy the equation

$$22(22 - \lambda - 1) = (100 - 22 - 1)\mu$$

from which we find

$$(\lambda, \mu) \in \{(0, 6), (7, 4), (14, 2), (21, 0)\}.$$

Among these pairs the condition (ii) from Lemma 4.13 is only fulfilled for

$$(\lambda, \mu) = (0, 6),$$

therefore $\mathcal{G}$ is of type $\mathrm{srg}(100, 22, 0, 6)$. From [**19**] we find that $\mathcal{G}$ must be isomorphic to the Higman-Sims graph with $\mathrm{Aut}(\mathcal{G}) = \mathrm{Aut}(\mathrm{HS})$, therefore A is a subgroup of Aut(HS). As HS and Aut(HS) are the only subgroups of Aut(HS) with subdegrees 1, 22, 77 we end up with $A = \mathrm{HS}$ or $A = \mathrm{Aut}(\mathrm{HS})$. □

A Polynomial with the sporadic Galois group HS.

Note that HS is an even and Aut(HS) an odd group. A computer calculation, e.g. in Magma, shows that the discriminant δ of

$$p(X) - tq(X) \in \mathbb{Q}(t)[X]$$

is given by

$$\delta = u^2 \cdot \frac{1}{-5} \cdot t(t-1)$$

for some $u \in \mathbb{Q}(t)$. As explained in section 2.3.4.2 we can deduce

$$\mathrm{HS} = \mathrm{Gal}\left(p(X) - \frac{1}{1+5r^2} q(X) \mid \mathbb{Q}(r)\right).$$

CHAPTER 5

Main results

We will now apply the computation method from chapter 4 to find more explicit Belyi maps with prescribed ramification. Note that all of the following triples and rational functions are also available in the ancillary Magma-readable files.

This chapter is taken over from [**10**, Chapter 4].

5.1. Belyi maps defined over $\mathbb{Q}$

Our first goal is to explicitly realize triples satisfying the conditions stated in Theorem 2.8. The following theorem contains a list of triples that theoretically lead to Belyi maps defined over $\mathbb{Q}$.

Theorem 5.1. *(a) Let G be a primitive and almost simple permutation group of degree between 50 and 250 not equal to a symmetric or alternating group. If there exists a rigid genus-0 triple $(\sigma_0, \sigma_1, \sigma_\infty)$ for G where each element is contained in a rational conjugacy class then G is isomorphic to one of the following groups:*

section	*group*	*primitive group identification*	*number of triples up to simultaneous S_n-conjugation*
5.1.1	$\mathrm{Aut}(\mathrm{PSL}(3,3))$	$[52,1]$	*1*
5.1.2	$\mathrm{PGL}(2,11)$	$[55,2]$, $[55,3]$	*2*
5.1.3	$N_{S_{56}}(\mathrm{PSL}(3,4))$	$[56,7]$	*1*
5.1.4	$\mathrm{Aut}(\mathrm{PSU}(3,3))$	$[63,3]$	*1*
5.1.5	$\mathrm{Aut}(\mathrm{M}_{22})$	$[77,2]$	*1*
5.1.6	$\mathrm{PSp}(4,4){:}2$	$[85,2]$	*1*
5.1.7	$\mathrm{Aut}(\mathrm{HS})$	$[100,36]$	*2*
5.1.8	$\mathrm{O}^+(8,2)$	$[135,2]$	*1*

(b) Let $(\sigma_0, \sigma_1, \sigma_\infty)$ be a genus-0 triple for a group G appearing in (a). Then there exists a Belyi map

$$f = \frac{p}{q} \in \mathbb{Q}(X)$$

with ramification triple $(\sigma_0, \sigma_1, \sigma_\infty)$ such that

$$p(X) - tq(X) \in \mathbb{Q}(t)[X]$$

defines a regular Galois extension of $\mathbb{Q}(t)$ with Galois group G.

Proof. (a) By using Magma and its database for finite primitive groups we are able to verify that the above almost simple groups are the only non-symmetric and non-alternating groups containing the desired genus-0 triples.

(b) This follows from Theorem 2.8. The cycle structure description of each triple is given in the respective sections. □

It is reasonable to expect that the above list is also complete if we ignore the upper permutation degree bound of 250.

Theorem 5.2. *The rational functions presented in the sections 5.1.1 - 5.1.8 are the Belyi maps from Theorem 5.1.*

Proof. In each case we again follow the verification process from section 4.5.

It is easy to see that the computed rational functions are indeed Belyi maps with ramification triple of the same cycle structure as the desired ramification triple.

The computation of the arithmetic and geometric monodromy groups A and G can be found in the following sections.

Finally, as all groups from Theorem 5.1 only contain one generating genus-0 triple up to simultaneous conjugation with the prescribed cycle structure any Belyi map with ramification triple of the same cycle structure as one of these triples must already coincide with it. □

Additionally, we will also explicitly realize all index-2 subgroups of the groups appearing in Theorem 5.1.

5.1.1. Aut(PSL(3, 3)) of degree 52. (ancillary file `52.txt`)
We start with the the genus-0 triple $(\sigma_0, \sigma_1, \sigma_\infty)$ for Aut(PSL(3, 3)) where

$$\begin{aligned}\sigma_0 = {} & (1,41,8,9,45,32,39,44)(2,13,29,21,50,26,34,6)(3,35,52,30)\\ & (4,7,22,18,33,43,10,38)(5,37,27,42,25,15,12,24)\\ & (11,51,17,47,36,31,49,40)(14,20,28,48)(16,19,23,46)\end{aligned}$$

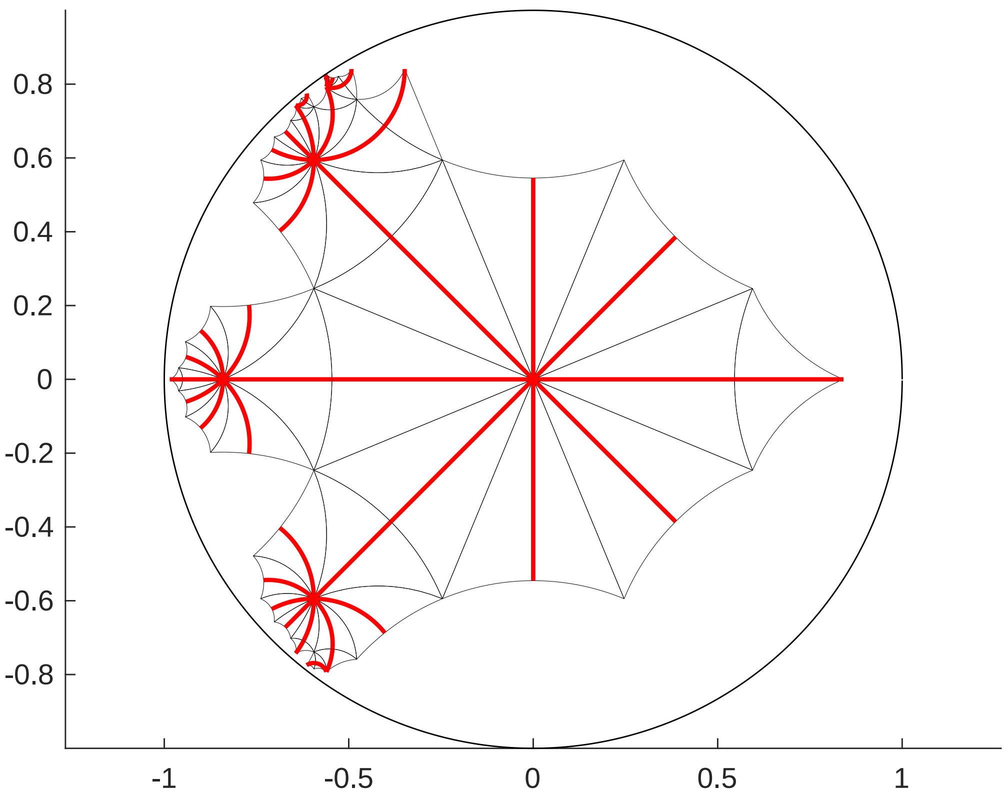
and

$$\begin{aligned}\sigma_1 = {} & (1,20)(2,34)(3,7)(4,16)(5,17)(8,41)(9,13)(10,52)(11,40)\\ & (12,23)(14,29)(15,25)(18,33)(19,47)(21,35)(22,43)(26,42)\\ & (27,45)(28,36)(31,39)(32,49)(37,51)(38,48)(46,50)\end{aligned}$$

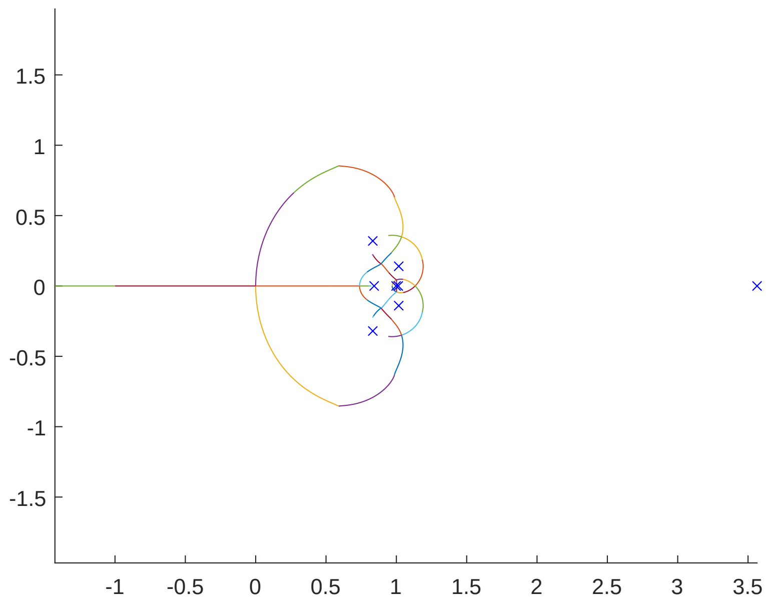
with cycle structure description:

	σ_0	σ_1	σ_∞
cycle structure	$8^5.4^3$	$2^{24}.1^4$	$4^{10}.2^4.1^4$

A fundamental domain is given by



with resulting approximate dessin:



The corresponding Belyi map $f = \frac{p}{q}$ is given by

$$\begin{aligned} p &= 3^3 \cdot (X+1)^8 \cdot (2X^2 - 8X - 1)^8 \cdot (2X^2+1)^4 \cdot (6X^2+4X+1)^8, \\ q &= 2^2 \cdot (4X^4 - 16X^3 - 24X^2 - 8X - 1)^2 \\ &\quad \cdot (4X^4 - 16X^3 - 18X^2 - 8X - 1) \cdot (4X^4 + 8X^3 + 36X^2 + 28X + 5)^4 \\ &\quad \cdot (4X^6 - 36X^5 - 24X^4 - 4X^3 + 9X^2 + 3X + 1)^4. \end{aligned}$$

Verification of monodromy.
The factorization of $p(X) - f(t)q(X)$ over $\mathbb{Q}(t)$ yields that A has subdegrees 1, 6, 18, 27, thus A is primitive by Lemma 4.11. As there is only one primitive group of degree 52 with these subdegrees we obtain $A = \mathrm{Aut}(\mathrm{PSL}(3,3))$. Since G is normal in A we have $G = \mathrm{PSL}(3,3)$ or $G = \mathrm{Aut}(\mathrm{PSL}(3,3))$. Since $\mathrm{PSL}(3,3)$ does not contain elements having the same cycle structure as σ_∞ we find $G = A = \mathrm{Aut}(\mathrm{PSL}(3,3))$. □

Realization of $\mathrm{PSL}(3,3)$.
Note that $\mathrm{PSL}(3,3)$ is the only index-2 subgroup of $\mathrm{Aut}(\mathrm{PSL}(3,3))$. From $\sigma_0 \in \mathrm{PSL}(3,3)$ and the character values of the class in H containing σ_0 we find $c = 2$, see section 2.3.4.1, therefore

$$\mathrm{PSL}(3,3) = \mathrm{Gal}(p(X) - (1+2s^2)q(X) \mid \mathbb{Q}(s)).$$

5.1.2. PGL(2, 11) of degree 55. (ancillary files `55a.txt` and `55b.txt`) The first genus-0 triple $(\sigma_0, \sigma_1, \sigma_\infty)$ for PGL(2, 11) is given by

$$\begin{aligned}\sigma_0 = &(1, 33, 17, 8, 41, 32)(2, 51, 45, 36, 55, 50)(3, 43, 47, 24, 44, 25)\\&(4, 40, 49, 52, 18, 6)(5, 7, 26, 16, 37, 23)(9, 21, 22, 38, 31, 20)\\&(10, 48, 29, 30, 35, 19)(11, 28, 54)(12, 15, 13, 34, 42, 27)\\&(39, 46, 53)\end{aligned}$$

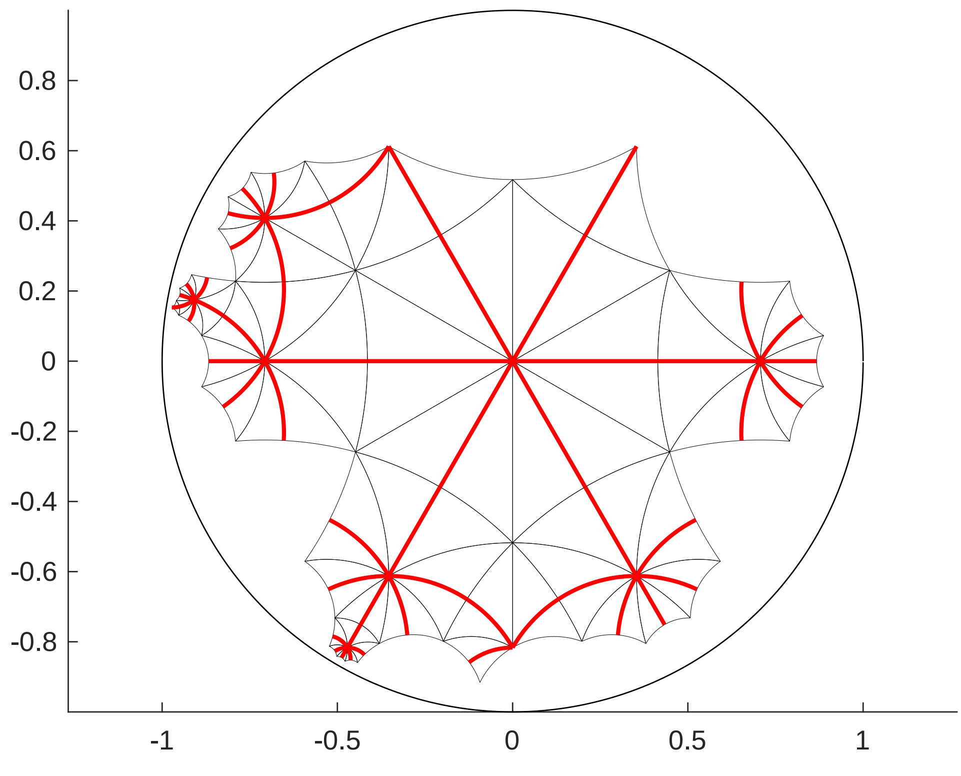
and

$$\begin{aligned}\sigma_1 = &(1, 26)(2, 41)(3, 51)(4, 32)(5, 29)(6, 37)(7, 20)(8, 47)(10, 44)(11, 50)\\&(12, 52)(13, 15)(14, 35)(17, 46)(18, 23)(19, 42)(21, 48)(24, 38)\\&(25, 45)(27, 30)(31, 39)(33, 53)(34, 36)(40, 54)(49, 55)\end{aligned}$$

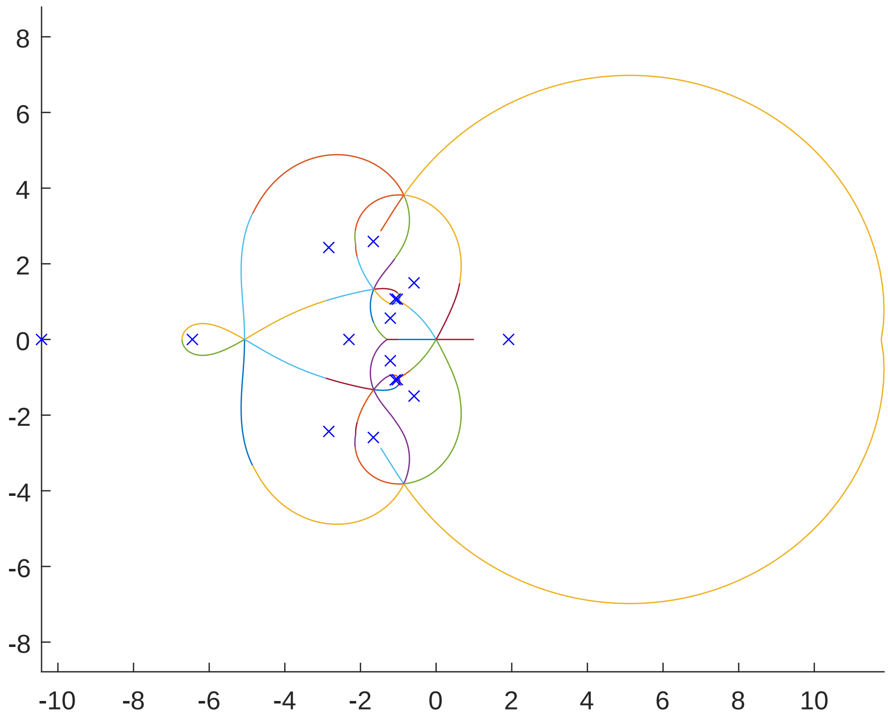
of type

	σ_0	σ_1	σ_∞
cycle structure	$6^8.3^2.1^1$	$2^{25}.1^5$	$4^{12}.2^3.1^1$

A fundamental domain is given by



with resulting approximate dessin:



This triple corresponds to the Belyi map $f = \frac{p}{q}$ where

$$\begin{aligned} p &= 2^4 \cdot (22X^2 - 11X - 2)^6 \cdot (44X^2 + 22X + 3)^6 \cdot (88X^2 + 55X + 1)^6 \\ &\quad \cdot (176X^2 + 44X + 5)^6 \cdot (704X^2 + 242X + 17)^3, \\ q &= 11^4 \cdot (2X + 1) \cdot (176X^3 + 1056X^2 + 330X + 31)^4 \\ &\quad \cdot (264X^3 + 154X^2 + 22X + 1)^4 \cdot (352X^3 + 264X^2 + 99X + 14)^4 \\ &\quad \cdot (704X^3 + 132X^2 + 1)^4 \cdot (1408X^3 + 693X^2 + 132X + 8)^2, \end{aligned}$$

see also file `55a.txt`.

Verification of monodromy.
The polynomial $p(X) - f(t)q(X)$ factorizes over $\mathbb{Q}(t)$ into irreducible polynomials of degree $1, 6, 12, 12, 12, 12$, thus A has the subdegrees 1, 6, 12, 12, 12, 12 and the primitivity of A follows from Lemma 4.11. The only primitive group of degree 55 having these subdegrees is $\mathrm{PGL}(2, 11)$. Since G is normal in $\mathrm{PGL}(2, 11)$ we have $G = \mathrm{PGL}(2, 11)$ or $G = \mathrm{PSL}(2, 11)$. The latter case can be ruled out because $\mathrm{PSL}(2, 11)$ does not contain elements with the cycle structure of σ_1. We find $G = A = \mathrm{PGL}(2, 11)$. $\square$

There is another degree-55 permutation representation of $\mathrm{PGL}(2,11)$ in which the above triple $(\sigma_0, \sigma_1, \sigma_\infty)$ is also of genus 0. It is given by

$$\begin{aligned}\sigma_0 = {} & (1,36,37,16,42,22)(2,31,17,19,30,9)(3,48,29,12,24,34)\\ & (4,44,28,46,18,8)(6,13,53,43,40,10)(7,26,33,15,27,35)\\ & (20,39,47,38,45,23)(21,52,50,41,55,32)(5,49,25)(11,14)(51,54)\end{aligned}$$

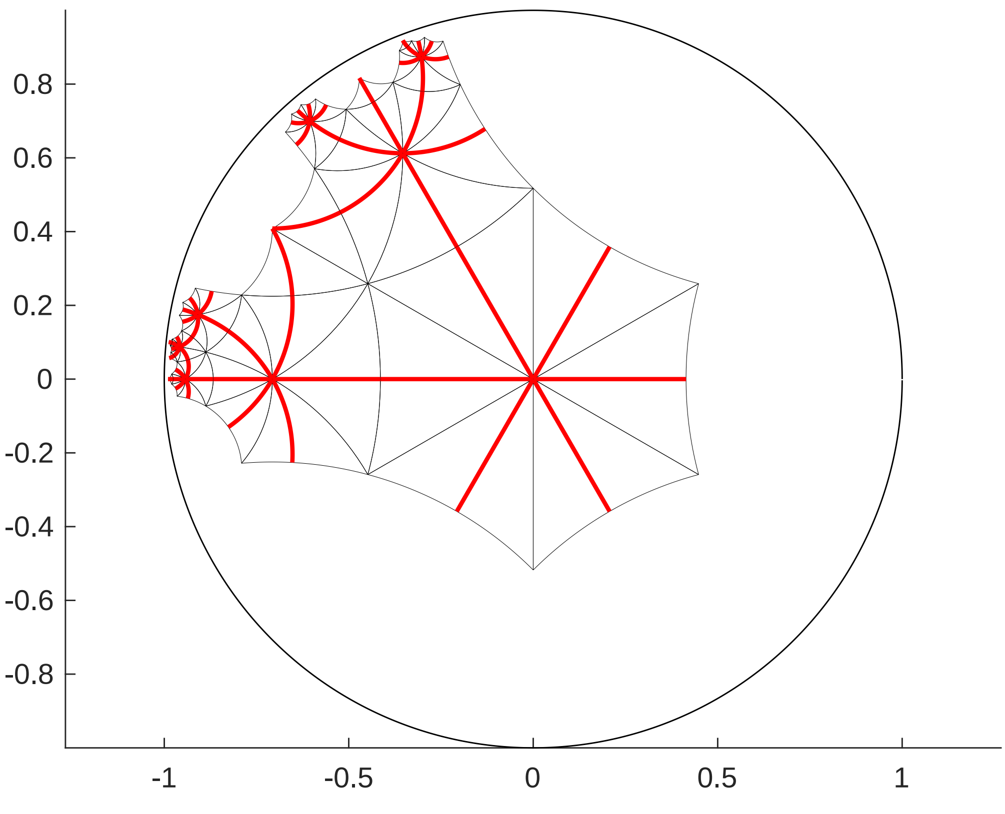
and

$$\begin{aligned}\sigma_1 = {} & (1,22)(2,34)(3,41)(4,39)(5,50)(6,52)(8,42)(9,54)(10,27)(11,23)\\ & (12,24)(14,44)(15,49)(16,33)(17,35)(18,37)(19,32)(21,40)(25,46)\\ & (26,47)(28,48)(29,45)(31,38)(43,53)(51,55)\end{aligned}$$

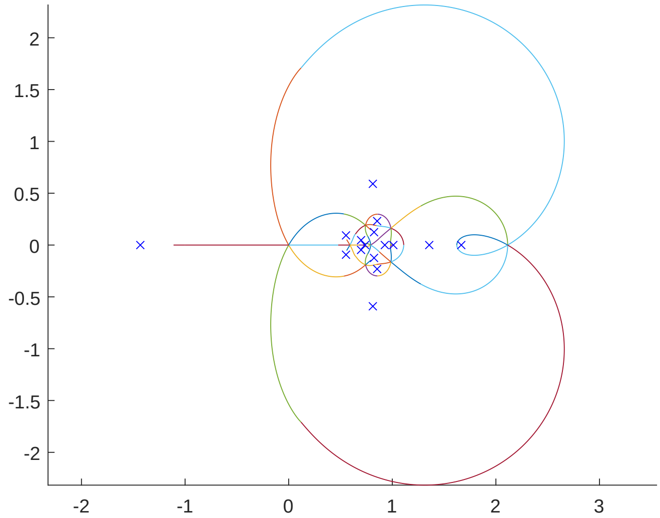
of type:

	σ_0	σ_1	σ_∞
cycle structure	$6^8.3^1.2^2$	$2^{25}.1^5$	$4^{13}.1^3$

A fundamental domain is given by



with resulting approximate dessin:



The computed results for this Belyi map $f = \frac{p}{q}$ are

$$\begin{aligned} p &= 2^4 \cdot (3X-1)^3 \cdot (2X^2-5X-1)^6 \cdot (3X^2-2X+4)^2 \\ &\quad \cdot (12X^2-8X+5)^6 \cdot (12X^4+6X^3+19X^2-3X+3)^6 \\ q &= 3^3 \cdot 11^5 \cdot (4X^3-4X^2+5X-3)^4 \cdot (6X^3+5X^2+2X+1)^4 \\ &\quad \cdot (12X^3-56X^2+15X-9) \\ &\quad \cdot (72X^6-144X^5+230X^4-134X^3+61X^2-14X+2)^4. \end{aligned}$$

Verification of monodromy.

The subdegrees of A are 1, 4, 6, 8, 12, 24. According to Lemma 4.10 the group A must be primitive. It follows $A = \mathrm{PGL}(2,11)$, therefore $G = \mathrm{PGL}(2,11)$ or $G = \mathrm{PSL}(2,11)$. Since there are no elements having the same cycle structure as σ_1 in $\mathrm{PSL}(2,11)$ we conclude $G = A = \mathrm{PGL}(2,11)$. $\square$

Note that both $\mathrm{PGL}(2,11)$-Belyi maps from this section define the same Galois extension of $\mathbb{Q}(t)$.

Realization of $\mathrm{PSL}(2,11)$.

Note that $\mathrm{PSL}(2,11) = \mathrm{PGL}(2,11) \cap A_{55}$ is the unique index-2 subgroup of the odd group $\mathrm{PGL}(2,11)$. For both polynomials $p(X) - tq(X)$ the square-free part of the discriminant is given by $-\frac{1}{33}(t-1)$. As described in section 2.3.4.2 we obtain

$$\mathrm{PSL}(2,11) = \mathrm{Gal}(p(X) - (1-33s^2)q(X) \mid \mathbb{Q}(s)).$$

5.1.3. $\mathbf{N_{S_{56}}(PSL(3,4))}$ of degree 56. (ancillary file `56.txt`)

We work with the genus-0 triple $(\sigma_0, \sigma_1, \sigma_\infty)$ for $N_{S_{56}}(\mathrm{PSL}(3,4))$ given by

$$\begin{aligned}\sigma_0 = &(1,36,2,5)(3,47)(7,45,33,22)(8,31,55,14)(9,21,50,48)\\ &(10,16,40,39)(11,54)(12,19,49,23)(13,41,42,15)\\ &(17,56,24,30)(18,53,44,25)(20,52,28,35)\\ &(26,29,46,37)(27,34,51,43)\end{aligned}$$

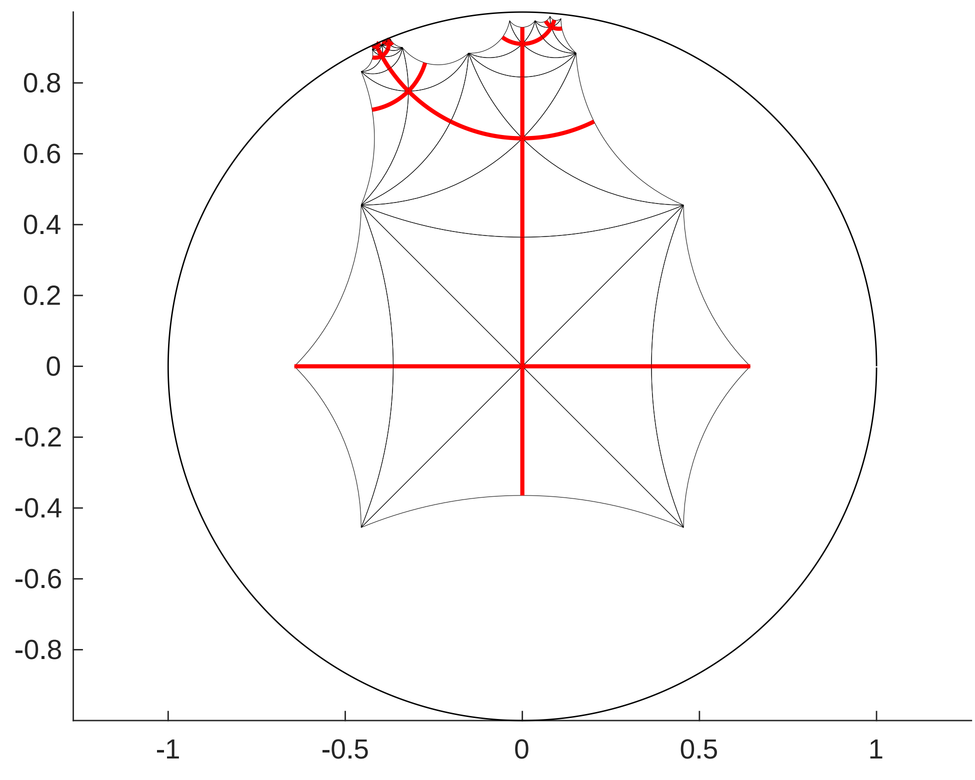
and

$$\begin{aligned}\sigma_1 = &(2,11)(3,29)(4,7)(5,27)(6,40)(8,9)(10,46)(12,35)(13,34)\\ &(14,21)(16,18)(17,20)(19,25)(23,41)(24,54)(26,33)(28,37)\\ &(31,44)(36,43)(38,47)(42,49)(45,53)(50,51)(52,55)\end{aligned}$$

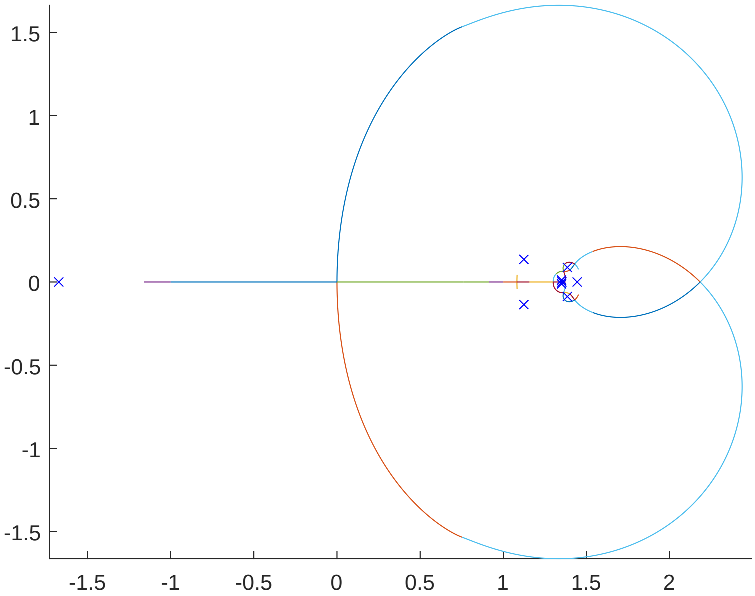
with cycle structure description

	σ_0	σ_1	σ_∞
cycle structure	$4^{12}.2^2.1^4$	$2^{25}.1^6$	$8^6.4^1.2^2$

A fundamental domain is given by



with resulting approximate dessin:



The Belyi map $f = \frac{p}{q}$ consists of

$$\begin{aligned} p \; =& \; (X^2 - 6X - 1)^2 \\ & \cdot (3X^4 - 468X^3 - 258X^2 - 60X - 5) \\ & \cdot (3X^4 + 36X^3 + 54X^2 + 60X + 19)^4 \\ & \cdot (3X^8 - 96X^7 - 12X^6 + 432X^5 + 1498X^4 \\ & \quad - 320X^3 - 348X^2 - 80X - 5)^4 \end{aligned}$$

and

$$\begin{aligned} q \; =& \; -2^2 \cdot 5^5 \cdot (X^2 + 2X + 3)^2 \cdot (3X^2 + 6X + 1)^8 \\ & \cdot (3X^4 - 12X^3 + 38X^2 + 12X + 3)^8. \end{aligned}$$

Verification of monodromy.

The subdegrees of A turn out to be 1, 10, 45, thus A is primitive by Lemma 4.11. There are five primitive groups having these subdegrees, all between $\mathrm{PSL}(3,4)$ and $N_{S_{56}}(\mathrm{PSL}(3,4))$. These five groups are also the only possibilities for the geometric monodromy group G from which only $N_{S_{56}}(\mathrm{PSL}(3,4))$ contains a generating triple with the desired cycle structures. It follows $A = G = N_{S_{56}}(\mathrm{PSL}(3,4))$. $\square$

Realization of all index-2 subgroups of $N_{S_{56}}(\mathrm{PSL}(3,4))$.
There are exactly three index-2 subgroups of $N_{S_{56}}(\mathrm{PSL}(3,4))$ given by their primitive group identification [56,3], $[56,4] = A_{56} \cap N_{S_{56}}(\mathrm{PSL}(3,4))$ and $[56,5] = \mathrm{P\Sigma L}(3,4)$. We treat all these groups separately:

- $[56,4]$: Note that $N_{S_{56}}(\mathrm{PSL}(3,4))$ is an odd and [56,4] an even group. The square-free factor of the discriminant of $p(X) - tq(X)$ is given by $\frac{1}{15}(t-1)$. The approach in section 2.3.4.2 then yields
$$[56,4] = \mathrm{Gal}(p(X) - (1+15s^2)q(X) \mid \mathbb{Q}(s)).$$
- $[56,5]$: We see $\sigma_\infty \in [56,5]$ and the computational approach established in section 2.3.4.3 suggests $c = -6$. This would imply
$$[56,5] = \mathrm{Gal}(p(X) - \tfrac{1}{1+6r^2}q(X) \mid \mathbb{Q}(r)).$$
In order to show that c is correct it suffices to proof for $r_0 := 1$ that $G_{r_0} := \mathrm{Gal}(p(X) - \frac{1}{1+6r_0^2}q(X) \mid \mathbb{Q})$ is a subgroup of $[56,5]$. Note that G_{r_0} is a transitive subgroup of $N_{S_{56}}(\mathrm{PSL}(3,4))$ by Dedekind's theorem. Let g be the irreducible degree-21 polynomial (obtained by using the Magma command `GaloisSubgroup` for an index-21 subgroup of $[56,5]$) from the ancillary file `56.txt` and α a root of g. A computer calculation shows that $p(X) - \frac{1}{1+6r_0^2}q(X)$ factors into irreducible polynomials of degree 16 and 40 over $\mathbb{Q}(\alpha)$, therefore G_{r_0} contains a proper subgroup of index dividing 21 with orbit lengths 16 and 40. This implies that G_{r_0} must be a subgroup of $[56,5]$.
- $[56,3]$: Note that $s_1 := \sqrt{\frac{1}{15}(t-1)}$ and $s_2 := \sqrt{-6t(t-1)}$ are $\mathbb{Q}(t)$-primitive elements for the fixed fields of $[56,4]$ and $[56,5]$. By using the fact that $N_{S_{56}}(\mathrm{PSL}(3,4))/\mathrm{PSL}(3,4)$ is a Klein four-group the element $s := \frac{s_2}{3s_1} = \sqrt{-10t}$ turns out to be a primitive element for the fixed field of the remaining index-2 subgroup $[56,3]$. We find
$$[56,3] = \mathrm{Gal}(p(X) + 10s^2q(X) \mid \mathbb{Q}(s)).$$

Via specializing in $p(X) - tq(X)$ we cannot reach the simple index-4 subgroup $\mathrm{PSL}(3,4)$, otherwise there would exist $s_1, s_2 \in \mathbb{Q}$ such that $1 + 15{s_1}^2 = -10{s_2}^2$, see the cases $[56,4]$ and $[56,3]$. This is obviously not possible.

5.1.4. Aut(PSU(3, 3)) of degree 63. (ancillary file `63.txt`)

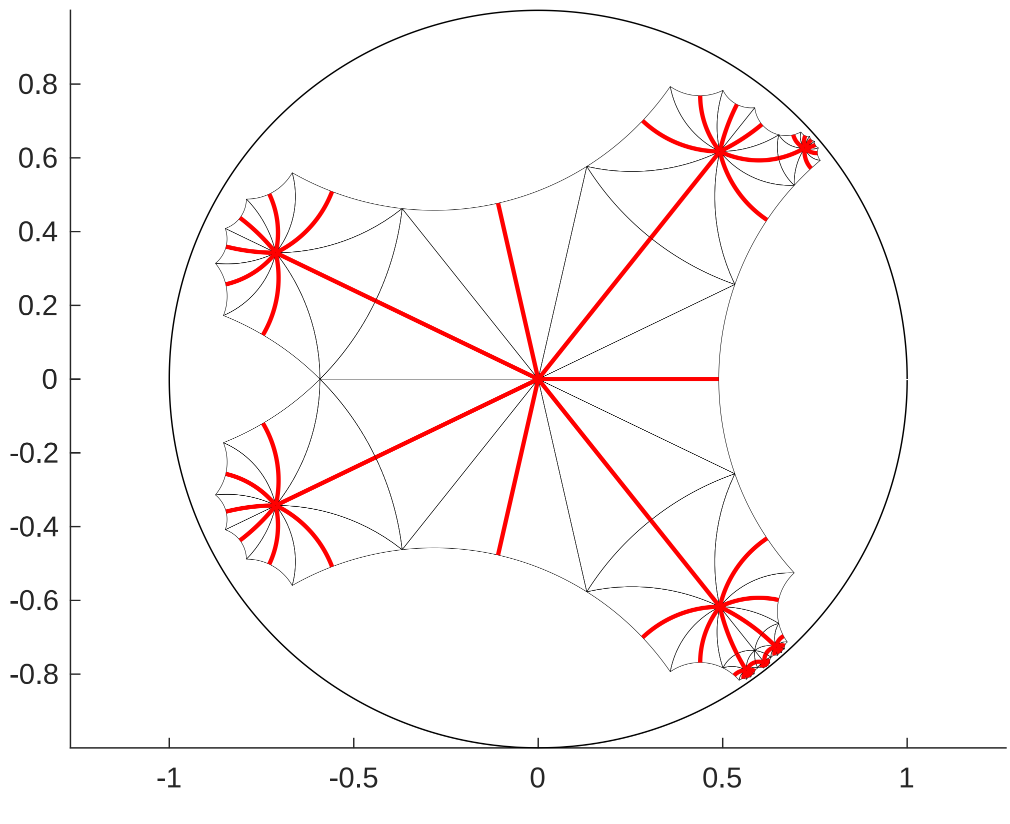
We now study the genus-0 triple $(\sigma_0, \sigma_1, \sigma_\infty)$ for Aut(PSU(3, 3)) given by

$$\begin{aligned}\sigma_0 = {} & (1,43,31,39,63,35,2)(3,17,24,21,20,55,53)(4,29,62,11,14,45,27)\\ & (5,38,23,32,48,18,51)(6,13,36,47,25,8,61)(7,9,15,56,34,28,42)\\ & (10,33,59,60,44,19,37)(12,26,52,30,54,49,41)\\ & (16,40,57,50,22,46,58),\\ \sigma_1 = {} & (2,53)(3,9)(4,38)(5,29)(6,51)(8,25)(10,19)(11,20)(12,44)(13,43)\\ & (14,21)(16,40)(17,32)(18,39)(22,62)(23,30)(24,26)(27,59)(33,54)\\ & (34,63)(35,56)(36,55)(41,49)(42,48)(45,60)(46,61)(47,50)(57,58)\end{aligned}$$

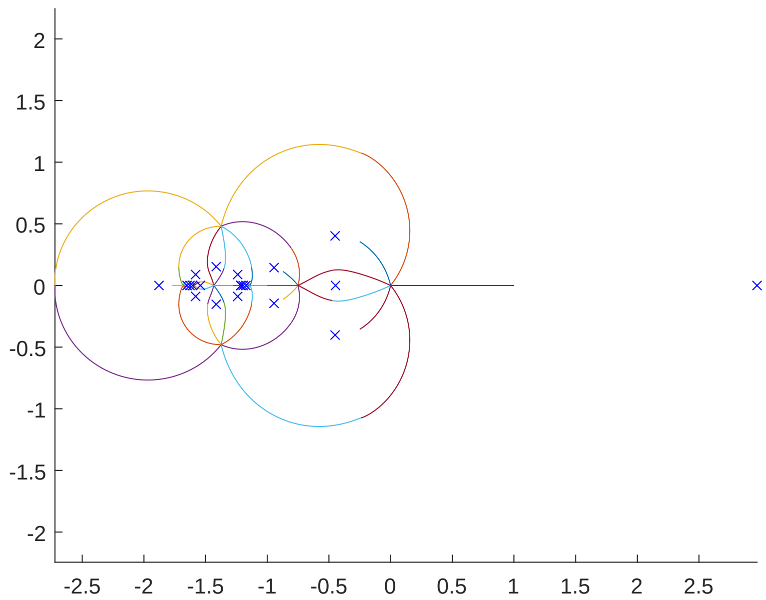
with cycle structure description

	σ_0	σ_1	σ_∞
cycle structure	7^9	$2^{28}.1^7$	$4^{12}.2^6.1^3$

A fundamental domain is given by



with resulting approximate dessin:



The corresponding Belyi map is given by $f = \frac{p}{q}$ where

$$
\begin{aligned}
p &= 2^8 \cdot 3^{12} \cdot (X^2 - X + 2)^7 \cdot (X^3 + 2X^2 - X - 1)^7 \cdot (X^3 + 9X^2 - X - 1)^7,\\
q &= (X^3 + 30X^2 + 27X + 6)\\
&\quad \cdot (X^6 + 18X^5 + 93X^4 + 169X^3 + 144X^2 - 75X - 62)^2\\
&\quad \cdot (X^{12} + 15X^{11} - 15X^{10} - 332X^9 - 2766X^8 + 4002X^7 + 2002X^6\\
&\qquad - 2496X^5 - 1215X^4 + 1047X^3 + 117X^2 - 108X + 36)^4.
\end{aligned}
$$

Verification of monodromy.

The subdegrees of A are 1, 6, 24, 32. By applying Lemma 4.10 we see that A is primitive and only three possibilities for A remain. Since G is normal in A there are four possibilities for G. Fortunately, among these groups only $\mathrm{Aut}(\mathrm{PSU}(3,3))$ contains elements having the same cycle structure as σ_1 and σ_∞. This yields $G = A = \mathrm{Aut}(\mathrm{PSU}(3,3))$. □

Realization of $\mathrm{PSU}(3,3)$.

The only index-2 subgroup of $\mathrm{Aut}(\mathrm{PSU}(3,3))$ is given by $\mathrm{PSU}(3,3)$. We have $\sigma_0 \in \mathrm{PSU}(3,3)$ and from the corresponding character values, see section 2.3.4.1, we obtain $c = 7$, hence

$$\mathrm{PSU}(3,3) = \mathrm{Gal}(p(X) - (1 + 7s^2)q(X) \mid \mathbb{Q}(s)).$$

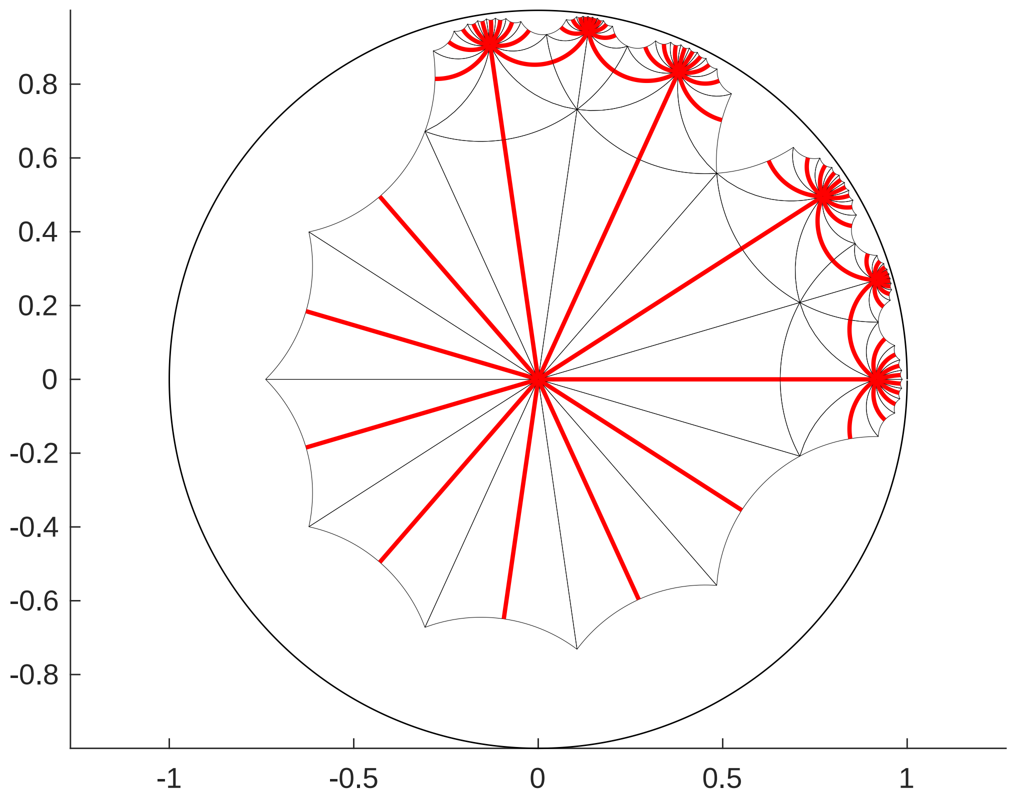
5.1.5. Aut($\mathbf{M_{22}}$) of degree 77. (ancillary file `77.txt`)
This time we work with the genus-0 triple $(\sigma_0, \sigma_1, \sigma_\infty)$ for $\mathrm{Aut}(M_{22})$ where

$$\begin{aligned}
\sigma_0 = {} & (1, 14, 3, 53, 31, 27, 71, 62, 10, 65, 61)(2, 50, 46, 29, 12, 7, 56, 19, 63, 28, 25) \\
& (4, 36, 38, 44, 17, 13, 66, 43, 39, 9, 72)(5, 49, 68, 51, 58, 59, 70, 15, 11, 23, 33) \\
& (6, 55, 42, 67, 32, 21, 45, 64, 48, 77, 57)(8,41,60,20, 26, 74, 76, 24, 69, 52, 40) \\
& (16, 22, 54, 35, 34, 37, 18, 73, 75, 30, 47), \\
\sigma_1 = {} & (1, 54)(2, 59)(3, 48)(4, 20)(6, 32)(7, 29)(11, 38)(13, 43)(14, 51)(15, 19) \\
& (18, 37)(21, 57)(22, 46)(24, 73)(30, 44)(31, 40)(33, 45)(34, 52)(35, 71) \\
& (36, 64)(39, 75)(47, 56)(49, 77)(50, 58)(53, 60)(62, 65)(63, 70)(72, 76).
\end{aligned}$$

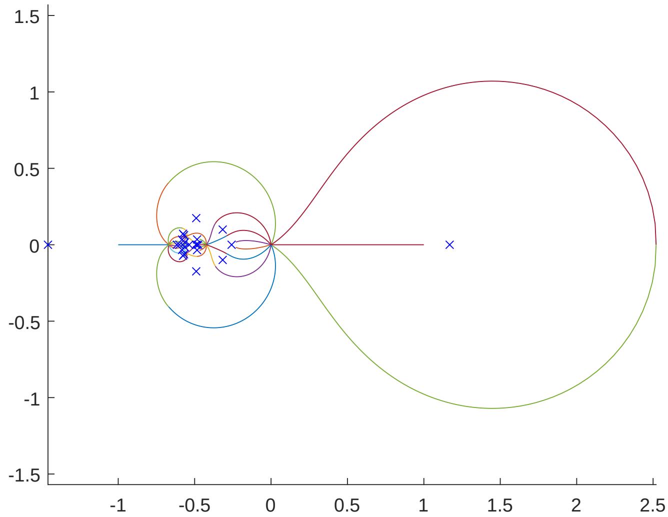
with cycle structures

	σ_0	σ_1	σ_∞
cycle structure	11^7	$2^{28}.1^{21}$	$4^{16}.2^6.1^1$

A fundamental domain is given by



with resulting approximate dessin:



The corresponding Belyi map is $f = \frac{p}{q}$ is given by

$$\begin{aligned} p = & \, 2^{22} \cdot (X^2 + X + 3)^{11} \\ & \cdot (X^5 - 3X^4 - 14X^3 + 15X^2 + X - 1)^{11}, \\ q = & -11^4 \cdot (X^4 + 2X^3 + 7X^2 - 16X - 2)^4 \\ & \cdot (X^6 + 14X^5 + 34X^4 + 8X^3 - 30X^2 + 60X + 16)^4 \\ & \cdot (4X^6 + X^5 + 15X^4 + 10X^3 - 10X^2 - 2X - 2)^4 \\ & \cdot (16X^6 - 29X^5 + 71X^4 - 136X^3 + 92X^2 - 8X - 8)^2. \end{aligned}$$

Verification of monodromy.

The subdegrees of A are 1, 16, 60, therefore A is primitive according to Lemma 4.11. The classification of finite primitive rank 3 groups yields $A = M_{22}$ or $A = \mathrm{Aut}(M_{22})$. Since M_{22} does not contain elements with the same cycle structure as σ_1, we conclude $G = A = \mathrm{Aut}(M_{22})$. $\square$

Note that a degree-22 polynomial having the same splitting field as $q(X) - (1-t)p(X)$ over $\mathbb{Q}(t)$ was already computed by Malle in [**29**].

Realization of M_{22}.

The only index-2 subgroup of $\mathrm{Aut}(\mathrm{M}_{22})$ is given by M_{22}. We have $\sigma_0 \in \mathrm{M}_{22}$ and using the method described in section 2.3.4.1 yields $c = 11$, thus

$$\mathrm{M}_{22} = \mathrm{Gal}(p(X) - (1 + 11s^2)q(X) \mid \mathbb{Q}(s)).$$

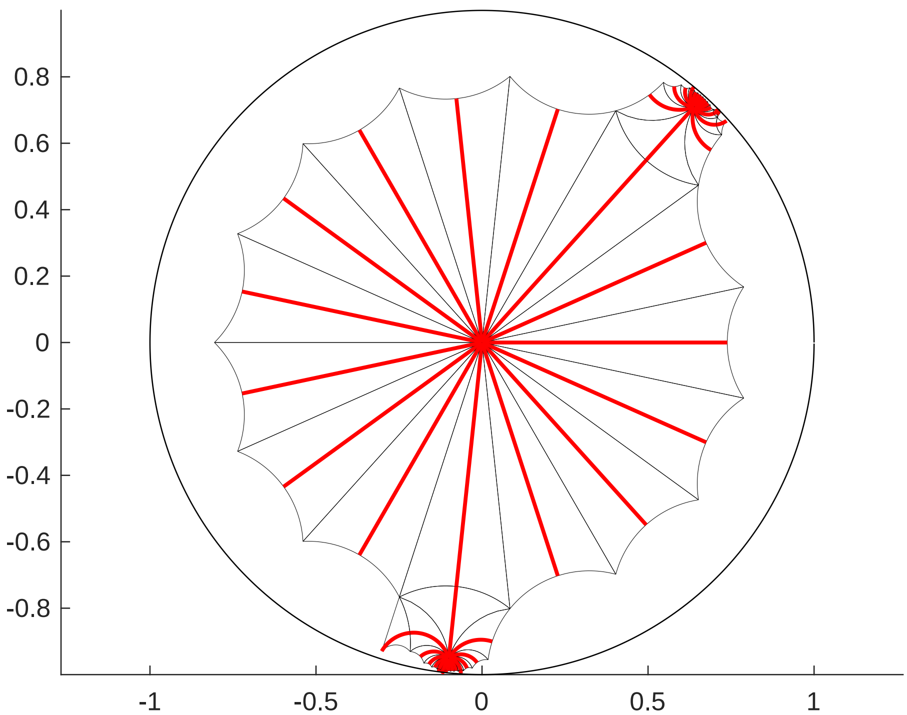
5.1.6. $\mathbf{PSp}(4,4):2$ of degree 85. (ancillary file `85.txt`)
We now consider the genus-0 triple $(\sigma_0, \sigma_1, \sigma_\infty)$ for $\mathrm{PSp}(4,4):2$ where

$$\begin{aligned}\sigma_0 = &(1,85,49,26,15,39,65,24,37,4,23,3,28,19,76)(9,57,83,78,17)\\&(2,82,64,74,52,58,20,70,43,7,68,12,53,40,16)(29,38,61,75,32)\\&(5,81,51,67,54,44,41,77,30,21,71,63,33,66,18)\\&(6,42,46,50,60,22,73,80,47,45,14,31,13,55,79)\\&(8,62,56,36,72,69,35,25,10,84,48,34,59,27,11),\\ \sigma_1 = &(3,70)(5,29)(6,17)(7,68)(8,27)(9,74)(14,23)(15,39)(16,57)(18,33)\\&(19,52)(20,58)(22,35)(24,31)(25,60)(26,65)(30,71)(32,81)(34,56)\\&(36,51)(38,53)(40,63)(41,78)(42,84)(43,61)(44,48)(45,75)(46,50)\\&(47,72)(49,55)(54,67)(64,82)(73,80)(76,79)(77,83).\end{aligned}$$

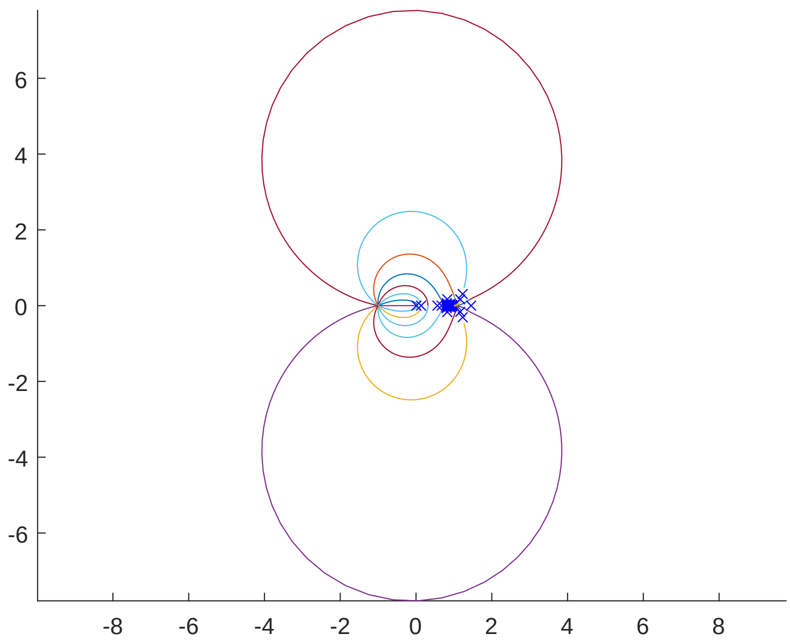
Its cycle structure description is given by

	σ_0	σ_1	σ_∞
cycle structure	$15^5.5^2$	$2^{35}.1^{15}$	$4^{16}.2^7.1^7$

A fundamental domain is given by



with resulting approximate dessin:



This leads to the Belyi map $f = \frac{p}{q}$ where

$$\begin{aligned} p &= 2^{24} \cdot (5X^2 + 5X + 2)^5 \cdot (5X^4 + 10X^3 - 5X - 1)^{15} \\ q &= X \cdot (5X^3 + 20X^2 + 20X + 6)^2 \cdot (5X^4 + 10X^3 - 14X - 10)^4 \\ &\quad \cdot (5X^4 + 10X^3 - 8X - 4)^2 \\ &\quad \cdot (5X^6 + 30X^5 + 60X^4 + 8X^3 - 48X^2 - 24X - 4) \\ &\quad \cdot (625X^{12} + 3750X^{11} + 7500X^{10} + 3500X^9 - 3750X^8 - 1500X^7 \\ &\qquad + 2700X^6 + 3000X^5 + 2100X^4 + 1040X^3 + 240X^2 - 8)^4. \end{aligned}$$

Verification of monodromy.

Again, A is a primitive rank 3 group with subdegrees 1, 20, 64. Thus, we either have $A = \mathrm{PSp}(4,4)$ or $A = \mathrm{PSp}(4,4){:}2$. As $\mathrm{PSp}(4,4)$ does not contain elements with the cycle structure of σ_1, we obtain $A = G = \mathrm{PSp}(4,4){:}2$. $\square$

Realization of $\mathrm{PSp}(4,4)$.

We find $\sigma_0 \in \mathrm{PSp}(4,4)$ and from the corresponding character values we obtain $c = -5$ as described in section 2.3.4.1, thus

$$\mathrm{PSp}(4,4) = \mathrm{Gal}(p(X) - (1 - 5s^2)q(X) \mid \mathbb{Q}(s)).$$

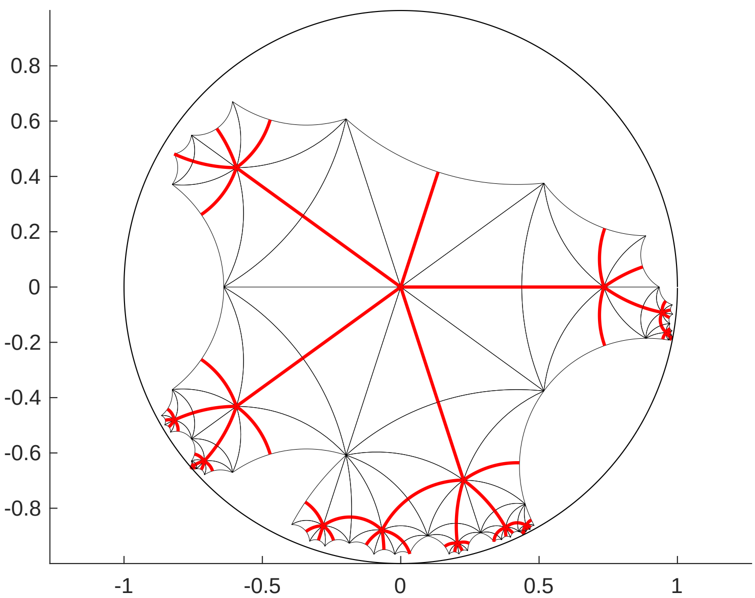
5.1.7. Aut(HS) of degree 100. (ancillary files `100a.txt` and `100b.txt`) The Belyi map for the first genus-0 triple $(\sigma_0, \sigma_1, \sigma_\infty)$ for Aut(HS) is given in Part 7 in the computation chapter and in file `100a.txt`. The second genus-0 triple $(\sigma_0, \sigma_1, \sigma_\infty)$ for Aut(HS) is given by

$$\begin{aligned}
\sigma_0 = {} & (1,64,8,54,37)(2,20,81,42,49)(3,98,32,73,89)(4,96,86,15,79) \\
& (5,22,28,78,48)(6,67,97,40,14)(7,58,82,59,18)(9,16,87,85,60) \\
& (10,70,41,56,55)(11,77,36,25,68)(12,17,19,21,80)(13,35,90,33,91) \\
& (23,50,66,84,27)(24,72,95,52,76)(26,99,100,57,93)(29,71,38,69,65) \\
& (30,74,94,53,51)(31,45,47,75,34)(43,63,44,46,62), \\
\sigma_1 = {} & (1,20)(2,64)(3,76)(4,45)(5,83)(6,26)(7,13)(8,74)(9,41)(10,63)(11,25) \\
& (12,66)(14,21)(15,52)(16,62)(17,33)(18,35)(19,42)(22,60)(23,58) \\
& (24,73)(28,98)(29,82)(30,53)(31,61)(32,59)(34,67)(36,95)(37,85) \\
& (38,47)(39,51)(40,80)(43,92)(44,78)(46,99)(48,55)(49,94)(50,91) \\
& (54,90)(65,88)(69,72)(71,75)(77,79)(81,87)(84,97)(86,100)(93,96)
\end{aligned}$$

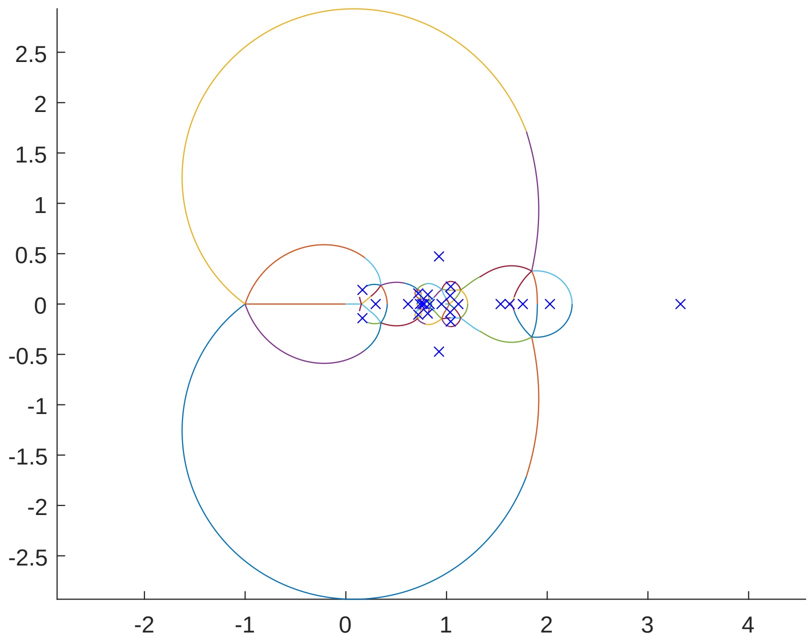
of type

	σ_0	σ_1	σ_∞
cycle structure	$5^{19}.1^5$	$2^{47}.1^6$	$6^{10}.3^{10}.2^5$

A fundamental domain is given by



with resulting approximate dessin:



The resulting Belyi map $f = \frac{p}{q}$ consists of

$$\begin{aligned} p &= 3^3 \cdot (X^4 - 8X^3 - 6X^2 + 8X + 1)^5 \cdot (X^5 - 5X^4 + 50X^3 + 70X^2 \\ &\qquad + 25X + 3)^5 \cdot (3X^5 - 5X^4 - 5X^3 + 35X^2 + 40X + 4) \\ &\quad \cdot (9X^{10} - 30X^9 + 55X^8 - 200X^7 + 210X^6 + 924X^5 \\ &\qquad - 890X^4 - 360X^3 + 1925X^2 - 1070X + 291)^5, \\ q &= (3X^5 - 35X^4 + 90X^3 - 50X^2 + 15X + 9)^2 \\ &\quad \cdot (9X^{10} - 120X^9 + 10X^8 - 1960X^7 - 1090X^6 + 3304X^5 \\ &\qquad - 760X^4 - 920X^3 + 145X^2 + 80X + 6)^3 \\ &\quad \cdot (3X^{10} - 10X^9 - 65X^8 + 160X^7 - 90X^6 - 932X^5 \\ &\qquad - 330X^4 + 880X^3 + 1255X^2 + 830X + 27)^6. \end{aligned}$$

Verification of monodromy.

One can use the exact same proof as for Theorem 4.12. □

Another realization of HS.

Up to squares the discriminant of $p(X) - tq(X)$ is equal to $\frac{1}{2}(t-1)$, therefore

$$\mathrm{HS} = \mathrm{Gal}(p(X) - (1 + 2s^2)q(X) \mid \mathbb{Q}(s)).$$

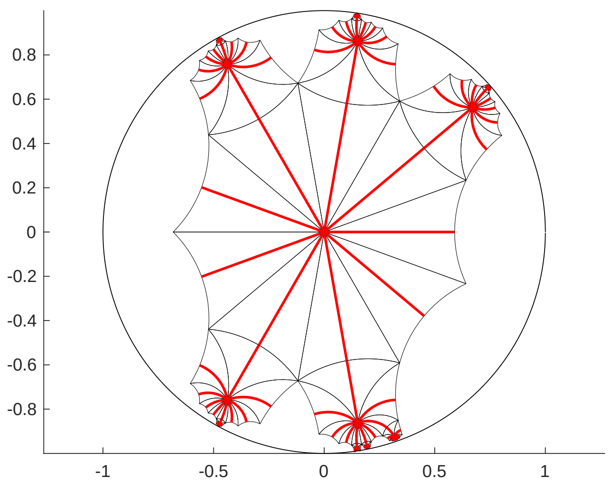
5.1.8. $\mathbf{O^+(8,2)}$ of degree 135. (ancillary file `135.txt`)
We work with the genus-0 triple $(\sigma_0, \sigma_1, \sigma_\infty)$ for $O^+(8,2)$ where

$$\begin{aligned}\sigma_0 = {} & (1,94,65,71,134,80,107,98,4)(2,104,58,121,97,116,88,8,23)\\ & (3,69,36,32,29,73,102,128,106)(5,14,124,105,67,18,49,117,34)\\ & (6,28,100,41,135,31,48,109,17)(7,133,112,53,91,15,25,122,129)\\ & (9,62,99,96,131,77,10,81,52)(11,56,110,13,115,111,95,89,54)\\ & (12,64,113,108,20,76,50,22,55)(16,61,83,118,75,66,39,35,132)\\ & (19,85,68,126,40,125,74,130,43)(21,47,79,78,72,84,24,37,57)\\ & (26,38,70,90,92,103,63,120,44)(27,119,127,42,87,82,101,93,45)\\ & (30,59,86,51,33,60,123,46,114),\\ \sigma_1 = {} & (3,118)(4,110)(5,132)(7,36)(9,33)(10,46)(12,112)(13,129)(16,65)\\ & (17,106)(20,113)(21,107)(22,55)(25,61)(26,27)(28,30)(29,37)(31,109)\\ & (32,98)(35,130)(40,42)(43,99)(44,125)(45,90)(47,49)(50,91)(51,93)\\ & (52,60)(54,56)(58,116)(59,128)(62,82)(63,73)(64,69)(66,96)(67,78)\\ & (71,117)(72,105)(74,124)(75,135)(76,83)(77,121)(80,134)(84,120)\\ & (85,87)(86,103)(88,104)(89,94)(95,122)(97,114)(100,131)(119,127).\end{aligned}$$

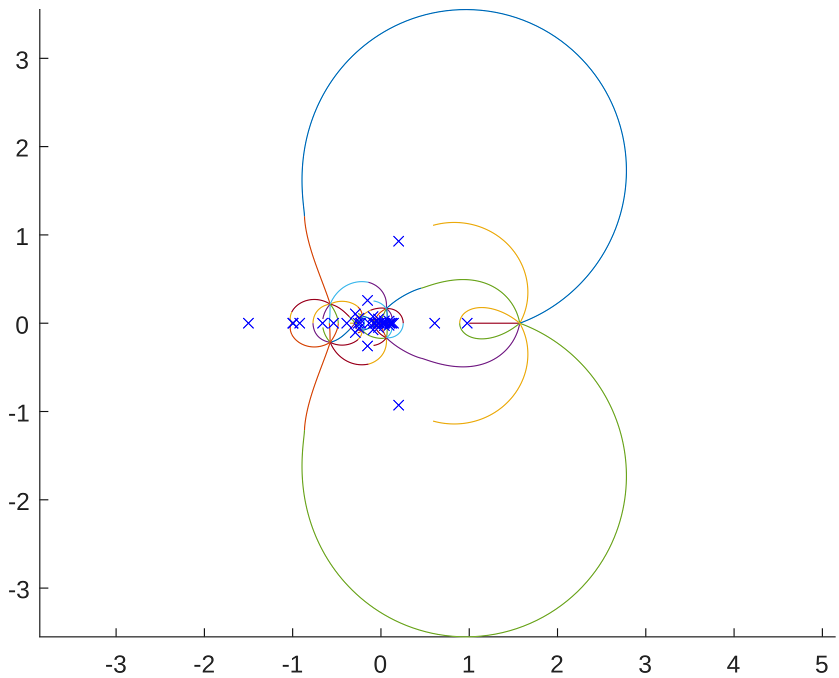
The cycle structure of $(\sigma_0, \sigma_1, \sigma_\infty)$ is given by

	σ_0	σ_1	σ_∞
cycle structure	9^{15}	$2^{52}.1^{31}$	$4^{30}.2^{6}.1^{3}$

A fundamental domain is given by



with resulting approximate dessin:



The computed Belyi map $f = \frac{p}{q}$ consists of

$$\begin{aligned}p = 2^{22} &\cdot (3X^3 - 9X^2 - 9X - 2)^9 \\ &\cdot (3X^3 + 9X^2 + 6X + 1)^9 \\ &\cdot (27X^9 + 243X^8 + 567X^7 + 513X^6 + 162X^5 \\ &\quad - 27X^4 + 9X^3 + 27X^2 + 9X + 1)^9\end{aligned}$$

and

$$\begin{aligned}q = (3X^3 &- 9X - 2) \cdot (6X^3 + 9X^2 - 1)^2 \\ &\cdot (3X^3 + 27X^2 + 27X + 7)^2 \\ &\cdot (36X^6 + 189X^5 + 189X^4 + 96X^3 + 36X^2 + 9X + 1)^4 \\ &\cdot (81X^{12} + 1944X^{11} + 11178X^{10} + 27648X^9 + 29403X^8 - 1944X^7 \\ &\quad - 39150X^6 - 44712X^5 - 25434X^4 - 8088X^3 - 1332X^2 - 72X + 4)^4 \\ &\cdot (648X^{12} + 3888X^{11} + 11907X^{10} + 15120X^9 + 13365X^8 + 14580X^7 \\ &\quad + 11772X^6 + 3240X^5 - 1782X^4 - 1632X^3 - 504X^2 - 72X - 4)^4.\end{aligned}$$

Verification of monodromy.
We find that A is a primitive rank-3 group with subdegrees 1, 64, 70. Therefore we have $A = \mathrm{O}^+(8,2)$ or $A = \mathrm{O}^+(8,2).2$ and due to normality $G = \mathrm{O}^+(8,2)$ or $G = \mathrm{O}^+(8,2).2$. Note that G is generated by permutations of cycle structure 9^{15}, $2^{52}.1^{31}$, $4^{30}.2^6.1^3$ and by inspecting the sizes of conjugacy classes of $\mathrm{O}^+(8,2)$ and $\mathrm{O}^+(8,2).2$ we can conclude that there are no elements with these cycle structures in $\mathrm{O}^+(8,2).2 \setminus \mathrm{O}^+(8,2)$. It follows $G = \mathrm{O}^+(8,2)$. Because $\mathrm{O}^+(8,2)$ contains only one genus-0 triple (up to simultaneous conjugation) having the desired cycle structure description the ramification triple is given by $(\sigma_0, \sigma_1, \sigma_\infty)$.

We will now prove that $p(X) - tq(X)$ defines a regular Galois extension of $\mathbb{Q}(t)$ with Galois group $\mathrm{O}^+(8,2)$: Let

$$f^* = \frac{p^*}{q^*} \in \mathbb{Q}(X)$$

be the map from Theorem 5.1 belonging to the genus-0 triple $(\sigma_0, \sigma_1, \sigma_\infty)$ for $\mathrm{O}^+(8,2)$. Since the normalizer of $\langle \sigma_1 \rangle$ in $\mathrm{O}^+(8,2)$ has a unique fixed point we may assume $f^*(\infty) = 1$ after applying an inner $\mathbb{Q}$-Möbius transformation due to Lemma 2.4 (a) and (c). Furthermore we find $\alpha, \beta \in \mathbb{Q}$ such that $f^*(\alpha X + \beta) = f(X)$.

> This is possible: Since ∞ is the unique rational pre-image of 1 under f and both f and f^* have the same ramification triple there exist $\alpha, \beta \in \mathbb{C}$ such that $f^*(\alpha X + \beta) = f(X)$ by the uniqueness property of Riemann's existence theorem, see Theorem 2.2. The explicit determination of α and β boils down to solving a linear system defined over $\mathbb{Q}$. As the existence of a complex solution guarantees the existence of a rational solution the assertion follows.

Since $p^*(X) - tq^*(X)$ defines a regular extension of $\mathbb{Q}(t)$ with Galois group $\mathrm{O}^+(8,2)$ the same must also hold for the translated polynomial $p(X) - tq(X)$. $\square$

5.2. A theorem of Magaard

Another natural question to ask is which groups occur as the monodromy group of rational functions, or equivalently, which transitive groups contain generating genus-0 tuples.

The Guralnick-Thompson conjecture [**20**] (proven by Frohardt and Magaard in [**18**]) states that only finitely many non-abelian and non-alternating simple groups occur as composition factors of monodromy groups of rational functions. In the special case for sporadic groups a complete list is delivered by Magaard [**28**]:

Theorem 5.3. *Let G be a sporadic finite simple group. Then there exists a rational function $f \in \mathbb{C}(X)$ such that G is a composition factor of the monodromy group of f if and only if :*

$$G \cong \begin{cases} \mathrm{M}_{11},\ \mathrm{M}_{12},\ \mathrm{M}_{22},\ \mathrm{M}_{23},\ \mathrm{M}_{24}, \\ \mathrm{HS},\ \mathrm{J}_1,\ \mathrm{J}_2, \mathrm{Co}_3 \end{cases}$$

As discovered by König [**24**, Proposition 9.2] the sporadic group HS is missing in Magaard's original publication.

Several explicit rational functions with monodromy groups isomorphic to the above Mathieu groups were calculated by Matzat [**33**], [**34**], König [**25**], Malle [**29**], Elkies [**17**], Hoyden-Siedersleben and Matzat [**21**].

For the remaining groups HS, J_1, J_2 and Co_3 there exist genus-0 triples for permutations groups of degree ≥ 100 having one of these sporadic groups as a composition factor.

5.2.1. The sporadic Higman-Sims group HS of degree 100.

Rational functions with Aut(HS) as a monodromy group are presented in Part 7 and section 5.1.7.

5.2.2. The sporadic Janko group J_1 of degree 266.

Let $(\sigma_0, \sigma_1, \sigma_\infty)$ be the genus-0 triple for J_1 of degree 266 where

$$\begin{aligned}\sigma_0 = {} & (1,263,66,201,120,150,247)(2,212,109,156,234,62,253)(3,93,226,139,259,160,92)\\ & (4,52,5,145,152,239,248)(6,246,88,233,173,111,209)(7,124,117,260,166,81,119)\\ & (8,188,86,37,32,112,170)(9,11,102,154,89,65,163)(10,101,182,223,13,158,115)\\ & (12,123,54,100,122,192,224)(14,80,28,171,230,255,262)(15,258,242,20,26,236,187)\\ & (16,219,227,206,237,58,129)(17,202,55,71,116,24,169)(18,252,159,198,34,82,238)\\ & (19,64,23,60,70,203,77)(21,106,235,245,43,50,229)(22,68,218,76,157,63,194)\\ & (25,72,178,134,103,228,213)(27,175,256,40,47,35,132)(29,97,33,79,240,56,176)\\ & (30,131,42,216,38,210,193)(31,225,46,45,67,78,121)(36,94,264,149,91,199,147)\\ & (39,251,104,174,204,87,177)(41,249,185,164,254,137,197)(44,143,110,221,191,184,135)\\ & (48,148,142,266,74,85,208)(49,114,167,217,61,244,220)(51,162,95,222,133,99,90)\\ & (53,126,181,232,195,138,141)(57,231,155,84,144,140,59)(69,205,250,113,257,168,151)\\ & (73,190,83,265,189,165,75)(96,215,172,108,118,214,180)(98,127,186,261,196,207,243)\\ & (105,130,128,107,136,125,179)(146,211,161,153,241,200,183);\\ \sigma_1 = {} & (1,170)(2,253)(3,243)(4,121)(5,208)(6,177)(7,97)(8,80)(9,254)(10,189)(11,108)\\ & (12,94)(13,266)(14,190)(15,145)(16,163)(17,134)(18,40)(19,158)(20,127)(21,131)\\ & (22,109)(23,105)(24,193)(25,144)(26,133)(27,132)(28,247)(29,119)(30,37)(31,198)\\ & (32,116)(33,176)(34,248)(35,46)(36,59)(38,244)(39,82)(41,212)(45,173)(47,111)\\ & (48,141)(49,245)(50,220)(51,246)(52,53)(54,213)(55,147)(56,240)(57,202)(58,218)\\ & (60,136)(61,250)(62,249)(63,156)(64,142)(65,195)(66,242)(67,241)(68,197)(69,169)\\ & (70,154)(71,196)(72,260)(73,188)(74,201)(75,235)(76,192)(77,217)(78,126)(81,124)\\ & (83,101)(84,166)(85,258)(86,106)(87,162)(88,160)(89,107)(90,92)(91,93)(95,236)\\ & (96,100)(98,99)(102,257)(103,151)(110,185)(112,261)(113,203)(114,165)(115,167)\\ & (117,155)(118,164)(120,223)(122,184)(123,140)(125,179)(128,138)(129,137)\\ & (130,148)(135,180)(143,214)(146,226)(149,211)(150,255)(152,174)(153,181)\\ & (157,191)(159,256)(161,227)(168,172)(171,230)(175,225)(178,231)(182,262)\\ & (183,259)(186,263)(187,204)(199,207)(200,233)(205,210)(206,264)(209,238)\\ & (215,228)(216,229)(219,232)(221,234)(224,237)(239,251)\end{aligned}$$

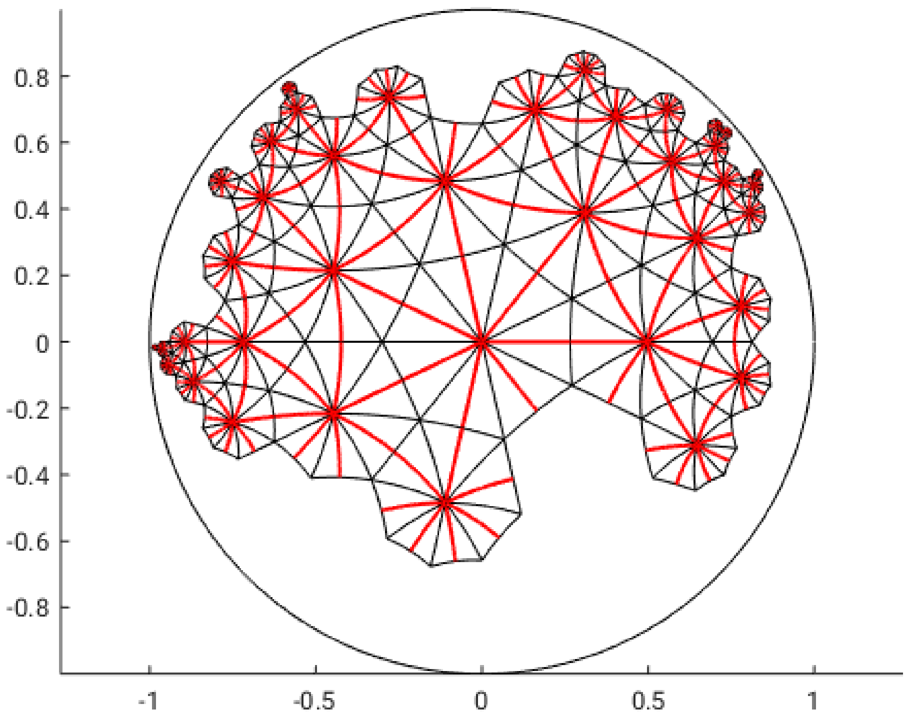
with cycle structure description

	σ_0	σ_1	σ_∞
cycle structure	7^{38}	$2^{128}.1^{10}$	$3^{87}.1^5$

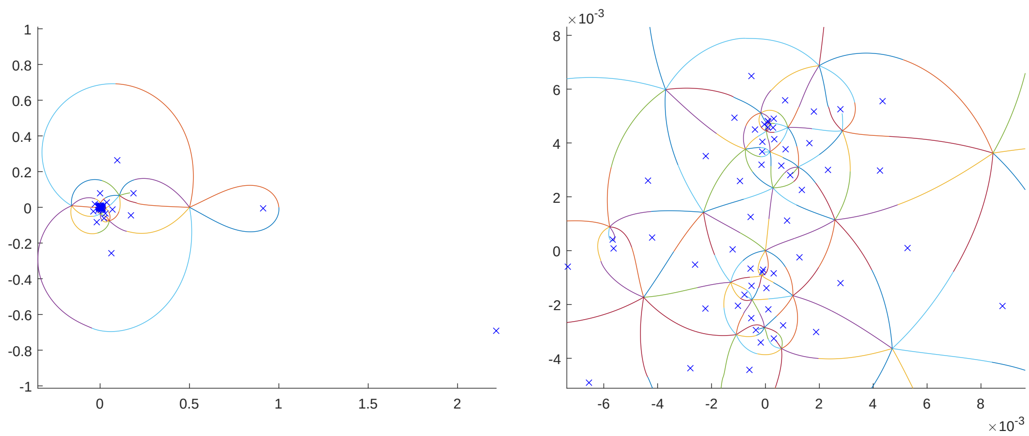
and the following properties:

- The permutations $\sigma_0, \sigma_1, \sigma_\infty$ are each contained in rational conjugacy classes of J_1.
- $(\sigma_0, \sigma_1, \sigma_\infty)$ is not rigid with $\ell(\sigma_0, \sigma_1, \sigma_\infty) = 7$.

A nice fundamental domain for $(\sigma_0, \sigma_1, \sigma_\infty)$ is given by



with resulting approximate dessin:



The computed Belyi map

$$f = \frac{p}{q} = 1 + \frac{r}{q}$$

defined over the number field $K = \mathbb{Q}(\alpha)$ where

$$\alpha^7 - \alpha^6 - 2\alpha^4 - \alpha^3 + 2\alpha^2 + 2\alpha + 2 = 0$$

is presented in file `266.txt`.

Theorem 5.4. *(a) $f = \frac{p}{q}$ is a Belyi map with ramification triple of the same cycle structure description as $(\sigma_0, \sigma_1, \sigma_\infty)$.*

(b) The arithmetic and geometric monodromy group A and G of f are both isomorphic to J_1. In particular,

$$p(X) - tq(X) \in K(t)[X]$$

defines a regular extension of $K(t)$ with Galois group J_1.

Proof. (a) The inseparability behaviour of p, q and r in combination with the Riemann-Hurwitz genus formula (Lemma 2.1 (d)) implies the assertion.

(b) According to Lemma 4.10 the group A is primitive as condition (10) in Proposition 4.3 from [**2**] is not satisfied modulo $\mathfrak{p} := (5, 2 + \alpha)$. By the classification of finite primitive groups A is isomorphic to J_1, A_{266} or S_{266}. Since

$$\frac{p(X) - f(t)q(X)}{X - t} \in K(t)[X]$$

has a degree-11 divisor, see ancillary file `266divisor.txt`, the group A cannot be 2-transitive and we remain with $A = \mathrm{J}_1$. We also find $G = \mathrm{J}_1$ because G is normal in A and J_1 is simple. $\square$

The degree-11 divisor of $p(X) - f(t)q(X)$ in the previous proof was computed via interpolation by using sufficiently enough specializations in t and factorizing the resulting polynomials.

5.2.3. The sporadic Janko group J_2 of degree 280.
Note that there exists a degree-100 genus-0 triple for J_2 which was first realized by Monien in [**36**].

There is also a degree-280 permutation representation of the group J_2 in which Monien's triple $(\sigma_0, \sigma_1, \sigma_\infty) \in {S_{280}}^3$ where

$$
\begin{aligned}
\sigma_0 = {} & (1,69,269,102,88,71,83)(2,254,143,226,117,238,134)(3,163,30,174,181,162,152)\\
& (4,131,154,160,230,249,158)(5,54,46,245,111,184,231)(6,256,132,201,200,15,271)\\
& (7,161,229,136,171,29,34)(8,141,209,133,223,240,66)(9,207,126,156,191,261,228)\\
& (10,185,129,100,97,58,147)(11,225,39,220,41,192,27)(12,265,241,110,144,42,151)\\
& (13,280,48,124,237,108,224)(14,187,266,193,72,67,85)(16,103,47,264,135,186,19)\\
& (17,114,227,20,53,148,80)(18,98,65,118,274,89,257)(21,244,157,276,250,32,62)\\
& (22,242,116,25,208,56,267)(23,87,127,79,45,221,239)(24,195,251,44,189,95,94)\\
& (26,64,168,40,278,199,155)(28,169,275,172,243,125,194)(31,93,81,183,253,204,37)\\
& (33,120,35,74,279,115,203)(36,232,59,259,91,113,61)(38,236,233,247,99,170,260)\\
& (43,175,121,202,57,196,78)(49,146,197,235,206,277,234)(50,137,86,263,128,217,55)\\
& (51,77,138,139,248,76,255)(52,164,212,112,214,123,198)(60,178,246,176,75,218,101)\\
& (63,268,122,145,166,177,130)(68,90,182,210,82,167,272)(70,153,92,150,159,215,252)\\
& (73,142,106,104,180,219,188)(84,119,258,179,273,216,211)\\
& (96,173,190,109,213,105,165)(107,222,262,140,270,205,149),\\
\sigma_1 = {} & (1,230)(2,48)(3,8)(4,77)(5,96)(6,46)(7,215)(9,122)(11,226)(13,100)(14,161)(15,52)\\
& (16,151)(17,80)(18,251)(19,265)(20,209)(21,24)(22,139)(23,74)(25,232)(26,118)\\
& (27,66)(28,54)(29,34)(30,128)(31,234)(32,195)(33,87)(35,120)(36,83)(37,146)\\
& (38,218)(39,229)(40,149)(41,217)(42,43)(44,266)(45,157)(47,95)(51,63)(53,248)\\
& (55,67)(56,114)(57,244)(58,119)(59,242)(60,237)(61,249)(64,104)(65,180)(68,185)\\
& (69,173)(70,81)(71,208)(72,137)(75,125)(76,141)(78,103)(79,188)(82,153)(84,182)\\
& (85,220)(86,89)(88,227)(90,147)(91,158)(92,264)(93,167)(94,196)(97,270)(98,250)\\
& (99,263)(101,199)(102,133)(105,177)(106,203)(107,279)(108,278)(111,238)(112,134)\\
& (115,168)(117,240)(121,262)(123,271)(124,212)(126,233)(127,142)(129,206)(130,131)\\
& (132,176)(135,210)(136,183)(138,259)(140,258)(143,204)(144,273)(145,191)(148,267)\\
& (150,189)(152,255)(154,213)(155,260)(156,172)(159,187)(160,190)(162,268)(163,192)\\
& (164,178)(165,169)(166,275)(170,274)(171,252)(174,247)(175,179)(181,207)(184,223)\\
& (186,211)(193,257)(194,256)(197,254)(200,246)(202,221)(205,224)(214,245)(216,241)\\
& (219,276)(222,239)(225,253)(228,261)(231,269)(235,280)(236,243)(272,277)
\end{aligned}
$$

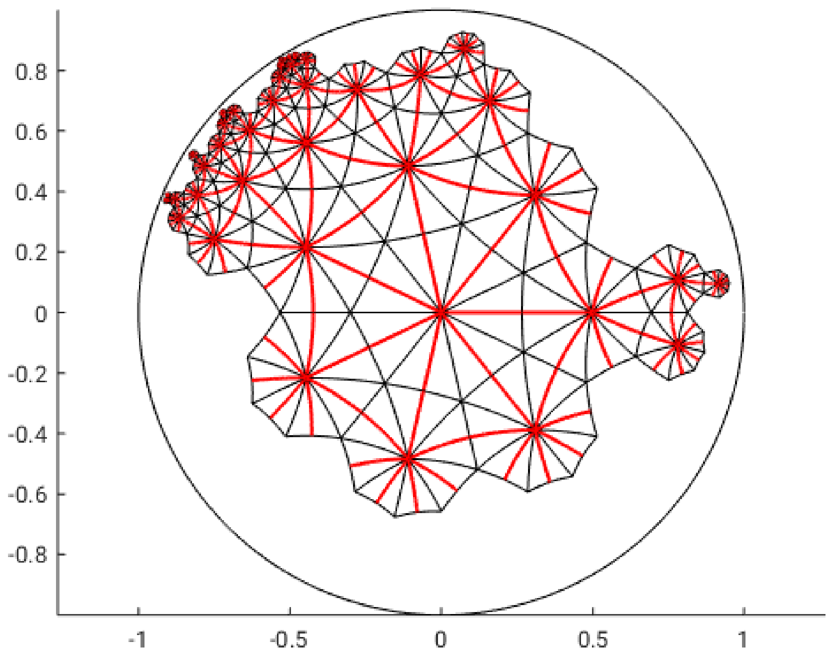
also turns out to be a genus-0 triple with cycle structure description

	σ_0	σ_1	σ_∞
cycle structure	7^{40}	$2^{134}.1^{12}$	$3^{92}.1^4$

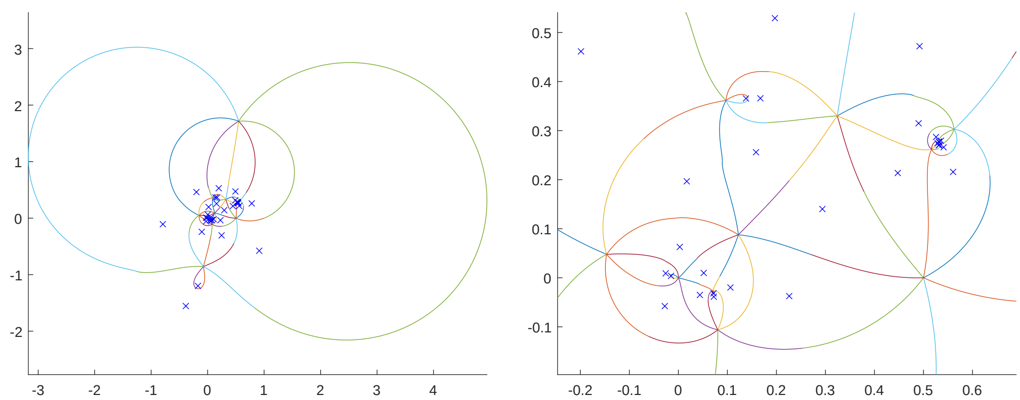
and the following properties:

- $\sigma_0, \sigma_1, \sigma_\infty$ are each contained in rational conjugacy classes of J_2.
- $(\sigma_0, \sigma_1, \sigma_\infty)$ is not rigid with $\ell(\sigma_0, \sigma_1, \sigma_\infty) = 10$.

A fundamental domain for $(\sigma_0, \sigma_1, \sigma_\infty)$ is given by



with resulting approximate dessin:



The computed Belyi map

$$f = \frac{p}{q} = 1 + \frac{r}{q},$$

see file `280.txt`, is defined over the number field $K = \mathbb{Q}(\alpha)$ where

$$\alpha^{10} - \alpha^9 - 4\alpha^8 - 2\alpha^7 + 65\alpha^6 + 27\alpha^5 + 11\alpha^4 - 89\alpha^3 + 25\alpha^2 - 13\alpha + 1 = 0.$$

Theorem 5.5. *(a) f is a Belyi map with ramification triple having same cycle structure description as $(\sigma_0, \sigma_1, \sigma_\infty)$.*

(b) The arithmetic monodromy group A of f is isomorphic to $\mathrm{Aut}(\mathrm{J}_2)$ *and the geometric monodromy group G of f is isomorphic to* J_2.

PROOF. (a) This follows from the inseparability behaviour of p, q and r in combination with the Riemann-Hurwitz genus formula, see Lemma 2.1 (d).

(b) Let $\mathcal{O}_K$ be the ring of integers in K and $\mathfrak{p} := (283, 167 + \alpha)$ the prime ideal in $\mathcal{O}_K$ lying over 283. Since all coefficients of p and q are contained in the localization of $\mathcal{O}_K$ at $\mathfrak{p}$ we can reduce them modulo $\mathfrak{p}$ to obtain polynomials $\bar{p}, \bar{q} \in \mathbb{F}_{283}[X]$ leading us to study

$$A_{\mathbb{F}} := \mathrm{Gal}(\bar{p}(X) - t\bar{q}(X) \mid \mathbb{F}_{283}(t)).$$

Computing the irreducible factors of

$$\bar{p}(X) - \frac{\bar{p}(t)}{\bar{q}(t)}\bar{q}(X) \in \mathbb{F}_{283}(t)[X]$$

enables us to determine the subdegrees of $A_{\mathbb{F}}$ which turn out to be 1, 36, 108 and 135. With the help of Corollary 4.11 we see that $A_{\mathbb{F}}$ must be primitive. According to the Magma database for finite primitive groups and the fact that the discriminant of $\bar{p} - t\bar{q}$ is not a square in $\mathbb{F}_{283}(t)$ only one possibility remains: $A_{\mathbb{F}} = \mathrm{Aut}(\mathrm{J}_2)$. Thanks to Dedekind reduction [**27**, Theorem VII.2.9] this implies: A is primitive and $\mathrm{Aut}(\mathrm{J}_2)$ is a subgroup of A, therefore $A = \mathrm{Aut}(\mathrm{J}_2)$ or $A = S_{280}$. Since

$$p(X) - \frac{p(t)}{q(t)}q(X) \in K(t)[X]$$

has a divisor of degree 36, see ancillary file `280divisor.txt`, A is not 2-transitive, thus $A = \mathrm{Aut}(\mathrm{J}_2)$.

Taking into account that G is normal in A and J_2 is simple we also find $G = \mathrm{J}_2$ or $G = \mathrm{Aut}(\mathrm{J}_2)$. Because G is generated by elements having the same cycle structures as σ_0, σ_1 and σ_∞ we can conclude that G must be an even group, therefore $G = \mathrm{J}_2$. $\square$

5.2.4. The sporadic Conway group Co_3 of degree 276. Complex approximations of a Belyi map with monodromy group Co_3 were independently computed Barth/W. and Monien.

A genus-0 triple for Co_3 is given by $(\sigma_0, \sigma_1, \sigma_\infty) \in {S_{280}}^3$ where

$$\begin{aligned}
\sigma_0 = {} & (1,59,221,174,190,137,96)(2,86,51,27,245,275,111)(3,104,11,131,136,155,25)\\
& (4,211,8,161,80,83,79)(5,23,20,239,168,145,147)(6,126,71,95,122,15,143)\\
& (7,19,31,16,201,246,40)(9,158,271,253,202,12,251)(10,276,100,97,84,42,175)\\
& (13,207,194,173,109,85,152)(14,219,115,261,274,159,146)(17,62,119,259,170,256,192)\\
& (18,216,26,61,183,267,151)(21,67,160,223,110,88,150)(22,196,156,181,72,265,225)\\
& (24,116,66,258,105,92,222)(28,209,44,117,269,74,58)(29,30,141,41,218,179,149)\\
& (32,198,235,208,262,182,164)(33,180,128,244,266,68,272)(34,212,247,242,103,91,125)\\
& (35,215,139,45,236,254,185)(36,121,98,101,78,50,176)(37,230,153,47,206,217,234)\\
& (38,65,132,184,273,166,227)(39,214,114,87,204,90,118)(43,210,135,228,270,241,76)\\
& (46,189,255,48,130,89,120)(49,56,197,205,144,70,123)(52,199,142,93,124,54,250)\\
& (53,113,112,73,224,107,69)(55,231,134,220,167,260,163)(57,193,243,94,129,63,77)\\
& (64,99,162,75,268,238,237)(81,178,106,108,229,169,102)(82,200,171,138,248,195,188)\\
& (127,187,264,249,252,148,263)(133,172,154,165,157,186,226)\\
& (177,240,191,203,233,232,257),\\
\sigma_1 = {} & (1,162)(2,96)(3,184)(4,248)(5,102)(6,26)(7,73)(8,149)(9,256)(10,222)(11,132)\\
& (12,63)(13,240)(14,85)(15,234)(16,116)(17,270)(18,126)(19,174)(20,84)(21,167)\\
& (23,125)(24,263)(25,115)(27,226)(28,197)(29,71)(30,151)(31,199)(32,154)(33,121)\\
& (34,169)(35,163)(36,44)(37,171)(38,276)(39,250)(40,53)(41,253)(42,91)(45,74)\\
& (46,193)(47,113)(48,141)(49,254)(50,271)(51,200)(52,66)(54,76)(55,67)(56,128)\\
& (58,144)(59,259)(60,107)(61,264)(62,135)(64,158)(65,92)(68,268)(69,224)(70,139)\\
& (72,232)(75,111)(77,189)(78,218)(80,164)(81,203)(82,133)(83,195)(86,230)(87,251)\\
& (88,260)(89,247)(90,136)(93,119)(94,155)(95,211)(97,177)(98,245)(99,170)(100,207)\\
& (101,186)(103,187)(105,131)(106,191)(108,152)(109,166)(110,215)(112,190)(114,192)\\
& (117,238)(118,258)(120,274)(122,138)(123,223)(124,210)(127,175)(129,204)(130,267)\\
& (134,213)(137,153)(140,235)(142,221)(143,249)(145,265)(146,229)(147,233)(148,201)\\
& (156,225)(157,179)(159,212)(160,185)(161,165)(168,181)(172,262)(176,237)(180,209)\\
& (182,188)(183,242)(194,227)(198,208)(202,255)(206,246)(214,241)(217,252)(219,273)\\
& (220,231)(236,244)(239,257)(243,261)(266,269)(272,275)
\end{aligned}$$

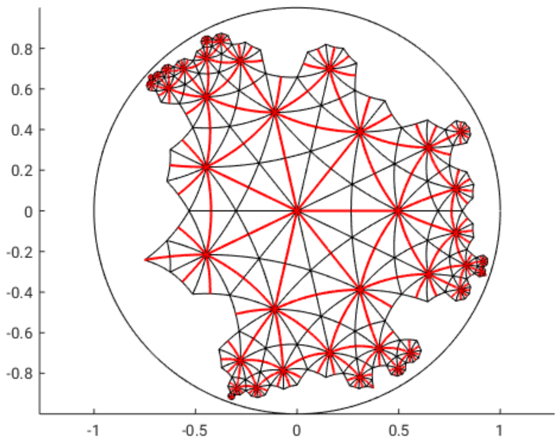
with cycle structure description

	σ_0	σ_1	σ_∞
cycle structure	$7^{39}.1^3$	$2^{132}.1^{12}$	$3^{92}.1^4$

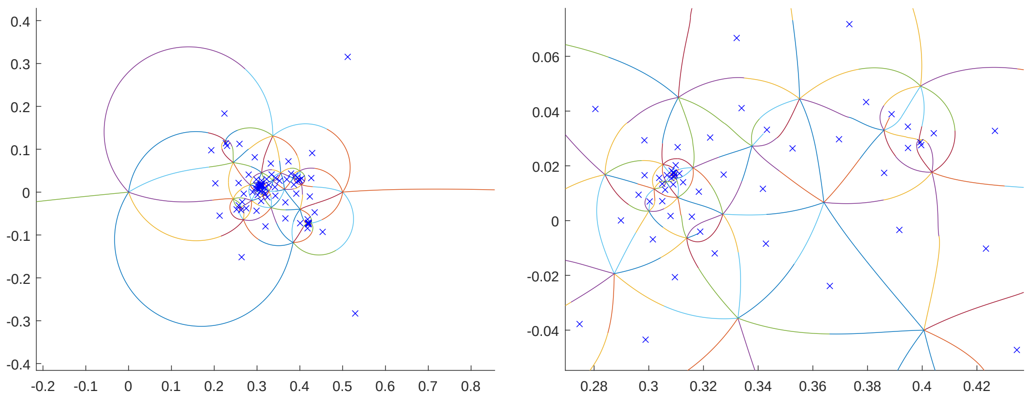
and the following properties:

- $\sigma_0, \sigma_1, \sigma_\infty$ are each contained in rational conjugacy classes of Co_3.
- $(\sigma_0, \sigma_1, \sigma_\infty)$ is not rigid with $\ell(\sigma_0, \sigma_1, \sigma_\infty) = 12$.

A fundamental domain is given by:



with resulting approximate dessin:



In [**37**] Monien presented such a Co_3-Belyi map of degree 276 with coefficients recognized in a degree-12 number field. A strict verification of the corresponding geometric and algebraic monodromy group by Barth/W. can be found in [**11**].

CHAPTER 6

Implementation

In this chapter we present the implementation files that are used in the computations from chapter 4, consisting of:

- Magma:
  - `Step1_ComputeFundamentalDomains.txt`
- Matlab:
  - `Step2_DrawDessin.m`
  - `Welding.m`
  - `CreateHyperbolicKite.m`
  - `CreatePolygonAndGluingData.m`

In the following sections we give an explanation on how to use the above files, discuss known issues and provide the code. The above files are available in the ancillary data of this disseration as well as in the accompanying files from [**10**].

Preparations.
In order to run the above codes download the Schwarz-Christoffel toolbox, also called *SC-toolbox*, for Matlab from

`https://github.com/tobydriscoll/sc-toolbox`,

extract the files and add the SC-toolbox folder to your Matlab path. Make sure that all Magma and Matlab files are contained in the same folder.

6.1. Instructions for use

The provided code computes an approximate dessin corresponding to a Belyi map with a prescribed hyperbolic ramification triple $(\sigma_0, \sigma_1, \sigma_\infty)$.

Step 1: Magma

- open Magma, load `Step1_ComputeFundamentalDomains.txt`
- define the permutations σ_0, σ_1 and apply the command

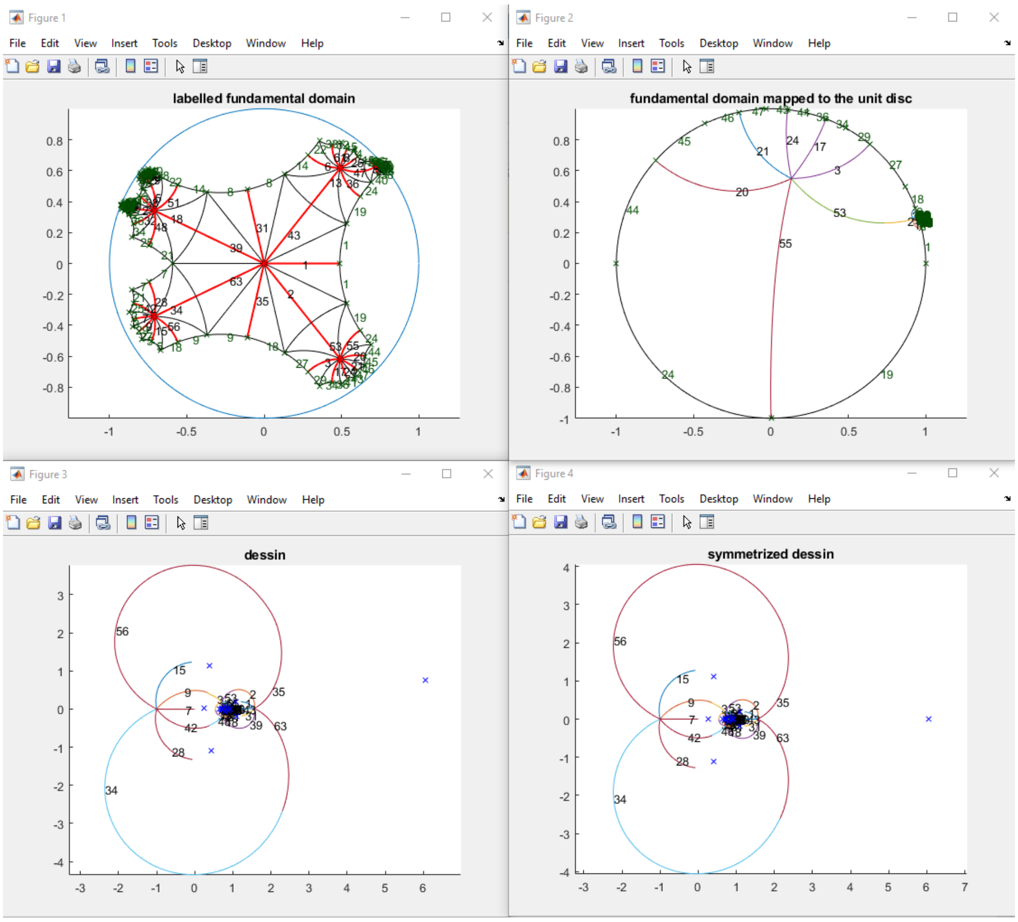
  `ComputeFundamentalDomains(`σ_0`,`σ_1`);`

- this will create the Matlab-readable file `matlab_input.m`

Step 2: Matlab

- open Matlab, load file `Step2_DrawDessin.m`
- run `Step2_DrawDessin.m`
- set the variable `index` to any allowed number
- Matlab will return three to four figures as well as the coordinates of the constructed dessin:
  - `figure(1)` visualizes the fundamental domain with the corresponding quotient structure
  - `figure(2)` shows the conformal image of the fundamental domain onto the unit disc with quotient structure
  - `figure(3)` presents the resulting dessin after the welding process, if possible the dessin is transformed in such a way that it appears to be symmetric to the real line, the coordinates of the zeroes, ones and poles are accessible via
    - `zeros_dessin`
    - `ones_dessin`
    - `poles_dessin`
  - `figure(4)` shows the symmetrized dessin from `figure(3)` (only if possible), the coordinates of the zeroes, ones and poles are available at
    - `zeros_dessin_sym`
    - `ones_dessin_sym`
    - `poles_dessin_sym`

The Matlab-output will be presented in the following form:



6.2. Known issues and solutions

Unfortunately, the above codes may occasionally run into some problems. We will discuss all of the known issues:

Computation problems in Magma.

problem: cannot compute a fundamental domain in Magma

solution: make sure that order(σ_1) $= 2$ and σ_0, σ_1 generate a transitive group

reason: our method for computing a fundamental domain only works if both of the above conditions are satisfied, if order(σ_1) is not equal to 2 one can still produce manually fundamental domains to be used in the Matlab file `Step2_DrawDessin.m`

Computation issues regarding the Schwarz-Christoffel map.

problem: - numerically singular matrices during the SC-map computation
- it takes too long to compute the SC-map
- a lot of `severe crowding` warnings during the computation of the SC-map
- computed SC-map is of low numerical precision
- it takes too long to compute the pre-images of points under the computed SC-map
- the computed dessins have zigzagways edges

solution: - try another index in the Matlab file `Step2_DrawDessin.m`
- re-compute the fundamental domains in Magma as the computed fundamental domains are constructed randomly
- instead of $(\sigma_0, \sigma_1, \sigma_\infty)$ switch the roles of σ_0 and σ_∞, this will lead to a different dessin with the same monodromy group

reason: computational issues coming with the SC-toolbox, one can instead work with any other method that conformally maps the interior of the fundamental domain to the unit disc

Odd-looking dessins.

problem: dessin appears to have a significant amount of false angles

solution: instead of $(\sigma_0, \sigma_1, \sigma_\infty)$ switch the roles of σ_0 and σ_∞, this will lead to a different dessin with the same monodromy group

reason: this problem occurs if the number of cycles of σ_0 of length order(σ_0) is 0 or very low, computer experiments have shown that some of these dessins still lead to a successful application of Newton's method when it comes to compute a complex approximation for the corresponding Belyi map

problem: in the symmetrized dessin there seems to exist an edge on the real line crossing all other edges meeting the real line

solution: nothing to worry about

reason: such edges appear if the approximated dessin is mapped via a Möbius transformation in such a way that an edge goes around infinity (on the Riemann sphere), the symmetrization process maps such edges to the real line, this is no issue when it comes to using the computed dessin for Newton's method

problem: symmetrized dessins from figure 4 have unwanted crossing edges

solution: - pick another index
- do not work with the symmetrized dessin (figure 4)

reason: a regular occurence when we force complex conjugate pairs of edges to be perfectly aligned

problem: there is no pole in the unbounded component in the symmetrized dessin (figure 4)

solution: - pick another index
- use only the data from figure 3
- manually adjust the coordinates of this bad pole

reason: this only happens if the pole in the unbounded component lies directly above or below the dessin, in a symmetrized setting such a pole is expected to have large value lying on the real line

6.3. Codes

`Step1_ComputeFundamentalDomains.txt`

```
/*
Applies the permutation x^word[1]*y^word[2]*x^word[3]*... to the element start.
*/
function calc(word,x,y,start);
ss := start;
if IsOdd(#word) then
    Append(~word,0);
end if;
for cc in [1..#word/2] do
    ss ^:= x^word[2*cc-1];
    ss ^:= y^word[2*cc];
end for;
return ss;
end function;

/*
Given permutations x,y generating a transitive permutation group and y having order 2,
this functions uses a special petalling approach to compute some fundamental domains
corresponding to the permutation triple x,y,(x*y)^-1.

Priority is given to x-petals having size order(x).
*/
function CompFD(x,y)

assert Parent(x) eq Parent(y);
assert Order(y) eq 2;
assert IsTransitive(sub<Parent(x) | x,y>);

deg := Degree(Parent(x));
Gx := sub<Sym(deg)|x>;
Orb := Orbits(Gx);

```

```
V := { Min(o) : o in Orb | #o eq Order(x) };
G := Graph< V | {{e1,e2} : e1, e2 in V | not IsDisjoint(Orbit(Gx, e1), Orbit(Gx,e2)^y) and e1 ne e2} >;

comps := Components(G);
Sort(~comps, func< x,y | #y-#x > );
if #comps ge 1 then
  Gmax,Vmax,E := sub< G | comps[1] >;
  Vset := Support(Gmax);
end if;

if #comps ge 1 then
  STARTLIST := Sort([a: a in Vset]);
else
  STARTLIST := Sort([ Min(o) : o in Orb ]);
end if;

FD_List := [];
for start in STARTLIST do

bfstree := [];

if #comps ge 1 then

// randomized breadth-first search in the graph G starting at the vertex start
done := { Minimum(Orbit(Gx,start))};
oldgen := [Minimum(Orbit(Gx,start))];
repeat
  newgen := [];
  for ele in [ oldgen[k] : k in [1..#oldgen]^Random(Sym(#oldgen)) ] do
    for kk in {i: i in Vset | Vmax!i in Neighbours(Vmax!ele)} do
      if kk notin done then
          Append(~newgen, Minimum(Orbit(Gx,kk)));
          done join:= {Minimum(Orbit(Gx,kk))};
                Append(~bfstree, [Minimum(Orbit(Gx,ele)),Minimum(Orbit(Gx,kk))]);
```

```
        end if;
      end for;
    end for;
    oldgen := newgen;
  until #newgen eq 0;
end if;

LISTWAY := [];

// first petal
for cc in [0..#Orbit(Gx,start)-1] do
  index := start^(x^cc);
  LISTWAY[index] := [cc];
end for;

// adds all possible x-petals to the first petal
for l in bfstree do
  a := l[1]; b := l[2];
  for cc in [0 .. #Orbit(Gx,a)-1] do
    if not IsDisjoint(Orbit(Gx,a^((x^cc)*y)),Orbit(Gx,b)) then
      c := a^(x^cc);
      break cc;
    end if;
  end for;
  for cc in [0..Order(x)-1] do
    index := c^(y*(x^cc));
    if not IsDefined(LISTWAY,index) then
      LISTWAY[index] := LISTWAY[c] cat [1,cc];
    end if;
  end for;
end for;

done := {calc(w,x,y,start) : w in LISTWAY};

// adds all remaining kites to the current collection regardless
    of the petal property
while #done lt deg do
  for ele in {1..deg} diff done do
```

```
    if ele^y in done then
      if IsOdd(#LISTWAY[ele^y]) then
        LISTWAY[ele] := LISTWAY[ele^y] cat [1];
      else
        LISTWAY[ele] := LISTWAY[ele^y] cat [0,1];
      end if;
      done join:= {ele};
    end if;
  end for;
  repeat
    stop := true;
    for ele in {1..deg} diff done do
      if ele^(x^-1) in done then
        if IsEven(#LISTWAY[ele^(x^(-1))]) then
          LISTWAY[ele] := LISTWAY[ele^(x^(-1))] cat [1];
        else
          LISTWAY[ele] := LISTWAY[ele^(x^(-1))] cat [0,1];
        end if;
        done join:= {ele};
        stop := false;
      end if;
    end for;
  until stop;
end while;

LISTWAY := [ m cat [0 : k in [0..Maximum([#m : m in LISTWAY])-#m]] : m in LISTWAY ];

FD := [];
for m in LISTWAY do
  FD[calc(m,x,y,start)] := m;
end for;

assert #FD eq deg and [calc(m,x,y,start) : m in FD] eq [1..deg];

Append(~FD_List,<start,FD>);

end for;
```

```

return FD_List;

end function;

/*
Given a fundamental domain FD (corresponding to permutations x,y and central kite start)
this function computes the border edges of the fundamental domain and the quotient structure on the border.
*/
function ComputeBorderAndGluingData(FD,x,y,start);

a := Order(x); b := Order(y); c:= Order(x*y);
RR := RealField(50);
CC<i> := ComplexField(50);
lambda := (Cos(Pi(RR)/a) * Cos(Pi(RR)/b) + Cos(Pi(RR)/c))/(Sin(Pi(RR)/a)*Sin(Pi(RR)/b));
mu := lambda + Sqrt(lambda^2 - 1);
DA := Matrix(RR,2,2,[Cos(Pi(RR)/a), Sin(Pi(RR)/a), -Sin(Pi(RR)/a), Cos(Pi(RR)/a)]);
DB := Matrix(RR,2,2,[Cos(Pi(RR)/b), mu*Sin(Pi(RR)/b), -(Sin(Pi(RR)/b))/mu , Cos(Pi(RR)/b)]);

// computes the "center" of the kite corresponding to word
function WordToKiteCenter(word)
  ss := Matrix(RR,2,2,[1,0,0,1]);
  if IsOdd(#word) then
      Append(~word,0);
  end if;
  for cc in [1..#word/2] do
      ss := ss * DA^word[2*cc-1];
    ss := ss * DB^word[2*cc];
  end for;
  mat := ss;
  c := (i+i*mu)/2;
  ss := (mat[1,1]*c + mat[1,2])/(mat[2,1]*c + mat[2,2]);
  return (ss-i)/(ss+i);
```

```
end function;

centerlist := [ WordToKiteCenter(word) : word in FD ];

border := [];
for ele in FD do

maxpos := Maximum([1] cat [ i : i in [1..#ele] | ele[i] ne 0 ]);
l1 :=ele; l2 :=ele; l3 :=ele; l4 :=ele;

if IsOdd(maxpos) then
  l1[maxpos]:=(ele[maxpos]+1) mod a;
  l2[maxpos]:=(ele[maxpos]-1) mod a;
  l3[maxpos+1]:=(ele[maxpos+1]+1) mod b;
  l4[maxpos+1]:=(ele[maxpos+1]-1) mod b;
else
  l1[maxpos]:=(ele[maxpos]+1) mod b;
  l2[maxpos]:=(ele[maxpos]-1) mod b;
  l3[maxpos+1]:=(ele[maxpos+1]+1) mod a;
  l4[maxpos+1]:=(ele[maxpos+1]-1) mod a;
end if;

if l1 notin FD and Abs(WordToKiteCenter(l1) - centerlist[calc(l1,x,y,start)]) gt 10^-10 then
  Append(~border, [calc(ele,x,y,start),calc(l1,x,y,start),maxpos mod 2,(-1)^(maxpos+1)]);
end if;
if l2 notin FD and Abs(WordToKiteCenter(l2) - centerlist[calc(l2,x,y,start)]) gt 10^-10 then
  Append(~border, [calc(ele,x,y,start),calc(l2,x,y,start),maxpos mod 2,-(-1)^(maxpos+1)]);
end if;
if l3 notin FD and Abs(WordToKiteCenter(l3) - centerlist[calc(l3,x,y,start)]) gt 10^-10 then
  Append(~border, [calc(ele,x,y,start),calc(l3,x,y,start),(maxpos+1) mod 2,(-1)^(maxpos)]);
end if;
```

```
if l4 notin FD and Abs(WordToKiteCenter(l4) - centerlist[calc(l4,x,y,start)]) gt 10^-10 then
  Append(~border, [calc(ele,x,y,start),calc(l4,x,y,start),(maxpos+1) mod 2,-(-1)^(maxpos)]);
end if;

end for;

result := [];
counter := 1;
while #border gt 0 do
  ele := border[1];
  Exclude(~border, ele);
  target := [ele[2],ele[1],ele[3],-ele[4]];
  Exclude(~border, target);
  Append(~result, ele cat [counter]);
  Append(~result, target cat [counter]);
  counter +:= 1;
end while;

return result;
end function;

/*
Given permutations x,y such that
- x,y generate a transitive permutation group
- x,y,z:=(x*y)^-1 is hyperbolic
- y has order 2
this function computes fundamental domains corresponding to the permutation triple x,y,z
as well as the gluing data on the border of the fundamental domains.

By default the output is written to the file matlab_input.m.
If the parameter WriteToFile is set to false or the file can not be written
then the output is instead printed in the console and can be saved to a file or
```

```
directly inserted into Matlab manually.
*/
procedure ComputeFundamentalDomains(x,y : WriteToFile := true)

assert Parent(x) eq Parent(y);
G := sub<Parent(x) | x,y >;
deg := Degree(G);
assert Order(y) eq 2;
assert IsTransitive(G);
z := (x*y)^-1;
a:= Order(x); b:= Order(y); c:= Order(z);
assert 1/a+1/b+1/c lt 1;

print "computing fundamental domains for a triple (x,y,z) with
    cycle structures:";
CycleStructure(x); CycleStructure(y); CycleStructure(z);

out := "%" cat Sprintf("CycleStructure(x) = '%o';\n",
    CycleStructure(x));
out cat:= "%" cat Sprintf("CycleStructure(y) = '%o';\n",
    CycleStructure(y));
out cat:= "%" cat Sprintf("CycleStructure(z) = '%o';\n\n",
    CycleStructure(z));

out cat:= Sprintf("permx = %o;\n", [j^x : j in [1..deg]]);
out cat:= Sprintf("permy = %o;\n", [j^y : j in [1..deg]]);
out cat:= Sprintf("permz = %o;\n\n", [j^z : j in [1..deg]]);

out cat:= "permx_cycles = {";
for orb in Orbits(sub<G |x>) do
    out cat:= Sprint([j : j in orb]);
    out cat:= ",";
end for;
out := Prune(out) cat "};\n";

out cat:= "permy_cycles = {";
for orb in Orbits(sub<G |y>) do
    out cat:= Sprint([j : j in orb]);
```

```
     out cat:= ",";
end for;
out := Prune(out) cat "};\n";

out cat:= "permz_cycles = {";
for orb in Orbits(sub<G |z>) do
     out cat:= Sprint([j : j in orb]);
     out cat:= ",";
end for;
out := Prune(out) cat "};\n\n";

out cat:= Sprintf("a = %o;\n",a);
out cat:= Sprintf("b = %o;\n",b);
out cat:= Sprintf("c = %o;\n\n",c);

Sigma := false;
NG := Normalizer(Sym(Degree(G)),G);
for a in [b: b in Normalizer(NG,sub<NG|x>) meet Normalizer(NG,sub<
    NG|y>) | Order(b) le 2] do
if x^a eq x^-1 and y^a eq y^-1 then
Sigma := a;
break a;
end if;
end for;
if Sigma cmpeq false then
   out cat:= Sprintf("ConjugateEdges = %o;\n\n",[ ]);
else
     if 1 in [#o : o in Orbits(sub<NG|Sigma>) cat Orbits(sub<NG|
         Sigma*y>) cat Orbits(sub<NG|Sigma*y*z>)] then
          out cat:= Sprintf("ConjugateEdges = %o;\n\n",[ j^Sigma : j
               in [1..deg] ]);
     else
          out cat:= Sprintf("ConjugateEdges = %o;\n\n",[ ]);
     end if;
end if;

FDList := CompFD(x,y);
print "number of computed fundamental domains:", #FDList;
```

```

counter := 1;
for obj in FDList do
  start, FD := Explode(obj);
  out cat:= Sprintf("FDList{%o}=%o;\n\n",counter,FD);
  out cat:= Sprintf("GluingDataList{%o}=%o;\n\n",counter,
      ComputeBorderAndGluingData(FD,x,y,start));
  counter +:= 1;
end for;

SetColumns(0);
printstring := true;
if WriteToFile then
  try
    Write("matlab_input.m",out:Overwrite := true);
    print "fundamental domains were saved to file matlab_input.m."
      ;
    print "please run Step2_DrawDessin.m in Matlab now.";
    printstring := false;
  catch e
    print "Error: Could not write to file matlab_input.txt.";
  end try;
end if;
if printstring then
  print "please copy the following output into a file called
      matlab_input.m and run Step2_DrawDessin.m in Matlab
      afterwards.";
  print "%============ BEGIN matlab_input.m ============ ";
  print out;
  print "%============ END matlab_input.m ============ ";
end if;

end procedure;
```

`Step2_DrawDessin.m`

This is the main file that requires all of the other Matlab files.

```
clear; close all; clc; warning ('on','all');

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% number of chosen fundamental domain
index = 1;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% output data
%
% figure(1): labelled fundamental domain
%            with the corresponding quotient structure
%
% figure(2): closure of fundamental domain mapped to the unit disc
%            with the corresponding quotient structure
%
% figure(3): dessin in the complex plane, the coordinates of the
%            zeroes, ones and poles:
%               zeros_dessin
%               ones_dessin
%               poles_dessin
%
% figure(4): (optional) symmetrized dessin in the complex plane,
%            coordinates of the zeroes, ones and poles:
%               zeros_dessin_sym
%               ones_dessin_sym
%               poles_dessin_sym
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% load Magma-computed fundamental domains, gluing data from file
clear('matlab_input.m')
run('matlab_input.m')

degree = size(permx,2);
fprintf('degree: %d\n',degree);
fprintf('number of computed fundamental domains: %d (choose index
    between 1 and %d)\n',size(FDList,2),size(FDList,2));
```

```

FD = FDList{index};
GluingData = GluingDataList{index};

%% Computing a fundamental domain
% Create hyperbolic kite in H
[z1,z2,z3,cz1,cz2] = CreateHyperbolicKite(a,b,c);

% define hyperbolic rotation matrices DA and DB
lambda = (cos(pi/a) * cos(pi/b) + cos(pi/c))/(sin(pi/a)*sin(pi/b))
    ;
mu      = lambda + sqrt(lambda^2 - 1);
DA = [cos(pi/a) sin(pi/a); -sin(pi/a) cos(pi/a)];
DB = [cos(pi/b) mu*sin(pi/b); -(sin(pi/b))/mu cos(pi/b)];

% Assemble transformation matrices (products of DA and DB)
% according to the chosen fundamental domain
[m,n] = size(FD);
T = cell(1,m);
for j=1:m
    T{j} = eye(size(DA));
    s = 1;
    for h=1:(n/2)
        T{j} = T{j}*DA^FD(j,s);
        T{j} = T{j}*DB^FD(j,s+1);
        s = s+2;
    end
end

% Apply transformation matrices to the hyperbolic kite to
% obtain the fundamental domain
moz1 = cell(1,m); moz2 = cell(1,m); cmoz1 = cell(1,m); cmoz2 =
    cell(1,m); cz3 = cell(1,m);
for j=1:m
   moz1{j}  = w(mo(T{j},z1));
   moz2{j}  = w(mo(T{j},z2));
   cmoz1{j} = w(mo(T{j},cz1));
   cmoz2{j} = w(mo(T{j},cz2));
```

```
    cz3{j}    = w(mo(T{j},z3));
end

% plot the labelled fundamental domain
figure(1)
clf
title('labelled fundamental domain')
hold on
axis equal
for j=1:m
   plot(real(moz1{j}),imag(moz1{j}) ,'black', 'LineWidth',.15)
   plot(real(moz2{j}),imag(moz2{j}), 'black','LineWidth',.15)
   plot(real(cmoz1{j}),imag(cmoz1{j}) ,'black','LineWidth',.15)
   plot(real(cmoz2{j}),imag(cmoz2{j}) , 'black','LineWidth',.15)
   plot(real(cz3{j}),imag(cz3{j}) , 'red','LineWidth',1.5)
   text(real(cz3{j}(100)),imag(cz3{j}(100)),num2str(j))
end
plot(cos(0:0.01:2*pi),sin(0:0.01:2*pi),'k')


%% mapping the fundamental domain to the unit disc

dessin_edges = zeros(m,size(z3,2));
for j = 1:m
     dessin_edges(j,:) = cz3{j};
end

matcmoz1 = cell2mat(cmoz1');
dessin_poles = matcmoz1(:,end);

border_edges = zeros(size(GluingData,1),size(z1,2)+1);
for j = 1:size(GluingData,1)
     edge = GluingData(j,:);
     if edge(3) == 0 && edge(4) == 1
          border_edge = cz2;
     end
     if edge(3) == 0 && edge(4) == -1
          border_edge = z2;
```

```
    end
    if edge(3) == 1 && edge(4) == 1
        border_edge = cz1;
    end
    if edge(3) == 1 && edge(4) == -1
        border_edge = z1;
    end
    border_edges(j,:) = [edge(5),w(mo(T{edge(1)},border_edge))];
end

[polygon_vertices, L] = CreatePolygonAndGluingData(border_edges);
poly = polygon(fliplr(polygon_vertices.'));

% plot quotient structure on the border of the fundamental domain
pv = polygon_vertices;
%plot(real(pv),imag(pv),'x','color',[0 0.3 0]);
q = size(pv,1);
for j = 1: size(L,1)
    for jj = [1,2]
    p1 = pv(L(j,jj));
    p2 = pv(mod(L(j,jj)-2,q)+1);
   m=(p1+p2)/2;
    text(real(m),imag(m),num2str(j),'color',[0 0.3 0])
    end
end

fprintf('computation of SC map for a polygon with %d edges starts ...\n',length(poly))
sc_map = diskmap(poly);
fprintf('accuracy of computed SC map: %d\n',accuracy(sc_map))

fprintf('mapping fundamental domain and dessin to D...\n')
polygon_vertices = evalinv(sc_map,polygon_vertices);
dessin_edges = evalinv(sc_map,dessin_edges);
dessin_poles = evalinv(sc_map,dessin_poles);

%% plot the closure of fundamental domain mapped to the
%% unit disc with the corresponding quotient structure
```

```
figure(2)
clf
title('fundamental domain mapped to the unit disc')
hold on
axis equal
% plot circle
plot(cos(0:0.01:2*pi),sin(0:0.01:2*pi),'Color','k')
% plot dessin
for j=1:size(dessin_edges,1)
    plot(real(dessin_edges(j,:)),imag(dessin_edges(j,:)))
    text(real(dessin_edges(j,end/2)),imag(dessin_edges(j,end/2)),
        num2str(j))
end
% plot poles
plot(real(dessin_poles(:,1)),imag(dessin_poles(:,1)),'bx')
% plot border of the fundamental domain
pv = polygon_vertices;
plot(real(pv),imag(pv),'x','color',[0 0.3 0]);
q = size(pv,1);
for j = 1:size(L,1)
    for jj = [1,2]
    p1 = pv(L(j,jj));
    p2 = pv(mod(L(j,jj)-2,q)+1);
    m = (p1+p2)/2;
    m = m/norm(m);
    text(real(m),imag(m),num2str(j),'color',[0 0.3 0])
    end
end

%% Welding process
disp('start welding')
[zz,poles] = Welding(polygon_vertices,dessin_edges,dessin_poles,L)
    ;
disp('welding finished')

%% Symmetrizing process (if possible)
if size(ConjugateEdges,2)>0
    real_edges = find(~(ConjugateEdges - (1:degree)))';
```

```
    nonreal_edges = find((ConjugateEdges - (1:degree)))';
    edges = [real_edges; nonreal_edges];
    if size(real_edges,1) > 0
        vektor = [edges(1),edges(1),edges(2)];
        r1 = vektor(1); r2 = ConjugateEdges(r1);
        A = (zz(r1,1)+zz(r2,1))/2;
        r1 = vektor(2); r2 = ConjugateEdges(r1);
        B = (zz(r1,end)+zz(r2,end))/2;
        r1 = vektor(3); r2 = ConjugateEdges(r1);
        C = (zz(r1,100)+zz(r2,100))/2;
        zz = moeb(zz,A,B,C,-1,0,1);
        poles = moeb(poles,A,B,C,-1,0,1);
    else
        r1 = edges(1); r2 = ConjugateEdges(r1);
        A = zz(r1,100);
        B = zz(r2,100);
        r1 = edges(end); r2 = ConjugateEdges(r1);
        C = (zz(r1,end)+zz(r2,end))/2;
        zz = moeb(zz,A,B,C,1i,-1i,1);
        poles = moeb(poles,A,B,C,1i,-1i,1);
    end
end

%% computing the poles
poles2 = poles;
for j = 1:size(permz_cycles,2)
    c = cell2mat(permz_cycles(j));
    for k = cell2mat(permz_cycles(j))
        poles2(k) = mean(poles(c));
    end
end
poles = poles2;
polesneu = zeros(size(permz_cycles,2),2);
for j = 1:size(permz_cycles,2)
     c = cell2mat(permz_cycles(j));
     polesneu(j,:) = [mean(poles(c)),size(c,2)];
end

```

```
%% plot the resulting dessin in the complex plane
figure(3);
clf
title('dessin')
hold on
axis equal

% plot dessin
for j=1:size(zz,1)
    plot(real(zz(j,:)),imag(zz(j,:)))
    text(real(zz(j,end/2)),imag(zz(j,end/2)),num2str(j))
end

% plot poles
plot(real(polesneu(:,1)),imag(polesneu(:,1)),'bx')

% output data
zeros_dessin = zz(:,1);
ones_dessin = zz(:,end);
poles_dessin = poles;

%% Smoothing process (if possible), i.e. averaging out complex
    conjugate points of the dessin
if size(ConjugateEdges,2) > 0
    zz_smooth = zeros(size(zz));
    for count = 1:degree
        A = zz(count,:); B = zz(ConjugateEdges(count),:);
        zz_smooth(count,:) = real(1/2 * (A + B)) + 1i*(imag(1/2 *
            (A - B)));
    end
    LC = 1:size(polesneu,1);
    poles2 = zeros(size(poles));
    for j = 1:size(polesneu,1)
        if ismember(j,LC)
            c = cell2mat(permz_cycles(j));
            tt = ConjugateEdges(permx(c(1)));
            for jj=1:size(permz_cycles,2)
                c = cell2mat(permz_cycles(jj));
```

```
                f = find(c==tt);
                if size(f,2) > 0
                    break
                end
            end
            if jj == j
                poles2(cell2mat(permz_cycles(j))) = real(polesneu(
                    j,1));
            else
                A = polesneu(j,1); B = polesneu(jj,1);
                rr = real(1/2 * (A + B)) + 1i*(imag(1/2 * (A - B)
                    ));
                poles2(cell2mat(permz_cycles(j))) = rr;
                poles2(cell2mat(permz_cycles(jj))) = conj(rr);
            end
            LC = setdiff(LC,[j,jj]);
        end
    end
    poles3 = zeros(size(permz_cycles,2),2);
    for j = 1:size(permz_cycles,2)
        c = cell2mat(permz_cycles(j));
        poles3(j,:) = [mean(poles2(c)),size(c,2)];
    end

%% plot the resulting symmetrized dessin in the complex plane (if
    possible)
    figure(4);
    clf
    title('symmetrized dessin')
    hold on
    axis equal
    % plot real dessin
    for j=1:size(zz_smooth,1)
        plot(real(zz_smooth(j,:)),imag(zz_smooth(j,:)))
        text(real(zz_smooth(j,end/2)),imag(zz_smooth(j,end/2)),
            num2str(j))
    end
    % plot poles
```

```
    for j=1:size(poles3,1)
        plot(real(poles3(j,1)),+imag(poles3(j,1)),'bx')
    end
    % output data
    zeros_dessin_sym = zz_smooth(:,1);
    ones_dessin_sym = zz_smooth(:,end);
    poles_dessin_sym = poles2;
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function res = mo(M,z)
% computes the image of z under the Moebius transformation given by a 2x2
% matrix M
    res = (M(1,1)*z + M(1,2)) ./ (M(2,1)*z + M(2,2));
end

function res = w(z)
% maps the upper half plane to the unit disc via z -> (z-i)/(z+i)
    res = (z-1i) ./ (z+1i);
end

function res = moeb(z,z1,z2,z3,w1,w2,w3)
% computes the image of z of a Moebius transformation mapping [z1,z2,z3] to [w1,w2,w3]
res = (w1*(-w2*(z1 - z2)*(z - z3*ones(size(z))) + w3*(z - z2*ones(size(z)))*(z1 - z3)) - ...
  w2* w3* (z - z1*ones(size(z)))* (z2 - z3))./(w3* (z1 - z2)* (z - z3*ones(size(z))) - ...
  w2* (z - z2*ones(size(z)))* (z1 - z3) + w1* (z - z1*ones(size(z)))* (z2 - z3));
end
```

Welding.m

```
% The following code is extracted from the weld.m file available at
% https://github.com/oelarnes/treeweld

function [zz,poles] =  Welding(z,zz,poles,L)

z = exp(pi/30*1i)*z;
poles = exp(pi/30*1i)*poles;
zz = exp(pi/30*1i)*zz;

n = size(z,1)/2;

% new degree data for welding
d = ones(2 * n, 1);

z = (z - 1) ./ (z + 1);
zz = (zz - 1) ./ (zz + 1);
poles = (poles - 1) ./ (poles + 1);

z = [z; 1];

%% the welding loop
for j = 1:size(L,1)-1
    % i1,i2,i3,i4 are the indices of the endpoints of the next interval to
    % be welded
    i1 = L(j, 1);
    i2 = mod(L(j, 1)-2, 2 * n) + 1;
    i3 = L(j ,2);
    i4 = mod(L(j, 2)-2, 2 * n) + 1;
    % the values on the circle
  x1 = z(i1);
  x2 = .5 * z(i2) + .5 * z(i3);
  x3 = z(i4);
  % the angle to evenly space the new vertex
  alpha = d(i4) / (d(i4) + d(i1));
  % perform the weld
```

```
    z = alpha_moeb(z, x1, x2, x3, alpha);
        zz = alpha_moeb(zz, x1, x2, x3, alpha);
        poles = alpha_moeb(poles, x1, x2, x3, alpha);
    z = slit_map(z, alpha);
    zz = slit_map(zz, alpha);
    poles = slit_map(poles, alpha);
    % degree of new vertex
    d(i1) = d(i1) + d(i4);
    d(i4) = d(i1);
end

% back to the unit circle
z = (z + 1) ./ (z - 1);
zz = (zz + 1) ./ (zz - 1);
poles = (poles + 1) ./ (poles - 1);

i1 = L(end, 1);
i2 = mod(L(end, 1)-2, 2 * n) + 1;
i3 = L(end ,2);
i4 = mod(L(end, 2)-2, 2 * n) + 1;

x1 = .5 * z(i1) + .5 * z(i4);
x2 = .5 * z(i2) + .5 * z(i3);

% two remaining endpoints map to -1, 1
z = (x1 + x2 + 1i * (z - x1) * abs(x1 + x2) - x1 * z * conj(x1 + x2)) ./ ...
    (x1 + x2 - 1i * (z - x1) * abs(x1 + x2) - x1 * z * conj(x1 + x2));

zz = (x1 + x2 + 1i * (zz - x1) * abs(x1 + x2) - x1 * zz * conj(x1 + x2)) ./ ...
    (x1 + x2 - 1i * (zz - x1) * abs(x1 + x2) - x1 * zz * conj(x1 + x2));

poles = (x1 + x2 + 1i * (poles - x1) * abs(x1 + x2) - x1 * poles * conj(x1 + x2)) ./ ...
```

```
    (x1 + x2 - 1i * (poles - x1) * abs(x1 + x2) - x1 * poles * conj(x1 + x2));

% weld the circle
z = z + ones(size(z)) ./ z;
zz = zz + ones(size(zz)) ./ zz;
poles = poles + ones(size(poles)) ./ poles;

x = z(2 * n + 1);

% normalize by resetting infinity
zz = ones(size(zz)) ./ (zz - x);
poles = ones(size(poles)) ./ (poles - x);
end

function value = slit_map(z, alpha)
% slit map used for welding
value = (z+1i*alpha).^(alpha) .* (z+1i*(alpha-1)).^(1-alpha);
end

function value = alpha_moeb(z, x1, x2, x3, alpha)
% alpha_moeb resets the points on the imaginary axis so that
% the points x1, x2, x3 map to i(1-alpha), 0, -i*alpha.
value = moeb(z,x1,x2,x3,-1i*(alpha-1),0,-1i*alpha);
end

function res = moeb(z,z1,z2,z3,w1,w2,w3)
% computes the image of z of a Moebius transformation mapping [z1, z2,z3] to [w1,w2,w3]
res = (w1*(-w2*(z1 - z2)*(z - z3*ones(size(z))) + w3*(z - z2*ones(size(z)))*(z1 - z3)) - ...
  w2* w3* (z - z1*ones(size(z)))* (z2 - z3))./(w3* (z1 - z2)* (z - z3*ones(size(z))) - ...
  w2* (z - z2*ones(size(z)))* (z1 - z3) + w1* (z - z1*ones(size(z)))* (z2 - z3));
end
```

CreateHyperbolicKite.m

```
function [z1,z2,z3,cz1,cz2] = CreateHyperbolicKite(a,b,c)
%% computes the first hyperbolic kite with parameters a,b,c

lambda = (cos(pi/a) * cos(pi/b) + cos(pi/c))/(sin(pi/a)*sin(pi/b));
mu     = lambda + sqrt(lambda^2 - 1);
G      = (mu^2-1)/(2*(cot(pi/a) + mu*cot(pi/b)));

% number of segments approximating the arcs of the kite
segments = 20;

t = linspace(0,1,segments);

% edge1 is the lower arc of the kite
edge1 = arc((mu-1)/(mu+1),w(G/2+1i*f2(G/2,a,b,mu)),w(G+1i*f1(G,a,b,mu)),segments);
% edge2 is the upper arc of the kite
edge2 = conj(fliplr(edge1));
% edge4 is the lower line of the kite
edge4 = fliplr(t*w(G+1i*f1(G,a,b,mu)));
% edge3 is the upper line of the kite
edge3 = conj(fliplr(edge4));
% edge5 is the diagonal of the kite
edge5 = linspace(0,(mu-1)/(mu+1),200);

% mapping the kite to the upper half-plane
z1 = w_inv(edge4);
z2 = w_inv(edge1);
z3 = w_inv(edge5);
cz1 = w_inv(edge3);
cz2 = w_inv(edge2);
end

function res = arc(A,B,C,n)
% Computes a circular arc containing the points A,B,C approximated by a
% polygonal chain of length n (counter clock-wise)
```

```

xA = real(A); xB = real(B); xC = real(C);
yA = imag(A); yB = imag(B); yC = imag(C);

Matrix = [yB-yC, -yA+yB; xC-xB, xA-xB];
b = 1/2*[xA-xC;yA-yC];
sol = Matrix\b;
tau = sol(1);

center = 1/2*(xB+xC)+tau*(yB-yC)+1i*(1/2*(yB+yC)+tau*(xC-xB));
radius = abs(B-center);

phiA = mod(angle(A-center),2*pi);
phiC = mod(angle(C-center),2*pi);

tAC = linspace(phiA,phiC,n);
res = center + radius*exp(1i*tAC);
end

function y = f1(x,a,~,~)
    y = sqrt((csc(pi/a))^2*ones(size(x)) - (x-cot(pi/a)).^2);
end

function y= f2(x,~,b,mu)
    y = sqrt(mu^2*csc(pi/b)^2*ones(size(x)) - (x + mu*cot(pi/b))
        .^2);
end

function res = w(z)
% maps upper half-plane to the unit disc via z -> (z-i)/(z+i)
    res = (z-1i) ./ (z+1i);
end

function res = w_inv(z)
% inverse of w
    res = 1i*(1+z) ./ (1-z);
end
```

CreatePolygonAndGluingData.m

```
function [vertices, L] = CreatePolygonAndGluingData(border_edges)
%% this function creates a polygon corresponding to border_edges and computes
%% the gluing data for the welding process

% rearranging border_edges
for j = 1:size(border_edges,1)-1
    [~,minimum_index] = min(abs(border_edges(:,2) - border_edges(j,end)));
    tmp = border_edges(j+1,:);
    border_edges(j+1,:) = border_edges(minimum_index,:);
    border_edges(minimum_index,:) = tmp;
end

% compute gluing data in L
L = zeros(size(border_edges,1)/2,2);
for j = 1:size(border_edges,1)/2
   L(j,:) = find(border_edges(:,1)==j);
end

L(1,:) = fliplr(L(1,:));
L2 = zeros(size(L));

% sort L with regard to distance of edges
ct = 1;
for j = 1:size(border_edges,1)/2
    for i = 1:size(L,1)
        if mod(L(i,1)-L(i,2),size(border_edges,1)) == mod(j,size(border_edges,1)) || mod(L(i,2)-L(i,1),size(border_edges,1)) == mod(j,size(border_edges,1))
            L2(ct,:) = fliplr(L(i,:));
            ct = ct + 1;
        end
    end
end
L = L2;
```

```
for j = 1:size(L,1)
    if mod(L(j,1)-L(j,2),size(border_edges,1)) > mod(L(j,2)-L(j,1)
        ,size(border_edges,1))
        L(j,:)=fliplr(L(j,:));
    end
end

vertices = border_edges(:,end);
end
```

Index of terms

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