Charles University in Prague
Faculty of Mathematics and Physics
Department of Mathematical Analysis

Julius-Maximilians-Universität Würzburg
Faculty of Mathematics and Computer Science
Institute of Mathematics

Weak Solutions to Mathematical Models of the Interaction between Fluids, Solids and Electromagnetic Fields

Dissertation Thesis

Cotutelle de Thèse

Jan Scherz
Prague/Würzburg 2023

Title:
Weak Solutions to Mathematical Models of the Interaction between Fluids, Solids and Electromagnetic Fields

Author:
Jan Scherz

Institutions:
– Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University in Prague
– Institute of Mathematics, Faculty of Mathematics and Computer Science, Julius-Maximilians-Universität Würzburg
– Mathematical Institute, Czech Academy of Sciences

Advisors:
– Barbora Benešová (Supervisor at Charles University in Prague)
– Šárka Nečasová (Advisor at Czech Academy of Sciences)
– Anja Schlömerkemper (Supervisor at Julius-Maximilians-Universität Würzburg)

Abstract:
We analyze the mathematical models of two classes of physical phenomena. The first class of phenomena we consider is the interaction between one or more insulating rigid bodies and an electrically conducting fluid, inside of which the bodies are contained, as well as the electromagnetic fields trespassing both of the materials. We take into account both the cases of the fluid being incompressible and the fluid being compressible. In both cases our main result yields the existence of weak solutions to the associated system of partial differential equations, respectively. The proofs of these results are built upon hybrid discrete-continuous approximation schemes: Parts of the systems are discretized with respect to time in order to deal with the solution-dependent test functions in the induction equation. The remaining parts are treated as continuous equations on the small intervals between consecutive discrete time points, allowing us to employ techniques which do not transfer to the discretized setting. Moreover, the solution-dependent test functions in the momentum equation are handled via the use of classical penalization methods.
The second class of phenomena we consider is the evolution of a magnetoelastic material. Here too, our main result proves the existence of weak solutions to the corresponding system of partial differential equations. Its proof is based on De Giorgi's minimizing movements method, in which the system is discretized in time and, at each discrete time point, a minimization problem is solved, the associated Euler-Lagrange equations of which constitute a suitable approximation of the original equation of motion and magnetic force balance. The construction of such a minimization problem is made possible by the realization that, already on the continuous level, both of these equations can be written in terms of the same energy and dissipation potentials. The functional for the discrete minimization problem can then be constructed on the basis of these potentials.

Keywords:
Fluid-structure interaction, Magnetoelasticity, Magnetohydrodynamics, Minimizing movements, Navier-Stokes equations, Rothe method

Contents

Chapter 1

Introduction

1.1 Motivation

The main objective of this thesis is the proof of the existence of weak solutions to systems of partial differential equations which model various physical phenomena. The phenomena we consider can be divided into two classes. The first class, which constitutes the major part of the thesis, is the interaction between electrically conducting fluids and insulating rigid bodies. The second class is the evolution of magnetoelastic materials.

The study of electrically conducting fluids interacting with solid materials is motivated by possible applications in e.g. the field of biomechanics. One outstanding example is a medical procedure known as capsule endoscopy, cf. for example [59]. This procedure constitutes a minimally invasive method for the detection of diseases by propelling small capsule shaped camera devices through parts of the human body such as veins or arteries. In the electrically conducting blood it is possible to generate the drive and control the navigation of these devices remotely by applying electromagnetic forces. In particular, the usage of moving mechanical parts can be avoided in the design of the capsules. In a similar fashion robots of a microscopic scale can be used for the transport of drugs through the blood stream in the human body. In this way, medication can be carried directly to the region of the body in which it is needed while damage through the medication to healthy tissue is avoided. This procedure is referred to as remote drug delivery, cf. [58, Section 4.4]. Further applications appear in the study of biological processes. Models of the interplay between solids and electrically conducting fluids can be used for the description of the interaction between either extracellular or intracellular fluids and the membranes of cells in living organisms.

The results in the present thesis provide an intermediate step in the analysis of the full-scale models describing these real-world applications: We focus on the setting of one or more insulating rigid bodies moving inside of an electrically conducting fluid, cf. Figure 1.1. Mathematically speaking, this constitutes a three-way interaction problem. The first kind of interaction occurring hereby falls into the realm of fluid-structure interactions (FSI). In all generality, the research field of FSI deals with interactions between rigid or deformable solids and fluids contained in, adjacent to or surrounding the solids, cf. [13, 20]. In our specific scenario, the motion of the rigid bodies exerts a force upon the surrounding fluid, which affects the fluid motion, and vice versa. The second kind of interaction in the described set-up takes place between the electrically conducting fluid and the electromagnetic fields living inside the fluid and the solids. The motion of the fluid is impacted by the electromagnetic fields. The electromagnetic fields, in turn, change according to the influence of the fluid motion. Interactions of this kind are studied in magnetohydrodynamics (MHD). In this research area, they are described via a coupling between the Navier-Stokes equations and the Maxwell system, cf. [21, 28, 83]. The equations in this coupling are simplified in comparison to the original systems under several physical assumptions. The latter procedure is commonly referred to as the magnetohydrodynamic approximation. A rigorous justification of this approximation is given in [74, 75]. Finally, the third kind of interaction happens between the electromagnetic fields and the rigid bodies. As the bodies are assumed to be insulating, there occurs no direct interaction between these objects. However, they interact indirectly due to their respective interplay with the fluid.

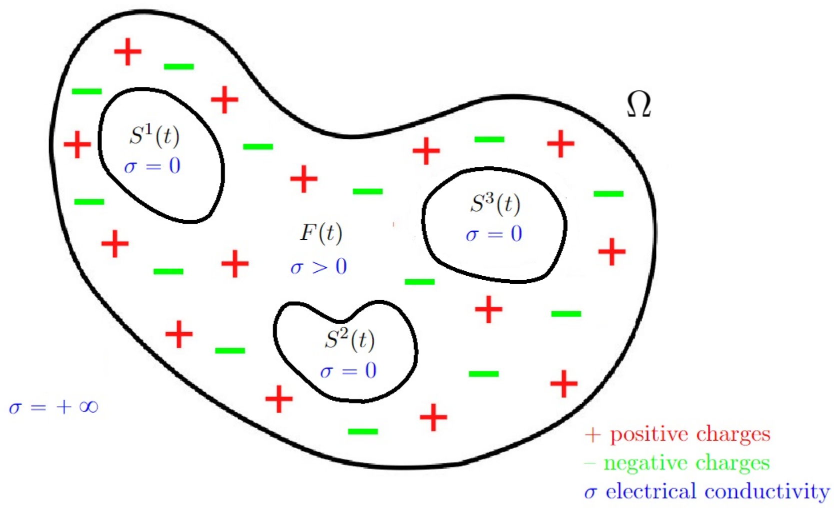

Figure 1.1: A domain Ω occupied by an electrically conducting fluid $F(t)$ and three insulating rigid bodies $S^1(t)$, $S^2(t)$ and $S^3(t)$. Figure taken from [105].

We study the fluid-rigid body interaction problem with an electrically conducting fluid in both the incompressible case - see the model presented in Section 1.3.1 below - and the compressible case, cf. the model introduced in Section 1.3.2. For the incompressible case we prove the existence of weak solutions to the problem in Chapter 3. In Chapter 4 we extend this result to the compressible case. In addition to this mathematical analysis we further take a glance at the derivation of (the electromagnetic part of) the models in Chapter 2. More precisely, we present a derivation of the magnetohydrodynamic approximation as well as the boundary and interface conditions for the electromagnetic fields from the original Maxwell system.

As mentioned above, the second class of phenomena we study is the evolution of magnetoelastic materials, cf. Figure 1.2. Magnetoelastic materials are ferromagnetic deformable materials, the magnetization and the deformation of which stand in a mutual relation with each other. More specifically, such materials undergo a deformation when exposed to a magnetic field, a behavior known as the magnetostrictive effect. The other way around, they experience a change in their magnetization when mechanical stress is applied to them, which is referred to as the inverse magnetostrictive effect. The use of magnetoelastic materials in real-word applications is twofold, both the magnetostrictive and the inverse magnetostrictive effect have various applications in engineering. The magnetostrictive effect is the basic principle used in the construction of magnetic actuators, cf. [16, 110]. Such actuators are transducers which convert changes in magnetic fields into mechanical energy. This, for example, leads to a connection between magnetoelasticity and the fluid-structure interaction problem with an electrically conducting fluid. Indeed, magnetic actuators provide an alternative option for the control of the microrobots in capsule endoscopy and remote drug delivery mentioned above, cf. [114]. The inverse magnetostrictive effect is exploited in sensors which measure mechanical stresses by converting them into alterations in their magnetic fields, cf. [10, 11, 16, 62]. This precise measurement technique is helpful for example in civil engineering, where it is used for monitoring damages, corrosion or fatigue and thus helps to prevent the collapse of civil buildings, see [3]. It further finds use in the field of bioelectronics, for example in monitoring the human cardiovascular system by attaching soft magnetoelastic generators to the body with the capability of transforming their deformation by the pulse into electric signals, cf. [116].

Magnetoelastic materials are investigated mathematically in the research field of magnetoelasticity. This field, in turn, constitutes a combination of the fields of micromagnetics and elasticity theory. The objective of micromagnetics is the description of the magnetic behavior of (ferromagnetic) materials at microscopically small length scales, see for example [19, 80]. This description is achieved in terms of the magnetization of the material in consideration. Elasticity theory, cf. [81, 85], is the study of materials

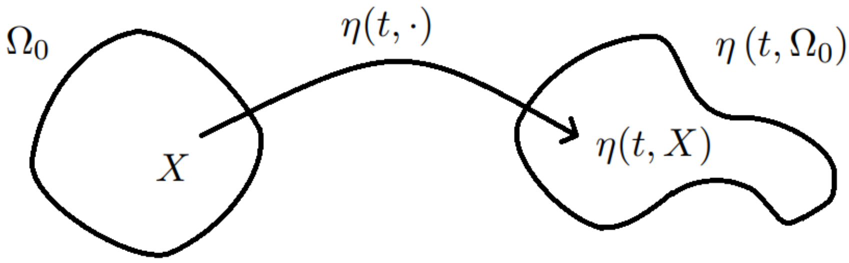

Figure 1.2: A deformation $\eta(t,\cdot)$ mapping a (magneto)elastic material from its reference configuration Ω_0 to its current configuration $\eta(t,\Omega_0)$.

with the property of being deformable under the application of mechanical stress and returning to their original shape once the stress applied to them is removed. In the theory of magnetoelasticity we see an interplay between these two fields. The magnetization and the deformation of magnetoelastic materials interact with each other via the magnetostrictive and the inverse magnetostrictive effect. For an introduction to magnetoelasticity we refer to [18]. The specific model of the interaction between the magnetization and the deformation of a magnetoelastic material we examine in the present thesis (cf. Section 1.3.3 below) is based on the model derived and analyzed in [6, 48]. In Chapter 5 we prove the existence of weak solutions to this model.

1.2 Related literature, mathematical challenges and methodology

In the following we present an overview of the mathematical literature related to this thesis and point out the advancements achieved through our results as well as the main difficulties and novelties in their proofs.

1.2.1 FSI in an electrically conducting fluid: Embedding of our results into the related literature

We begin with the fluid-rigid body interaction problem with an electrically conducting fluid studied in Chapter 3 and Chapter 4. The existence of weak solutions to fluids modeled via the Navier-Stokes equations without any solid bodies or electromagnetic quantities involved is well-studied, cf. for example [86] for the incompressible case and [87, 94] for the compressible case. A first introduction to the interaction problem between a fluid and a rigid body can be found in [51, 109]. In the early years the investigation of the problem was mostly focused on the incompressible case. In this setting, the existence of weak solutions up to the first time at which a collision occurs between the body and the boundary of the domain was proved for example in [25, 65, 71] in both two and three spatial dimensions. The case of several rigid bodies was studied for example in [33], wherein the existence of weak solutions in the two- and the three-dimensional setting was proved up to the first contact between a body and the domain boundary or between the bodies themselves.
After the achievement of such local-in-time results the investigation of the global-in-time existence was not long in coming. In [103] the global-in-time existence of weak solutions to the interaction problem between an incompressible fluid and several rigid bodies was proved in two spatial dimensions. The authors of this article moreover addressed the issue of the possibility of contacts between the bodies or a body and the domain boundary. It turned out that such contacts are possible, however, only under the condition of vanishing relative velocity and acceleration between the colliding objects. In three spatial dimensions the global-in-time existence of weak solutions to the same problem was proved in [44].
Besides weak solutions, also strong solutions to the problem are of great interest and the question about their existence was investigated e.g. in [54, 111, 113]. Moreover, the uniqueness of solutions has been a subject of research. In the article [89] it was shown that strong solutions to the fluid-rigid

body interaction problem with an incompressible fluid are unique within the classes of both strong and weak solutions. This property of the problem is referred to as weak-strong uniqueness. Finally we point out that the study of different boundary conditions (for the fluid on the domain boundary) and interface conditions (between the fluid and the rigid bodies) instead of the classical no-slip condition has started to move towards the center of attention nowadays. As a specific example we mention the article [4], wherein the local-in-time existence of a weak solution to a system modeling the interaction between an incompressible fluid and a rigid body coupled via the Coulomb friction law boundary and interface condition could be proved.

For the fluid-rigid body interaction problem with a compressible fluid similar results are available. The local-in-time existence of weak solutions to the interaction problem between multiple rigid bodies and a compressible fluid for both the 2D and the 3D case was achieved in [34]. An extension to the global-in-time existence of weak solutions was subsequently obtained in [43]. The existence of strong solutions was investigated for example in [14, 66, 70, 100] and weak-strong uniqueness of solutions could be proved in [79]. Moreover, alternative boundary and interface conditions under which the problem was studied include the Navier-slip condition. Under this condition the proof of the local-in-time existence of weak solutions was achieved in [91].

Besides the plenty results for the fluid-rigid body interaction problem, there is also a comprehensive existence theory for the MHD coupling between the Navier-Stokes equations and the Maxwell system without the involvement of any solid bodies. A proof for global-in-time existence of weak solutions to the MHD problem for the case of an incompressible fluid can be found in [55]. A corresponding result for the case of a compressible fluid was proved in [104]. Moreover, we mention the article [12], wherein the model in consideration has further been expanded by the assumption of the fluid being thermally conducting. For the resulting system of partial differential equations, the global-in-time existence of weak solutions is proved and, in addition, the question about the existence of strong solutions is addressed.

Despite the extensively worked out theory in both FSI and MHD, not many results on the combination between these two research areas appear to be available. A first exploration of this uncharted territory was made in [63, 64]. In these articles the flow of an electrically conducting incompressible fluid around an insulating rigid body is studied in two and three spatial dimensions respectively and the existence of weak solutions to the corresponding models is proved. While the rigid body therein is non-movable, we extend these articles in Chapter 3 by proving the (local-in-time) existence of weak solutions to a system modeling the interaction between an incompressible electrically conducting fluid, a moving insulating rigid body and the electromagnetic fields present in both materials, cf. Section 1.3.1 for the model and Theorem 3.1.1 for the result. This result, which is joint work of Barbora Benešová, Šárka Nečasová, Anja Schlömerkemper and the author of this thesis, has been published in the article [8]. In Chapter 4, we in turn extend this result to the proof of the global-in-time existence of weak solutions to a model of the interaction between a compressible electrically conducting fluid, finitely many insulating rigid bodies and the electromagnetic fields present in these materials, see the model presented in Section 1.3.2 and the result stated in Theorem 4.1.1. This result has been published by the author of this thesis in the article [105]. The proofs which we here present for these results are essentially identical with the proofs given in the articles [8] and [105], respectively. However, we include additional details in our analysis, hopefully facilitating some of the more technical parts of the proofs.

1.2.2 FSI in an electrically conducting fluid: Mathematical challenges and methodology

The main difficulty in the proofs of both these results arises from the dependence of the test functions in the variational form of the induction equation on the solid domain. Indeed, reflecting the non-conductivity of the solid region, these test functions are chosen curl-free in this part of the domain. The crucial difference to [63, 64], wherein similar test functions were used without causing any serious trouble, lies in the movability of the solid domain and thus its dependence on the solution to the problem in our setting. Consequently, in our case, the test functions in the induction equation also depend on the overall solution to the system. Our idea for handling the resulting high coupling of the problem consists of the usage of a time discretization via the Rothe method (cf. [99, Section 8]). More precisely, we split the original time interval into a finite sequence of discrete times in order to decouple the system via the use of time-lagging functions. At each fixed discrete time, this procedure allows us to first determine the position of the solid body and subsequently choose the test functions for the induction equation accordingly. With these test functions at hand we may then solve the induction equation at that specific discrete time via classical methods.

The discretization with respect to the time variable, however, leads to further problems, as it is not compatible with certain methods developed for the continuous Navier-Stokes equations. The first one of these problems concerns the non-negativity of the density, which is needed to obtain the uniform bounds from the energy inequality necessary for the limit passage in the approximate problem. In the continuous incompressible Navier-Stokes equations in Chapter 3 the density evolves according to a transport equation and thus its bound away from zero follows from a corresponding bound for the initial data. For the discretized equations, this argumentation does not hold true anymore. In order to recover the boundedness of the density away from zero in the discrete system we thus borrow the technique used to derive non-negativity of the density in the continuous compressible system, cf. [94, Section 7.6.5]. This classical technique consists of a regularization of the continuity equation through the addition of a Laplacian to the right-hand side and turns out to be still applicable in the discretized incompressible setting. A similar problem arises in the transport equation for the characteristic function of the solid body, which is used for the description of the solid domain in the incompressible setting. In order to be able to precisely identify the solid domain at each time we want this function to take only the values 0 and 1. Again, in the continuous system this property follows immediately from the transport theory but it gets lost once we discretize the transport equation. In the spirit of [56] we solve this problem by, in fact, not discretizing the equation but instead considering it as a continuous equation on the small intervals between two consecutive discrete times. Consequently, our approach does not consist of a full discretization of the problem, but we rather consider a hybrid approximation to the system, in which some equations are discretized while others are treated as continuous equations on small time intervals.

The latter idea further turns out to play an important role in the compressible setting in Chapter 4, in which a discretization of the equations causes some additional problems: The author has not been able to discretize the compressible Navier-Stokes equations in such a way that non-negativity of the density can be derived via the classical regularization of the continuity equation. Instead, we again use a hybrid approximation scheme, in which this time the whole mechanical part of the problem is treated as a continuous system on the small intervals between discrete time points, while only the induction equation is actually discretized. This allows us to prove the existence of a non-negative density by the classical arguments and, via a suitable choice of the coupling terms in the hybrid approximation, we are able to combine the continuous mechanical subsystem and the discrete induction equation into a meaningful energy inequality, from which we again obtain the uniform bounds required for the limit passage in the approximate system.

We moreover point out that in both the incompressible and the compressible setting we use test functions which depend on the solid domain not only in the induction equation but also in the momentum equation. This is standard in fluid-structure interaction problems and we can make use of well-known techniques developed to deal with this situation. More precisely, in the incompressible case we apply the Brinkman penalization (cf. [15]). In the approximate problem generated by this method the fluid is extended into the solid region, so that only a classical Navier-Stokes system with test functions independent of the solid domain needs to be solved. The solid domain is determined separately via a transport equation with a velocity field given as a rigid projection of the velocity obtained from the fluid equations. In the limit of the penalization the fluid velocity and its rigid projection in fact coincide in the solid domain, which allows us to recover a solution to the original fluid-rigid body interaction problem.

In the compressible case we exploit a similar penalization method. More specifically, we use the same method as for example in [43, 103], in which an approximate fluid-only problem with classical test functions is constructed and solved. We then proceed by letting the viscosity of the fluid tend to infinity in the later solid domain. In the limit, this procedure again yields the desired solution to the original fluid-rigid body interaction problem.

1.2.3 Magnetoelasticity: Embedding of our result into the related literature

Next, we turn to the development of the mathematical side of the theory of magnetoelasticity. The first mathematical result on magnetoelasticity we mention is [101]. In this article the existence of a deformation and a magnetization of a magnetoelastic material as a minimizing pair to the associated energy functional was shown in the steady-state case in three spatial dimensions. This result was obtained under several restrictive assumptions, the most striking of which is the dependence of the elastic energy on the second gradient of the deformation. Moreover, while the result was not limited to the case of incompressible materials, a saturation constraint close to incompressibility had to be imposed, namely the condition of the product of the absolute value of the magnetization and the determinant of the deformation gradient being equal to one. An approach to relax the latter condition by adding a penalization term for the compressibility worked out for the authors of [9], leading to an existence result in the case of so-called nearly incompressible materials. An existence result in which the dependence of the energy functional on the second deformation gradient could be circumvented was achieved in [82], as a compensation the authors imposed stronger growth conditions on the elastic energy density and restricted their investigation to the case of strictly incompressible materials. Moreover, in the same article, the authors studied the quasi-static setting, in which the problem is turned into an evolutionary problem but inertial effects remain neglected, and were able to prove a corresponding existence result also in this case. A generalization of these results to the setting of compressible materials satisfying the aforementioned saturation condition close to incompressibility was obtained in [77] in both the static and the quasi-static case. Furthermore, an extension to the setting of fully compressible materials can be found in [5] in the static case and in [17] in both the static and the quasi-static case.

Another approach to the evolutionary problem consists of restricting to the setting of small strain, in which large deformations are excluded. This simplification of the problem in turn allows to also take into account inertial effects. Existence results relying on this approach can be found for example in [22, 23, 38]. The latter articles further differ from the works cited above in that they do not formulate the problem as a minimization problem but rather as a system of partial differential equations. Such systems of PDEs are composed of an equation of motion for the description of the deformation and a magnetic force balance - typically (a version of) the Landau-Lifshitz-Gilbert equation, cf. [72, Section 3.2.7] - determining the evolution of the magnetization.

A further evolutionary model in the form of a system of PDEs was derived (via a variational approach) and analyzed in [6, 48]. A specialty in these works is that the equation of motion is not formulated, as usual in elasticity theory, in the reference configuration but in the current configuration. At first sight this complicates the problem: In the current configuration the equation is solved for the velocity instead of the deformation so that in particular the deformation gradient appears to be unavailable. This obstacle, however, is overcome by determining the deformation gradient separately - in the current configuration - via an additional transport equation. On closer inspection we then realize that this approach even bears an advantage in the analysis: Due to the fact that the magnetic force balance (the Landau-Lifshitz-Gilbert equation) is formulated in the current configuration as well, the necessity to guarantee invertibility of the deformation falls away and the accompanying mathematical difficulties are eliminated.

For the proof of two existence results the authors of these works studied a simplified version of this model. In this simplified model, the deformation gradient is replaced - for regularization purposes - by an approximate deformation gradient satisfying a regularized version of the transport equation and the setting is restricted to deformations which do not change the surface of the material in consideration and thus preserve its shape. Additionally, the material is assumed to be incompressible and the stray field, among other quantities, is neglected. The first existence result in [48], in which the Landau-Lifshitz-Gilbert equation is further replaced by a gradient flow equation, guarantees the global-in-time existence of weak solutions in two and three spatial dimensions. In [106], uniqueness of these weak solutions in the 2D case as well as weak-strong uniqueness in the 3D case was shown. Moreover, the global-in-time existence and uniqueness of strong solutions in the 2D setting was proved in [53].

The second existence result in [48], a proof of which is also given in [6], assures the global-in-time existence of weak solutions in two spatial dimensions under a smallness assumption on the initial data in case of the full Landau-Lifshitz-Gilbert equation. This result received an extension in [76, 78]: The authors thereof included the stray field, relaxed the conditions on the elastic energy density and still managed to prove the global-in-time existence of weak solutions in 2D under a corresponding smallness assumption. They further proved the local-in-time existence of strong solutions and weak-strong uniqueness for the problem. The global-in-time existence of weak solutions to the same problem in a two dimensional periodic domain without any smallness assumptions was achieved in [29]. The weak solutions constructed in the latter article belong to (and are, in fact, unique in) a class of functions which are smooth except for in a finite number of time points. Finally, as another extension to [76, 78], we also mention [37], wherein the local-in-time existence as well as the uniqueness of strong solutions is obtained in the 3D case.

In Chapter 5 of the present thesis we prove the existence of weak solutions to another modification of the model from [48]: In three spatial dimensions we study the setting in which the magnetic force balance consists of a gradient flow equation with the stray field included in the micromagnetic energy. As an additional simplification, neglecting the inertia term in the equation of motion, we restrict ourselves to the quasi-static case and we regularize the problem by adding the second deformation gradient to the elastic energy. Moreover, by including the reciprocal of the determinant of the deformation gradient into the elastic energy and thus guaranteeing invertibility of the deformation, we are able to formulate the equation of motion in the reference configuration while we keep the magnetic force balance expressed in the current configuration as it is typical in magnetoelasticity. In this setting we achieve two main novelties in comparison to the previous works cited above. Firstly, we are able to solve for the deformation gradient itself instead of for a regularized approximation. Secondly, we include deformations changing the shape of the magnetoelastic material into our investigation. In addition, we take into account compressible materials and, while we choose the elastic energy to be convex with respect to the second deformation gradient, we do not require it to be convex with respect to the deformation itself. The specific model we study is presented in Section 1.3.3, the result we achieve (in Chapter 5) is stated in Theorem 5.1.1.

1.2.4 Magnetoelasticity: Mathematical challenges and methodology

Mathematically the differences in the considered problem (cf. the previous paragraph) manifest themselves in a different approach to the proof of our result. While the typical approach to solve the system of PDEs in the works cited above consists of a Galerkin approximation, we here employ the implementation of De Giorgi's minimizing movements method (see [30]) which was already used in [7] for the construction of weak solutions to (purely mechanical) FSI problems. In this variational method the evolutionary problem is discretized with respect to the time variable, similarly as in the proofs of our results on the fluid-rigid body interaction problem explicated above. However, the aim of this discretization is not to decouple the equations and solve them successively. Instead, at each discrete time, a minimization problem is solved via the direct method, the associated Euler-Lagrange equations of which constitute a suitable discrete approximation to the original system. The desired solution to the original problem is then obtained by passing to the limit in the discretization.

The main reason why we opt for the minimizing movements scheme lies in the non-convexity of the energy. Indeed, solving the problem via a Galerkin method typically involves solving the coupled equations directly by a fixed point argument. Such fixed point arguments in turn often require convexity of the energy, making the Galerkin method unfeasible in our setting. Further, discretizing the system in order to decouple the equations and solve them successively but still directly is also not compatible with the non-convexity of the energy: In this approach, the necessary a priori estimates for the limit passage in the discretization require a discrete form of the chain rule, which again relies on the (unavailable) convexity of the energy. The minimizing movements method instead provides a discrete energy estimate in a rather natural way, it is obtained by comparing, at each discrete time point, the value of the minimized functional in its minimizer to its value in the solution from the previous discrete time.

The main difficulty in the application of the minimizing movements scheme in our case lies in the right choice of the minimization problem on the discrete level. It needs to be chosen in such a way that the variation of the functional to be minimized with respect to the deformation yields a suitable approximation of the equation of motion while a variation with respect to the magnetization leads to a suitable approximation of the magnetic force balance. The possibility to do this comes along with the realization that the transport terms in the magnetic force balance in the considered model can be expressed via a dissipation potential which does not contribute to the equation of motion, i.e. which vanishes when differentiated with respect to the time derivative of the deformation. This allows us to derive, on the continuous level, both the equation of motion and the magnetic force balance from the same energy and dissipation. The functional for the discrete minimization problem can then be built after this energy and dissipation. We remark that, since in the discrete setting we cannot differentiate the functional with respect to the time derivative of the deformation but only with respect to the deformation itself, the transport terms from the magnetic force balance in fact do give a contribution to the equation of motion in the approximate system. This contribution, however, vanishes when we pass to the limit in the discretization. Further difficulties include the proof of strong convergence of the stray field and the magnetization gradient. Both of these convergences are obtained by showing first weak convergence and subsequently convergence of the norm of the respective quantity. In case of the stray field, convergence of the norm is deduced under exploitation of the definition of the stray field as the solution to a Poisson problem. Convergence of the norm of the magnetization gradient is obtained via a comparison between the magnetic force balance tested by the magnetization on the continuous level and the limit of the discrete magnetic force balance tested by the discrete magnetization.

1.3 Models

In this section we introduce the mathematical models studied throughout this thesis.

1.3.1 Fluid-rigid body interaction in an incompressible electrically conducting fluid

In Chapter 3 we study the movement of an insulating rigid body through an electrically conducting viscous non-homogeneous incompressible fluid as well as the electromagnetic fields present in these materials. The result we achieve is joint work with Barbora Benešová, Šárka Nečasová and Anja Schlömerkemper and has been published in [8]; the associated model, which we present in the following, can be found in this article as well. We consider a time $T > 0$, a bounded spatial domain $\Omega \subset \mathbb{R}^3$ and define the time-space domain $Q := (0,T) \times \Omega$. At each fixed time $t \in [0,T]$, Ω is filled with the rigid body, occupying a domain $S(t) \subset \Omega$ and the fluid, occupying the domain $F(t) := \Omega \backslash \overline{S(t)}$. The initial position of the body is denoted by $S_0 = S(0)$. We further split the time-space domain Q into its solid part Q^s and its fluid counterpart Q^f,

$$Q^s := \{(t,x) \in Q : \; x \in S(t)\}, \qquad Q^f := \{(t,x) \in Q : \; x \in F(t)\}.$$

Correspondingly we divide any function defined on Q into its restriction to the fluid region - marked by the superscript f - and its restriction to the solid region - marked by the superscript s. The mechanical processes in the considered situation are described by the density $\rho : Q \to \mathbb{R}$ of the fluid and the rigid body, the velocity field $u : Q \to \mathbb{R}^3$ of the fluid and the body as well as the pressure $p = p^f : Q^f \to \mathbb{R}$ of the fluid. The electromagnetic effects come into play via the magnetic induction $B : Q \to \mathbb{R}^3$, the magnetic field $H : Q \to \mathbb{R}^3$, the electric field $E : Q \to \mathbb{R}^3$ and the electric current density $j : Q \to \mathbb{R}^3$. The evolution of these quantities is described by the system of partial differential equations

$$\operatorname{curl} H = j + J \qquad \text{in } Q^f, \tag{1.3.1}$$
$$\operatorname{curl} H = 0 \qquad \text{in } Q^s, \tag{1.3.2}$$
$$\partial_t B + \operatorname{curl} E = 0 \qquad \text{in } Q^f \text{ and } Q^s, \tag{1.3.3}$$
$$\operatorname{div} E = 0 \qquad \text{in } Q^s, \tag{1.3.4}$$
$$\operatorname{div} B = 0 \qquad \text{in } Q^f \text{ and } Q^s, \tag{1.3.5}$$
$$\operatorname{div} u = 0, \quad \partial_t \rho + u \cdot \nabla \rho = 0 \qquad \text{in } Q^f, \tag{1.3.6}$$
$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \operatorname{div}\mathbb{T} + \rho g + \frac{1}{\mu}\operatorname{curl} B \times B \qquad \text{in } Q^f, \tag{1.3.7}$$
$$m\frac{d}{dt}V(t) = \frac{d}{dt}\int_{S(t)} \rho u \, dx = \int_{\partial S(t)} [\mathbb{T} - p \operatorname{id}]\,\mathrm{n}\, dA + \int_{S(t)} \rho g \, dx, \quad t \in [0,T], \tag{1.3.8}$$
$$\begin{aligned}\frac{d}{dt}\left(\mathbb{J}(t)w(t)\right) &= \frac{d}{dt}\int_{S(t)} \rho\,(x - X) \times u \, dx \\ &= \int_{\partial S(t)} (x - X) \times [\mathbb{T} - p \operatorname{id}]\,\mathrm{n}\, dA + \int_{S(t)} \rho\,(x - X) \times g \, dx, \quad t \in [0,T],\end{aligned} \tag{1.3.9}$$

together with the relations

$$j = \sigma(E + u \times B) \quad \text{in } Q^f \text{ and } Q^s, \qquad \sigma = \begin{cases} \sigma^f > 0 & \text{in } Q^f, \\ \sigma^s = 0 & \text{in } Q^s, \end{cases} \tag{1.3.10}$$
$$B = \mu H, \qquad \mu > 0 \quad \text{in } Q \tag{1.3.11}$$

and the boundary and interface conditions

$$B(t) \cdot \mathrm{n} = 0 \quad \text{on } \partial\Omega, \qquad B^f(t) - B^s(t) = 0 \quad \text{on } \partial S(t), \tag{1.3.12}$$
$$E(t) \times \mathrm{n} = 0 \quad \text{on } \partial\Omega, \qquad \left(E^f(t) - E^s(t)\right) \times \mathrm{n} = 0 \quad \text{on } \partial S(t), \tag{1.3.13}$$
$$u(t) = 0 \quad \text{on } \partial\Omega, \qquad u^f(t) - u^s(t) = 0 \quad \text{on } \partial S(t) \tag{1.3.14}$$

for any $t \in [0,T]$. The system (1.3.1)–(1.3.14) splits into an electromagnetic subsystem (1.3.1)–(1.3.5), (1.3.10)–(1.3.13) and a mechanical subsystem (1.3.6)–(1.3.9), (1.3.14). In the electromagnetic part we recognize a modified version of the Maxwell system (see [63, 64]) in the equations (1.3.1)–(1.3.5): In the solid domain this system is customized to the assumption of the rigid body being insulating while in the fluid domain it is simplified in accordance with the classical magnetohydrodynamic approximation. Additionally, as in [63, 64], Ampère's law (1.3.1) in the fluid region contains an external forcing term $J : Q \to \mathbb{R}^3$. The equation (1.3.10), which is commonly known as Ohm's law and in which σ represents the electrical conductivity of the respective material, couples the electromagnetic part of the problem to the mechanical part. It expresses the influence of the fluid velocity on the electromagnetic fields and further shows that these fields are not directly affected by the insulating solid body, in which it holds that $\sigma = \sigma^s = 0$. The linear relation (1.3.11) constitutes a common constitutive assumption in both Q^f and Q^s, which relates B to H via the magnetic permeability μ of the respective material, see for example [73, Section 5.8]. However, we point out that in our setting we specifically assume μ to be constant in the whole domain,

$$\mu = \mu^s = \mu^f > 0 \qquad \text{in } Q.$$

This simplifying assumption needs to be made, despite its physical impreciseness, for mathematical reasons. Indeed, in the mathematical analysis carried out in Chapter 3 we require B to be a Sobolev function over the whole domain Ω. This can be achieved provided that, as stated in (1.3.12), B is continuous across $\partial S(t)$. The latter condition, however, is a stronger assumption than the typically imposed continuity of the normal component of B across $\partial S(t)$. Nevertheless, for μ being constant across $\partial S(t)$, it can be seen as a consequence of the relation (1.3.11) and the standardly assumed

continuity of the tangential component of H. The remaining boundary and interface conditions imposed in (1.3.12) and (1.3.13), in which n denotes the outer unit normal vector on $\partial\Omega$ and $\partial S(t)$, are classical.
In the mechanical part of the above system, which can essentially be found for example in [44], we see the incompressible Navier-Stokes system, consisting of the continuity equation and the incompressibility constraint (1.3.6) as well as the momentum equation (1.3.7). In the latter of these relations the stress tensor $\mathbb{T}$ is defined by the formula

$$\mathbb{T} = \mathbb{T}(u) := 2\nu\mathbb{D}(u), \qquad \mathbb{D}(u) := \frac{1}{2}\nabla u + \frac{1}{2}(\nabla u)^T,$$

in which the constant $\nu > 0$ is called a viscosity coefficient. The momentum equation further contains two forcing terms: On the one hand, the given function $g : Q \to \mathbb{R}^3$ constitutes an external forcing term. The reduced - in accordance with the magnetohydrodynamic approximation - Lorentz force $\frac{1}{\mu}\mathrm{curl}B \times B$, on the other hand, expresses the force exerted by the electromagnetic fields on the fluid. It thus couples the mechanical part of the problem to the electromagnetic part. We remark that, in accordance with the non-conductivity of the rigid body, the Lorentz force does not appear in the balance of linear momentum (1.3.8) and the balance of angular momentum (1.3.9) of the body, which determine its translational velocity V and its rotational velocity w. Consequently, the movement of the solid body remains unaffected by the electromagnetic fields and is entirely driven by the Cauchy stress $\mathbb{T} - p$ id exerted by the fluid and the external forcing term g. The quantities

$$m := \int_{S(t)} \rho(t,x)\, dx, \quad X(t) := \frac{1}{m}\int_{S(t)} \rho(t,x)x\, dx,$$
$$\mathbb{J}(t)a \cdot b := \int_{S(t)} \rho(t,x)\left[a \times (x - X(t))\right] \cdot \left[b \times (x - X(t))\right]\, dx, \quad a, b \in \mathbb{R}^3,$$

in (1.3.8) and (1.3.9) are the total mass m, the center of mass X and the inertia tensor $\mathbb{J}$ of the solid body. The overall velocity of the body is then given as the rigid velocity field

$$u(t,x) = u^s(t,x) := V(t) + w(t) \times (x - X(t)) \qquad \text{for } t \in [0,T],\ x \in S(t).$$

Finally, the system comes full circle in the relation (1.3.14), which shows that this solid velocity, in turn, also exerts an effect on the fluid motion. The no-slip boundary and interface conditions described in this relation form a standard set of boundary and interface conditions for fluid-structure interaction problems, cf. for example [44, 103].

1.3.2 Fluid-rigid body interaction in a compressible electrically conducting fluid

In Chapter 4 we study the compressible pendant to the model presented in the previous Section 1.3.1, which we additionally generalize to the setting of several instead of only one solid body. More precisely, we study a model of multiple insulating rigid bodies traveling through an electrically conducting viscous non-homogeneous compressible fluid as well as the electromagnetic fields trespassing both the solids and the fluid. The mathematical result we achieve as well as the model, which we present in the following, can also be found in the article [91] by the author of this thesis. Let $T > 0$, let $\Omega \subset \mathbb{R}^3$ denote a bounded domain and set $Q := (0,T) \times \Omega$. Let the initial positions of the rigid bodies be given as subsets $S_0^i = S^i(0) \subset \Omega$, $i = 1, ..., N \in \mathbb{N}$, and denote by $S^i(t) \subset \Omega$ the position of the i-th body at an arbitrary time $t \in [0,T]$. By $F(t) := \Omega \backslash \overline{S(t)}$, where $S(t) := \bigcup_{i=1}^N S^i(t)$, we denote the region occupied by the fluid at time t and we split the time-space domain Q into its solid and its fluid part,

$$Q^s := \{(t,x) \in Q :\ x \in S(t)\}, \qquad Q^f := \{(t,x) \in Q :\ x \in F(t)\}.$$

The restriction of any function defined on Q to Q^s or Q^f is indicated by the superscript s or f, respectively. The interplay between the fluid, the rigid bodies and the electromagnetic fields is described via the density $\rho : Q \to \mathbb{R}$, the velocity field $u : Q \to \mathbb{R}^3$, the pressure $p = p^f : Q^f \to \mathbb{R}$, the magnetic induction $B : Q \to \mathbb{R}^3$, the magnetic field $H : Q \to \mathbb{R}^3$, the electric field $E : Q \to \mathbb{R}^3$ and the electric current density $j : Q \to \mathbb{R}^3$. The evolution of these quantities is described by the system of partial differential equations

$$\mathrm{curl} H = j + J \quad \text{in } Q^f, \tag{1.3.15}$$

$$\mathrm{curl} H = 0 \quad \text{in } Q^s, \tag{1.3.16}$$

$$\partial_t B + \mathrm{curl} E = 0 \quad \text{in } Q^f \text{ and } Q^s, \tag{1.3.17}$$

$$\mathrm{div} E = 0 \quad \text{in } Q^s, \tag{1.3.18}$$

$$\mathrm{div} B = 0 \quad \text{in } Q^f \text{ and } Q^s, \tag{1.3.19}$$

$$\partial_t \rho + \mathrm{div}(\rho u) = 0 \quad \text{in } Q^f, \tag{1.3.20}$$

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p = \mathrm{div}\mathbb{T} + \rho g + \frac{1}{\mu}\mathrm{curl} B \times B \quad \text{in } Q^f, \tag{1.3.21}$$

$$m^i \frac{d}{dt} V^i(t) = \frac{d}{dt}\int_{S^i(t)} \rho u \, dx = \int_{\partial S^i(t)} [\mathbb{T} - p \, \mathrm{id}] \, \mathrm{n} \, dA + \int_{S^i(t)} \rho g \, dx, \quad t \in [0,T], \; i = 1, ..., N, \tag{1.3.22}$$

$$\frac{d}{dt}\left(\mathbb{J}^i(t) w^i(t)\right) = \frac{d}{dt}\int_{S^i(t)} \rho \left(x - X^i\right) \times u \, dx$$
$$= \int_{\partial S^i(t)} (x - X^i) \times [\mathbb{T} - p \, \mathrm{id}] \, \mathrm{n} \, dA + \int_{S^i(t)} \rho \left(x - X^i\right) \times g \, dx, \quad t \in [0,T], \; i = 1, ..., N \tag{1.3.23}$$

together with the relations

$$j = \sigma(E + u \times B) \quad \text{in } Q^f \text{ and } Q^s, \qquad \sigma = \begin{cases} \sigma^f > 0 & \text{in } Q^f, \\ \sigma^s = 0 & \text{in } Q^s, \end{cases} \tag{1.3.24}$$

$$B = \mu H, \qquad \mu > 0 \quad \text{in } Q \tag{1.3.25}$$

and the boundary and interface conditions

$$B(t) \cdot \mathrm{n} = 0 \quad \text{on } \partial\Omega, \qquad B^f(t) - B^s(t) = 0 \quad \text{on } \partial S(t), \tag{1.3.26}$$

$$E(t) \times \mathrm{n} = 0 \quad \text{on } \partial\Omega, \qquad \left(E^f(t) - E^s(t)\right) \times \mathrm{n} = 0 \quad \text{on } \partial S(t), \tag{1.3.27}$$

$$u(t) = 0 \quad \text{on } \partial\Omega, \qquad u^f(t) - u^s(t) = 0 \quad \text{on } \partial S(t) \tag{1.3.28}$$

for any $t \in [0,T]$. The electromagnetic part (1.3.15)–(1.3.19), (1.3.24)–(1.3.27) of this model, in which $J : Q \to \mathbb{R}^3$ denotes an external force, σ denotes the electrical conductivity, μ denotes the magnetic permeability and n denotes the outer unit normal vector on $\partial\Omega$ and $\partial S(t)$, coincides with the electromagnetic subsystem (1.3.1)–(1.3.5), (1.3.10)–(1.3.13) from the incompressible case. For the mechanical part (1.3.20)–(1.3.23), (1.3.28) the mechanical subsystem (1.3.6)–(1.3.9), (1.3.14) has been adjusted to the assumption of the fluid being compressible, cf. [43]. More precisely, the divergence-free condition and the associated simplified continuity equation (1.3.6) have been replaced by the more general form (1.3.20) of the continuity equation. The momentum equation (1.3.21), while having the same form as its incompressible counterpart (1.3.7), bears two important differences: Firstly, as the velocity field u is not assumed to be divergence-free, the stress tensor $\mathbb{T}$ takes the more general form

$$\mathbb{T} = \mathbb{T}(u) := 2\nu\mathbb{D}(u) + \lambda \mathrm{id} \; \mathrm{div}\, u, \qquad \mathbb{D}(u) := \frac{1}{2}\nabla u + \frac{1}{2}(\nabla u)^T,$$

where the viscosity coefficients ν, λ satisfy $\nu > 0$ and $\lambda + \nu \geqslant 0$. Secondly, the pressure $p : Q \to \mathbb{R}$ is assumed to be a function of only the density, prescribed by the isentropic constitutive relation

$$p = p\left(\rho^f\right) = a\left(\rho^f\right)^\gamma, \quad a > 0, \; \gamma > \frac{3}{2}. \tag{1.3.29}$$

The reason for the latter assumption is of mathematical nature and also lies in the missing of the divergence-free condition on u in the present setting. In the incompressible setting, due to the divergence-free condition (1.3.6) on the velocity field, it is natural to choose divergence-free test functions in the variational formulation of the problem, cf. Definition 3.1.1 below. In particular, this causes the pressure term to vanish from the variational formulation and thus simplifies the mathematical analysis of the problem. In the compressible case, however, we cannot restrict to divergence-free test functions and therefore have to deal with the pressure term explicitly. The isentropic relation (1.3.29) is a common assumption which allows us to handle this issue, see for example [94, Section 1.2.18]. The function $g : Q \to \mathbb{R}^3$ in the momentum equation (1.3.21) constitutes an external forcing term. The balance of linear momentum (1.3.22) and angular momentum (1.3.23), as opposed to their pendants (1.3.8) and (1.3.9) in the incompressible setting, are formulated for multiple solid bodies, indexed by $i = 1, ..., N$. In these equations the quantities

$$m^i := \int_{S^i(t)} \rho(t,x)\, dx, \quad X^i(t) := \frac{1}{m^i}\int_{S^i(t)} \rho(t,x)x\, dx,$$
$$\mathbb{J}^i(t)a \cdot b := \int_{S^i(t)} \rho(t,x)\left[a \times \left(x - X^i(t)\right)\right] \cdot \left[b \times \left(x - X^i(t)\right)\right]\, dx, \quad a, b \in \mathbb{R}^3,$$

constitute the mass, the center of mass and the inertia tensor of the i-th body respectively. The relations (1.3.22) and (1.3.23) determine the translational velocity V^i and the rotational velocity w^i of the i-th rigid body. Consequently, the overall velocity of the i-th body is given as

$$u(t,x) = u^{s^i}(t,x) := V^i(t) + w^i(t) \times \left(x - X^i(t)\right) \qquad \text{for } t \in [0,T],\ x \in S^i(t).$$

Finally, as its counterpart (1.3.14) in the incompressible model, the relation (1.3.28) constitutes the classical no-slip boundary and interface condition.

1.3.3 Evolution of a magnetoelastic material

In Chapter 5 we study the evolution of a (solid) magnetoelastic material. The model we consider goes back to the model introduced in [48]. We assume the reference configuration of the material to be a bounded domain $\Omega_0 \subset \mathbb{R}^3$, the boundary of which is divided into a free part N and a part P with a prescribed deformation,

$$N \subset \partial\Omega_0, \qquad P := \partial\Omega_0 \backslash N.$$

The deformation of the material is characterized by a mapping $\eta : (0,\infty) \times \Omega_0 \to \mathbb{R}^3$, the current configuration $\Omega(t)$ of the material at any time $t \in (0,\infty)$ is expressed as

$$\Omega(t) := \eta\left(t, \Omega_0\right)$$

and we define the time-space domain

$$Q := \left\{(t,x) \in (0,\infty) \times \mathbb{R}^3 :\ x \in \Omega(t)\right\}.$$

Further, for each fixed $t \in (0,\infty)$ the mapping $\eta(t,\cdot) : \Omega_0 \to \mathbb{R}^3$ is assumed to be injective and by $\eta^{-1}(t,\cdot) : \Omega(t) \to \Omega_0$ we denote its inverse function. The magnetization of the material is described by a function $M : Q \to \mathbb{R}^3$. It can also be expressed in the reference configuration by the mapping $\tilde{M} : (0,\infty) \times \Omega_0 \to \mathbb{R}^3$, related to the magnetization M via the formula

$$M(t,x) := M_\eta\left[\tilde{M}\right](t,x) := \frac{1}{\det\left(\left[\nabla_X \eta(t)\right]\left(\eta^{-1}(t,x)\right)\right)}\tilde{M}\left(\eta^{-1}(t,x)\right), \qquad (t,x) \in Q,$$

where ∇_X denotes the gradient with respect to the Lagrangian variable $X \in \Omega_0$. The evolution of the deformation and the magnetization is described by the system

$$\operatorname{div}\sigma + \rho f(\eta) + \mu\left[\left(\nabla_X\left(H_{\text{ext}}(\eta)\right)\left(\nabla_X \eta\right)^{-1}\right)^T \tilde{M}\right] = 0 \qquad \text{in } (0,\infty) \times \Omega_0, \quad (1.3.30)$$

$$\tilde{\Psi}_M\left(\left[\nabla_X \eta\right]\left(\eta^{-1}\right), \det\left(\left[\nabla_X \eta\right](\eta)^{-1}\right) M\right) - \mu H\left[\tilde{M}, \eta\right] - 2A\Delta M$$
$$+ \frac{1}{\beta^2}\left(|M|^2 - 1\right) M + \partial_t M + (v \cdot \nabla) M + (\nabla \cdot v) M - \mu H_{\text{ext}} = 0 \qquad \text{in } Q, \qquad (1.3.31)$$

supplemented by the boundary condition

$$\eta(t) = \gamma \qquad \text{on } P \tag{1.3.32}$$

for any $t \in [0, \infty)$. In this system the relation (1.3.30) represents the balance of momentum or the equation of motion (cf. [7, Section 1.1], [48, Section 2.6, Section 2.7]) and the relation (1.3.31) constitutes the magnetic (microscopic) force balance (cf. [48, Section 2.8]) of the material. The equation of motion (1.3.30) is formulated, as it is usual in elasticity theory, in Lagrangian coordinates $X \in \Omega_0$. It contains the density $\rho > 0$ and the magnetic permeability $\mu > 0$ of the material, as well as an external forcing term $f : (0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ and an external magnetic field $H_{\text{ext}} : (0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3$, both of which are defined in Eulerian coordinates. It further contains the first Piola-Kirchhoff stress tensor σ of the material, determined by the relation

$$\operatorname{div} \sigma = \tilde{E}_\eta\left(\eta, \tilde{M}\right) + \tilde{R}_{\partial_t \eta}\left(\eta, \partial_t \eta, \partial_t \tilde{M}\right) \tag{1.3.33}$$

where $\tilde{E}$ denotes the energy potential

$$\tilde{E}\left(\eta, \tilde{M}\right) := \begin{cases} \tilde{E}_{\text{el}}(\eta) + \tilde{E}_{\text{mag}}\left(\eta, \tilde{M}\right) & \text{if } \det(\nabla_X \eta) > 0 \text{ a.e. in } \Omega_0 \\ +\infty & \text{otherwise} \end{cases} \tag{1.3.34}$$

composed of the elastic energy $\tilde{E}_{\text{el}}$ and the micromagnetic energy $\tilde{E}_{\text{mag}}$,

$$\tilde{E}_{\text{el}}(\eta) := \int_{\Omega_0} W(\nabla_X \eta) + \frac{1}{(\det(\nabla_X \eta))^a} + \frac{1}{q}\left|\nabla_X^2 \eta\right|^q \, dX, \tag{1.3.35}$$

$$\begin{aligned} \tilde{E}_{\text{mag}}\left(\eta, \tilde{M}\right) := \int_{\Omega_0} & \tilde{\Psi}\left(\nabla_X \eta, \tilde{M}\right) - \frac{\mu}{2} \tilde{M} \cdot H\left[\tilde{M}, \eta\right](\eta) \\ & + A\left|\nabla_X\left(\frac{1}{\det(\nabla_X \eta)} \tilde{M}\right)(\nabla_X \eta)^{-1}\right|^2 \det(\nabla_X \eta) \\ & + \frac{1}{4\beta^2}\left(\left|\frac{1}{\det(\nabla_X \eta)} \tilde{M}\right|^2 - 1\right)^2 \det(\nabla_X \eta)\, dX, \end{aligned} \tag{1.3.36}$$

$\tilde{R}$ denotes the dissipation potential

$$\tilde{R}\left(\eta, \partial_t \eta, \partial_t \tilde{M}\right) := \int_{\Omega_0} \nu\left|\nabla_X \partial_t \eta\,(\nabla_X \eta)^{-1}\right|^2 \det(\nabla_X \eta) + \frac{1}{2}\left|\frac{1}{\det(\nabla_X \eta)} \partial_t \tilde{M}\right|^2 \det(\nabla_X \eta)\, dX, \tag{1.3.37}$$

and $\tilde{E}_\eta$ and $\tilde{R}_{\partial_t \eta}$ denote the Fréchet derivatives of $\tilde{E}$ and $\tilde{R}$ with respect to the first and the second argument respectively. Materials for which the stress tensor is characterized via an energy potential and a dissipation potential as in the formula (1.3.33) are known as generalized standard materials, cf. [67, 81, 93]. We point out that technically, since the Fréchet derivative of a functional is defined as a bounded linear operator on the domain of the functional, the form (1.3.30) of the equation of motion constitutes a mix between a classical and a weak formulation. The precise form of the Fréchet derivatives $\tilde{E}_\eta$ and $\tilde{R}_{\partial_t \eta}$ is written out explicitly during our presentation of the weak formulation of the model (1.3.30)–(1.3.32) in Section 5.1.2 below, cf. the formula 5.1.21. The condition $\tilde{E}(\eta, \tilde{M}) = +\infty$ if the determinant of the deformation gradient takes non-positive values guarantees that the deformation is orientation preserving. In the elastic energy $\tilde{E}_{\text{el}}$ the quantity $W : \mathbb{R}^{3\times 3} \to \mathbb{R}_0^+$ denotes the elastic energy density. In the expression

$$\frac{1}{(\det(\nabla_X \eta))^a} + \frac{1}{q}\left|\nabla_X^2 \eta\right|^q, \tag{1.3.38}$$

in which $q > 3$ and $a > \frac{3q}{q-3}$ (cf. [7]), the first term bears both a physical and a mathematical meaning. It causes the determinant of the deformation gradient to be bounded away from zero, cf. Lemma A.7.1 in the appendix, and so, physically speaking, it prevents the material from being compressed infinitely.

Its mathematical meaning becomes clear in combination with the second term in (1.3.38). Indeed, the mostly mathematical significance of the latter term consists of guaranteeing C^1-regularity of the deformation. This in combination with the bound of the determinant of the deformation gradient away from zero allows for unproblematic transformations between the reference configuration and the current configuration. As a side note, we remark that the first term in (1.3.38) further causes the energy $\tilde{E}$ to be non-convex with respect to the deformation. In the micromagnetic energy $\tilde{E}_{\mathrm{mag}}$ the quantity $\tilde{\Psi} : \mathbb{R}^{3\times3} \times \mathbb{R}^3 \to \mathbb{R}_0^+$ represents the anisotropy energy density, cf. [32, Section 2.1]. The stray field $H[\tilde{M}, \eta] : (0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3$, defined in the current configuration, is given as $H[\tilde{M}, \eta] := -\nabla\phi$, where ∇ denotes the gradient in the current configuration and $\phi = \phi[\tilde{M}, \eta]$ denotes the solution to a Poisson equation with the right-hand side $\operatorname{div} M = \operatorname{div} M_\eta[\tilde{M}]$, cf. [52, Section 1]. More precisely, for each fixed $t \in (0, \infty)$ the function $\phi(t, \cdot) : \mathbb{R}^3 \to \mathbb{R}$ denotes the solution to the problem

$$\Delta\phi(t,\cdot) = \operatorname{div} M(t,\cdot) \quad \text{in } \mathbb{R}^3, \qquad \begin{cases} \left(\phi^{\mathrm{int}}(t,\cdot) - \phi^{\mathrm{ext}}(t,\cdot)\right) = 0 & \text{in } \partial\Omega(t), \\ \left(\nabla\phi^{\mathrm{int}}(t,\cdot) - \nabla\phi^{\mathrm{ext}}(t,\cdot)\right) \cdot \mathrm{n} = -M \cdot \mathrm{n} & \text{in } \partial\Omega(t), \end{cases} \tag{1.3.39}$$

where $M(t, \cdot)$ has been extended by 0 outside of $\Omega(t)$, $\phi^{\mathrm{int}}(t, \cdot)$ and $\phi^{\mathrm{ext}}(t, \cdot)$ denote the restriction of $\phi(t, \cdot)$ to the interior and the exterior of $\Omega(t)$ respectively and n denotes the outer unit normal vector on $\partial\Omega(t)$. The expression

$$A \left| \nabla_X \left(\frac{1}{\det(\nabla_X \eta)} \tilde{M} \right) (\nabla_X \eta)^{-1} \right|^2 \det(\nabla_X \eta),$$

in which $A > 0$ is called the exchange stiffness constant, represents the exchange energy. The magnetization of a ferromagnet has the tendency to align in a constant direction, this expression penalizes the deviation from such a behavior, cf. [72, Section 3.2.2], [84, §39]. Another penalization term in the micromagnetic energy is given by

$$\frac{1}{4\beta^2} \left(\left| \frac{1}{\det(\nabla_X \eta)} \tilde{M} \right|^2 - 1 \right)^2 \det(\nabla_X \eta), \tag{1.3.40}$$

cf. for example [23]. For a small value of the constant $\beta > 0$ this term penalizes the magnitude of the magnetization (in the current configuration) taking values far away from one. In general, the magnetization of a ferromagnetic material is considered to be constantly equal in magnitude to the saturation magnetization (cf. [32], [39, Section 9.2]), which for simplicity can be set equal to one by a scaling argument. We point out that the quantity (1.3.40) further implies non-convexity of the energy $\tilde{E}$ also with respect to the magnetization. In the dissipation potential $\tilde{R}$ the first quantity models the viscosity of the material. The second quantity, being independent of $\partial_t\eta$, has no influence on the equation of motion (1.3.30) and thus its inclusion into the dissipation potential does not change the overall system (1.3.30)–(1.3.32). Instead it is included for the reason that it allows us to express the magnetic force balance (1.3.31) alternatively via an energy and a dissipation potential similarly to the equation of motion (1.3.30): Indeed, turning to the equation (1.3.31), we first notice that it is formulated, as it is common for the description of magnetic effects, in Eulerian coordinates $x \in \Omega(t)$, $t \in (0, \infty)$. The quantity $\tilde{\Psi}_M$ in this equation denotes the derivative of $\tilde{\Psi}$ with respect to the second argument and the velocity field v is related to the deformation η via the formula

$$v(t, x) = [\partial_t \eta(t)] \left(\eta^{-1}(t, x) \right), \qquad (t, x) \in Q.$$

Now, in order to express the equation (1.3.31) through an energy and a dissipation potential we introduce versions of the corresponding potentials $\tilde{E}$ and $\tilde{R}$ formulated in the current configuration. More precisely, we define the energy potential in the current configuration

$$E(\eta, M) := \begin{cases} E_{\mathrm{el}}(\eta) + E_{\mathrm{mag}}(\eta, M) = \tilde{E}\left(\eta, \tilde{M}\right) & \text{if } \det(\nabla_X \eta) > 0 \text{ a.e. in } \Omega_0, \\ +\infty & \text{otherwise} \end{cases}$$

with

$$E_{\mathrm{el}}(\eta) := \int_{\eta(\Omega)} \frac{1}{\det\left(\left[\nabla_X \eta\right]\left(\eta^{-1}\right)\right)} W\left(\left[\nabla_X \eta\right]\left(\eta^{-1}\right)\right) + \frac{1}{\left(\det\left(\left[\nabla_X \eta\right]\left(\eta^{-1}\right)\right)\right)^{a+1}} + \frac{1}{q \det\left(\left[\nabla_X \eta\right]\left(\eta^{-1}\right)\right)} \left|\left[\nabla_X^2 \eta\right]\left(\eta^{-1}\right)\right|^q \, dx,$$

$$E_{\mathrm{mag}}(\eta, M) := \int_{\eta(\Omega)} \frac{1}{\det\left(\left[\nabla_X \eta\right]\left(\eta^{-1}\right)\right)} \tilde{\Psi}\left(\left[\nabla_X \eta\right]\left(\eta^{-1}\right), \det\left(\left[\nabla_X \eta\right]\left(\eta^{-1}\right)\right) M\right) - \frac{\mu}{2} M \cdot H\left[\det\left(\nabla_X \eta\right) M, \eta\right] + A\left|\nabla_X M\right|^2 + \frac{1}{4\beta^2}\left(|M|^2 - 1\right)^2 \, dx,$$

and the dissipation potential in the current configuration

$$R\left(\eta, v, D_t M\right) := \int_{\eta(\Omega_0)} \nu\,|\nabla v|^2 \det\left(\nabla_X \eta\right) + \frac{1}{2}\left|D_t M\right|^2 \, dX = \tilde{R}\left(\eta, \partial_t \eta, \partial_t \tilde{M}\right), \tag{1.3.41}$$

where $D_t M$ constitutes as an extended material derivative of the magnetization in the current configuration, consisting of the transport terms from the equation (1.3.31),

$$D_t M := \partial_t M + (v \cdot \nabla) M + (\nabla \cdot v) M. \tag{1.3.42}$$

With this notation at hand the magnetic force balance (1.3.31) can be written in the form

$$E_M(\eta, M) + R_{D_t M}\left(\eta, v, D_t M\right) - \mu H_{\mathrm{ext}} = 0 \qquad \text{in } Q, \tag{1.3.43}$$

where E_M and $R_{D_t M}$ denote the Fréchet derivatives of E and R with respect to the third and the second argument respectively. The possibility to express the magnetic force balance in this form also has mathematical importance, it is in fact essential to our proof of the main result Theorem 5.1.1 of Section 5. As explained above in Section 1.2, this proof is built upon De Giorgi's minimizing movements scheme: The system (1.3.30)–(1.3.32) is discretized with respect to time and discrete (weak) solutions are constructed via minimization of a functional chosen on the basis of the energy potential $\tilde{E}$ and the dissipation potential $\tilde{R}$ in the reference configuration. Discrete versions of the equation of motion (1.3.30) and the magnetic force balance (1.3.31) are then obtained as the Euler-Lagrange equations of this functional. Finally, in the boundary condition (1.3.32), the function $\gamma : P \to \mathbb{R}^3$ denotes a given boundary deformation, cf. [7].

1.4 Results beyond the scope of the thesis

The work on this thesis has further lead to some results which go beyond the scope of the thesis itself. We mention the article [90], which is joint work of Šárka Nečasová, Justyna Ogorzały and the author of this thesis. In this article the global-in-time existence of weak solutions to the compressible Navier-Stokes equations subject to the slip boundary condition of friction type is proved. This boundary condition, which describes the property of a fluid slipping on the domain boundary if the tangential component of the stress tensor is large enough, is particularly interesting for fluid structure interaction problems. Potential goals for future research include the proof of the existence of weak solutions to fluid-rigid body interaction problems - similar to the ones in Chapter 3 and Chapter 4 of this thesis - in which a corresponding condition is imposed additionally as an interface condition between the fluid and the solid bodies.

1.5 Outline of the thesis

The thesis is organized as follows. Chapter 2 deals with the derivation of the models for the interaction between an electrically conducting fluid, insulating rigid bodies and the electromagnetic fields therein presented in Chapter 1 in the exemplary case of one rigid body and an incompressible fluid as in Section 1.3.1. In Section 2.1 we briefly summarize the incompressible Navier-Stokes equations, the balances of linear and angular momentum for the rigid body and the Maxwell system in their original forms and adjust them to the material assumptions in our specific setting. In Section 2.2 we derive the boundary and interface conditions for the electromagnetic fields from the Maxwell system. In Section 2.3 we identify several insignificant terms, which we subsequently drop from the equations in order to simplify the model. Applying a few final adjustments to the system resulting from this procedure in Section 2.4, we arrive at the desired model from Section 1.3.1.

In Chapter 3 we prove the local-in-time existence of a weak solution to the interaction problem between an electrically conducting incompressible fluid and an insulating rigid body. After introducing the weak formulation and presenting the main result in Section 3.1, we explain the idea of the proof, which consists of a three level approximation scheme, in Section 3.2. In Section 3.3 we prove the existence of solutions to the approximate system on the highest approximation level and in Sections 3.4—3.6 we pass to the limit in all of the approximation levels in order to recover a solution to the original problem.

In Chapter 4 we prove the corresponding result for the case of a compressible fluid. We further generalize the result (in a minor way) to global-in-time existence and the setting of finitely many rigid bodies. Thus the layout of the chapter strongly resembles the one of the previous chapter with the difference that five instead of only three approximation levels for the proof of the main result are required. This existence result is stated in Section 4.1 after providing the definition of weak solutions. In Section 4.2 we introduce the approximation scheme and give an explanation of the main ideas of the proof. The approximate system is solved in Section 4.3 and the desired solution to the original system is obtained by passing to the limit in all approximation levels throughout Sections 4.4–4.8.

Finally, in Chapter 5 we prove the existence of weak solutions to the model for the evolution of a magnetoelastic material presented in Section 1.3.3. Again, we first give a definition of weak solutions and state the main result, see Section 5.1. Subsequently we set up an approximate system, which in this case consists of only one level, and explain the main ideas behind it in Section 5.2. We solve this approximate system in Section 5.3 and pass to the limit in Section 5.4, concluding the proof of the existence of weak solutions to the original problem.

Finally, in Chapter 6 we provide a summary of the thesis together with an outlook on potential future research topics and in the Appendix A we collect miscellaneous auxiliary results finding use throughout the thesis.

Chapter 2

Modeling of the fluid-rigid body interaction problem in an electrically conducting fluid

In this chapter we provide a mathematical derivation of the two fluid-rigid body interaction models already outlined in Section 1.3.1 and Section 1.3.2, respectively. The first model describes the interplay between an electrically conducting viscous non-homogeneous incompressible fluid, an insulating rigid body and the electromagnetic fields within both materials. It is analyzed mathematically in Chapter 3, cf. also [8]. The second one models the same scenario for the setting of a compressible fluid and multiple rigid bodies and is analyzed in Chapter 4, see also [105]. The individual partial differential equations in these models are well-known. In particular, there exists plenty of literature on magnetohydrodynamics (i.e. the coupling between fluids and electromagnetic fields), cf. for example [21, 28, 83], which lies in the center of interest of the present chapter. We thus emphasize that the findings of this chapter do not constitute a new result. Instead, our aim is to provide deeper mathematical insight in the derivation than we were able to find in the literature.
In our examination here we focus on the case of an incompressible electrically conducting fluid interacting with one insulating rigid body, i.e. the model from Section 1.3.1. The derivation of the model from Section 1.3.2, however, can be achieved by the exact same procedure.
We derive the model from Section 1.3.1 from a system composed of the Maxwell equations, the incompressible Navier-Stokes equations and the balances of linear and angular momentum of a rigid body in their most universal forms. This system models the interaction between a fluid, a rigid body and the electromagnetic fields inside of both materials in full generality. After a first adjustment of the system to the properties of the materials we consider in our specific setting - in particular the non-conductivity of the solid domain - our procedure consists of two main steps: Firstly, we derive boundary and interface conditions for the electromagnetic fields from the Maxwell equations. Secondly, we simplify the system by identifying several insignificant terms and dropping them from the equations. This step, which is achieved via a nondimensionalization, in particular constitutes the classical magnetohydrodynamic approximation in the fluid part of the domain. The final system we achieve through this approach is a slightly more general version of the model from Section 1.3.1. The latter system is then obtained after a few more (mathematical) adjustments discussed at the end of this chapter.

2.1 General Model

We study an insulating rigid body moving through an electrically conducting viscous non-homogeneous incompressible fluid. Additionally, we assume the fluid to be a linear magnetic material and a linear dielectric, cf. the relations (2.1.26) below. The latter conditions are in particular satisfied for diamagnetic linear dielectrics such as for example blood, cf. [50, 96]. We also assume the fluid to be surrounded by a (rigid) perfect conductor. In the present section we showcase the mathematical model of this setup in full generality. Let $T > 0$ and let $\Omega \subset \mathbb{R}^3$ be a bounded domain. At any time

$t \in [0,T]$ we denote by $S(t) \subset \Omega$ the domain occupied by the body. The remainder $F(t) := \Omega \backslash \overline{S(t)}$ of the domain is filled with the fluid; the perfect conductor is located in the time-independent exterior domain $\mathbb{R}^3 \backslash \overline{\Omega}$. Moreover, by $Q := (0,T) \times \Omega$ we denote the time-space domain, which we split into a solid part Q^s and its fluid counterpart Q^f,

$$Q^s := \{(t,x) \in Q : \; x \in S(t)\}, \qquad Q^f := \{(t,x) \in Q : \; x \in F(t)\}.$$

The exterior of Q in $(0,T) \times \mathbb{R}^3$ is denoted by

$$Q^{\text{ext}} := \left((0,T) \times \mathbb{R}^3\right) \backslash \overline{Q}.$$

We label the restriction of any function on $(0,T) \times \mathbb{R}^3$ to Q^f, Q^s and Q^{ext} by the superscripts f, s and ext, respectively. The interaction between the fluid, the solid and the electromagnetic fields trespassing and surrounding these materials is described by the mass density $\rho : Q \to \mathbb{R}$, the velocity field $u : Q \to \mathbb{R}^3$, the pressure $p : Q^f \to \mathbb{R}$, the magnetic induction $B : (0,T) \times \mathbb{R}^3 \to \mathbb{R}^3$, the magnetic field $H : (0,T) \times \mathbb{R}^3 \to \mathbb{R}^3$, the electric field $E : (0,T) \times \mathbb{R}^3 \to \mathbb{R}^3$, the electric induction $D : (0,T) \times \mathbb{R}^3 \to \mathbb{R}^3$, the electric current density $j : (0,T) \times \mathbb{R}^3 \to \mathbb{R}^3$, the density of electric charges $\rho_c : (0,T) \times \mathbb{R}^3 \to \mathbb{R}$, the magnetization $M : (0,T) \times \mathbb{R}^3 \to \mathbb{R}^3$ and the polarization $P : (0,T) \times \mathbb{R}^3 \to \mathbb{R}^3$. In full generality, the evolution of these quantities is modeled by the following system of partial differential equations

$$\begin{aligned}
\operatorname{curl} H &= \partial_t D + j + J && \text{in } Q^f,\ Q^s \text{ and } Q^{\text{ext}}, && (2.1.1)\\
\partial_t B + \operatorname{curl} E &= 0 && \text{in } Q^f,\ Q^s \text{ and } Q^{\text{ext}}, && (2.1.2)\\
\operatorname{div} D &= \rho_c && \text{in } Q^f,\ Q^s \text{ and } Q^{\text{ext}}, && (2.1.3)\\
\operatorname{div} B &= 0 && \text{in } Q^f,\ Q^s \text{ and } Q^{\text{ext}}, && (2.1.4)\\
\operatorname{div} u &= 0, \quad \partial_t \rho + u \cdot \nabla \rho = 0 && \text{in } Q^f, && (2.1.5)\\
\partial_t (\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p &= \operatorname{div} \mathbb{T} + \rho g + \rho_c E + (j + J) \times B && \text{in } Q^f, && (2.1.6)\\
m \frac{d}{dt} V(t) &= \frac{d}{dt} \int_{S(t)} \rho u \; dx = \int_{\partial S(t)} [\mathbb{T} - p \text{ id}] \mathrm{n} \; dA + \int_{S(t)} \rho g \; dx, && && (2.1.7)\\
\frac{d}{dt} \left(\mathbb{J}(t) w(t)\right) &= \frac{d}{dt} \int_{S(t)} \rho \, (x - X) \times u \; dx \\
&= \int_{\partial S(t)} (x - X) \times [\mathbb{T} - p \text{ id}] \mathrm{n} \; dA + \int_{S(t)} \rho \, (x - X) \times g \; dx, && && (2.1.8)
\end{aligned}$$

complemented by the relations

$$j = \begin{cases} \sigma(E + u \times B) & \text{in } Q^f \text{ and } Q^s, \\ \sigma E & \text{in } Q^{\text{ext}}, \end{cases} \qquad \sigma = \begin{cases} \sigma^f > 0 & \text{in } Q^f, \\ \sigma^s := 0 & \text{in } Q^s, \\ \sigma^{\text{ext}} := +\infty & \text{in } Q^{\text{ext}}, \end{cases} \tag{2.1.9}$$

$$H = \frac{1}{\mu_0} B - M \quad \text{in } Q^f,\ Q^s \text{ and } Q^{\text{ext}}, \qquad D = \epsilon_0 E + P \quad \text{in } Q^f,\ Q^s \text{ and } Q^{\text{ext}} \tag{2.1.10}$$

and the boundary and interface conditions

$$u(t) = 0 \quad \text{on } \partial\Omega, \qquad u^f(t) - u^s(t) = 0 \quad \text{on } \partial S(t). \tag{2.1.11}$$

In the electromagnetic part (2.1.1)–(2.1.4), (2.1.9), (2.1.10) of this model Ampère's law (2.1.1), the Maxwell-Faraday equation (2.1.2), Gauss's law (2.1.3) and Gauss's law for magnetism (2.1.4) constitute the Maxwell system in its general form. The quantity $J : (0,T) \times \mathbb{R}^3 \to \mathbb{R}^3$ in Ampère's law (2.1.1) denotes a given external force onto which we impose the assumptions

$$\operatorname{div} J = 0 \quad \text{in } Q^f, \qquad J = 0 \quad \text{in } Q^s \text{ and } Q^{\text{ext}}. \tag{2.1.12}$$

In Ohm's law (2.1.9) the quantity σ denotes the electrical conductivity. We point out that the identity $\sigma^s = 0$ corresponds to the fact that the solid body is insulating while the identity $\sigma_{\text{ext}} = +\infty$ reflects upon the assumption of $\mathbb{R}^3\backslash\overline{\Omega}$ being a perfect conductor. We also remark that the absence of u in the exterior domain in Ohm's law (2.1.9) is explained by the fact that this part of the domain is immovable. Further, in the constitutive relations (2.1.10) the constants $\mu_0 > 0$ and $\epsilon_0 > 0$ denote the magnetic permeability and the dielectric permittivity in a vacuum, respectively. In the mechanical part (2.1.5)–(2.1.8), (2.1.11) of the model the incompressibility constraint and continuity equation (2.1.5) together with the momentum equation (2.1.6) constitutes the Navier-Stokes system. In the momentum equation (2.1.6) the stress tensor $\mathbb{T}$ is defined by

$$\mathbb{T} = \mathbb{T}(u) := 2\nu\mathbb{D}(u), \qquad \mathbb{D}(u) := \frac{1}{2}\nabla u + \frac{1}{2}(\nabla u)^T,$$

wherein $\nu > 0$ denotes a viscosity coefficient. The function $g : Q \to \mathbb{R}^3$ denotes another given external forcing term. Further, in form of the Lorentz force $\rho_c E + (j + J) \times B$, the momentum equation contains an additional forcing term, which describes the impact of the electromagnetic fields on the fluid motion. The balance of linear momentum (2.1.7) and the balance of angular momentum (2.1.8) determine the translational velocity V and the rotational velocity w of the rigid body. In these equations the quantities

$$m := \int_{S(t)} \rho(t,x)\ dx, \qquad X(t) := \frac{1}{m}\int_{S(t)} \rho(t,x)x\ dx$$

denote the total mass and the center of mass of the body respectively. The inertia tensor $\mathbb{J}$ is defined by the relation

$$\mathbb{J}(t)a \cdot b := \int_{S(t)} \rho(t,x)\,[a \times (x - X(t))] \cdot [b \times (x - X(t))]\ dx$$

for any $a, b \in \mathbb{R}^3$. The overall velocity of the body is then given as the rigid velocity field

$$u(t,x) = u^S(t,x) := V(t) + w(t) \times (x - X(t)) \qquad \text{in } Q^S.$$

Finally, we note that, as opposed to the no-slip boundary and interface condition (2.1.11) on the velocity field, there are no conditions imposed on the behavior of the electromagnetic fields on $\partial\Omega$ and $\partial S(t)$. This is due to the fact that these conditions are contained implicitly in the Maxwell system itself. We will deduce them explicitly in Section 2.2 below.

2.1.1 Mathematical assumptions

For the mathematical calculations in the following sections we impose certain regularity assumptions on the involved functions and domains. These assumptions can be summarized as

$$\text{The domains } \Omega \text{ and } S(t) \text{ are of class } C^{2,1} \text{ at each time } t \in [0,T], \tag{2.1.13}$$

$$\rho, u, B, H, D, E \text{ and } J \text{ are twice continuously differentiable in } Q^f, Q^s \text{ and } Q^{\text{ext}} \text{ and} \tag{2.1.14}$$

$$\rho, u, B, H, D, E, J \text{ and all their derivatives are bounded.} \tag{2.1.15}$$

We mainly require these assumptions for the derivation of the interface conditions of the electromagnetic fields in Section 2.2 below. These conditions are derived from integrated versions of the Maxwell equations. The assumptions (2.1.13)–(2.1.15) are used for passing to the limit in the equations when the domain of integration shrinks to a point.

2.1.2 Adjustments to the material assumptions

The physical properties assumed for the materials in consideration lead to some immediate simplifications of certain aspects of the model (2.1.1)–(2.1.11). In the perfect conductor in the exterior domain we may assume, in accordance with the physical literature (cf. [27, Chapter 1, Part A, §4.2.4.3]), that

$$B = E = j = 0 \quad \text{in } Q^{\text{ext}}. \tag{2.1.16}$$

Indeed, in the (immovable) exterior domain we can set $u = 0$. Moreover, by definition of the electrical conductivity σ in (2.1.10), we know that $\sigma = \sigma^{\text{ext}} = +\infty$ in Q^{ext}. Thus, due to the boundedness of E assumed in (2.1.15), Ohm's law (2.1.9) implies that $E = j = 0$. The Maxwell-Faraday equation (2.1.2) then further shows that $\partial_t B = 0$. Assuming, without loss of generality, that $B = B(0) = 0$ at the initial time $t = 0$, we thus infer that $B = B(t) = 0$ at all times $t \in [0, T]$. As a consequence of the trivial relations (2.1.16) the equations for B, E and j in the exterior domain become superfluous in our system. In particular, under exploitation of the assumption (2.1.12) on J, the Maxwell system (2.1.1)–(2.1.4) in the exterior domain reduces to the equations

$$\operatorname{curl} H = \partial_t D \qquad \text{in } Q^{\text{ext}}, \tag{2.1.17}$$

$$\operatorname{div} D = \rho_c \qquad \text{in } Q^{\text{ext}}. \tag{2.1.18}$$

In the insulating solid domain the electrical conductivity $\sigma = \sigma^s$ vanishes, cf. (2.1.10). Consequently, by Ohm's law (2.1.9), it holds that $j = 0$ in Q^s. The non-conductivity of the solid also allows us to regard this region as a vacuum from the electromagnetic point of view, within which there exist no electric charges. In particular this means that the electric charge density vanishes in the solid region, i.e. it holds that

$$\rho_c = 0 \qquad \text{in } Q^s. \tag{2.1.19}$$

Further, by the assumptions (2.1.12), we know that $J = 0$ in Q^s. Under consideration of these facts, the Maxwell system (2.1.1)–(2.1.4) in the solid domain reduces to

$$\operatorname{curl} H = \partial_t D \qquad \text{in } Q^s, \tag{2.1.20}$$

$$\partial_t B + \operatorname{curl} E = 0 \qquad \text{in } Q^s, \tag{2.1.21}$$

$$\operatorname{div} D = 0 \qquad \text{in } Q^s, \tag{2.1.22}$$

$$\operatorname{div} B = 0 \qquad \text{in } Q^s. \tag{2.1.23}$$

The fact that the solid domain is considered as a vacuum from the electromagnetic point of view moreover means that the magnetization and the polarization vanish in this region,

$$M = P = 0 \qquad \text{in } Q^s. \tag{2.1.24}$$

In the fluid domain, the assumption of the fluid being a linear (magnetic) material and a linear dielectric imply a linear relation between the magnetization and the magnetic field as well as between the polarization and the electric field, respectively. Namely, it holds that

$$M = \chi_m^f H \qquad \text{in } Q^f, \qquad\qquad P = \epsilon_0 \chi_e^f E \qquad \text{in } Q^f, \tag{2.1.25}$$

where $-1 \ll \chi_m^f < 0$ and $0 < \chi_e^f$ denote the magnetic and the electric susceptibility of the fluid respectively, cf. [61, Section 4.4.1, Section 6.4.1]). As a consequence of the relations (2.1.24) and (2.1.25), the nonlinear identities (2.1.10) in the fluid and the solid domain reduce to the linear relations

$$B = \mu H \qquad \text{in } Q^f \text{ and } Q^s, \qquad\qquad D = \epsilon E \qquad \text{in } Q^f \text{ and } Q^s, \tag{2.1.26}$$

where μ and ϵ denote the magnetic permeability and the dielectric permittivity of the respective material,

$$\mu = \begin{cases} \mu^f := \mu_0 \mu_r^f > 0 & \text{in } Q^f, \\ \mu^s := \mu_0 > 0 & \text{in } Q^s, \end{cases} \qquad \epsilon = \begin{cases} \epsilon^f := \epsilon_0 \epsilon_r^f > 0 & \text{in } Q^f, \\ \epsilon^s := \epsilon_0 > 0 & \text{in } Q^s. \end{cases} \tag{2.1.27}$$

Here, the quantities $\mu_r^f := (1 + \chi_m^f)$, $\epsilon_r^f := (1 + \chi_e^f)$ represent the relative permeability and the relative permittivity of the fluid respectively. We point out that the relations $\mu = \mu^s = \mu_0$ and $\epsilon = \epsilon^s = \epsilon_0$ in the insulating solid region are reasonable as μ_0 and ϵ_0 constitute the magnetic permeability and the dielectric permittivity in vacuum.

2.2 Interface conditions for the electromagnetic fields

The goal of this section is the derivation of the interface conditions for the electromagnetic fields on $\partial\Omega$ and $\partial S(t)$, $t \in [0,T]$. We emphasize, that here we do not refer to the conditions on $\partial\Omega$ as boundary conditions. This is because we study the electromagnetic fields not only inside of Ω but also in the perfect conductor $\mathbb{R}^3\backslash\overline{\Omega}$. The full set of interface conditions for the electromagnetic fields reads

$$E^f(t) \times \mathrm{n} = 0 \quad \text{on } \partial\Omega, \qquad \left(E^f(t) - E^s(t)\right) \times \mathrm{n} = 0 \quad \text{on } \partial S(t), \tag{2.2.1}$$

$$\left(H^{\text{ext}}(t) - H^f(t)\right) \times \mathrm{n} = k(t) \quad \text{on } \partial\Omega, \qquad \left(H^f(t) - H^s(t)\right) \times \mathrm{n} = 0 \quad \text{on } \partial S(t), \tag{2.2.2}$$

$$B^f(t) \cdot \mathrm{n} = 0 \quad \text{on } \partial\Omega, \qquad \left(B^f(t) - B^s(t)\right) \cdot \mathrm{n} = 0 \quad \text{on } \partial S(t), \tag{2.2.3}$$

$$\left(D^{\text{ext}}(t) - D^f(t)\right) \cdot \mathrm{n} = \omega(t) \quad \text{on } \partial\Omega, \qquad \left(D^f(t) - D^s(t)\right) \cdot \mathrm{n} = \omega(t) \quad \text{on } \partial S(t), \tag{2.2.4}$$

for the surface current density k on $\partial\Omega$ and the surface charge density ω on $\partial\Omega$ and $\partial S(t)$. We remark that here the surface charge density ω stands for the charge per unit area, whereas the surface current density k is defined as the current per unit length. The latter definition becomes clear in the derivation of the condition (2.2.2) below, cf. the formula (2.2.17). We further remark that the fields E^{ext} and B^{ext} do not appear in the conditions on $\partial\Omega$ in (2.2.1) and (2.2.3), which is due to the fact that these fields vanish in the exterior domain, cf. (2.1.16). In the following we deduce the above conditions from the Maxwell equations. In this procedure we restrict ourselves to the conditions (2.2.1)–(2.2.3), since these cover the conditions which we presume in the models presented in Section 1.3.1 and Section 1.3.2 for the analytical work in Chapter 3 and Chapter 4. The final condition (2.2.4), however, can be derived by essentially the same arguments as the condition (2.2.3). We first derive the conditions (2.2.1) for the tangential component of E. At some arbitrary but fixed time $t \in [0,T]$ we pick an arbitrary point $x \in \partial S(t)$ and consider it the origin $x = (0,0,0)^T$ of a local coordinate system with axes θ, ζ and η, where ζ is orthogonal to $\partial S(t)$ at x while θ and η are tangential to $\partial S(t)$ at x. We point out that in this construction the axis ζ is determined uniquely (except for its orientation). The axis η may be chosen as an arbitrary axis intersecting ζ orthogonally in x and subsequently the axis θ is obtained as the unique axis intersecting both ζ and η orthogonally in x. Further, we introduce the outer unit normal vector $\mathrm{n}_x = (0,1,0)^T$ on $\partial S(t)$ at x and the unit vector $\mathrm{n}'_x = (0,0,1)^T$, normal to the $\theta\zeta$-plane. Due to the smoothness assumptions (2.1.13) on $\partial S(t)$ there exists some small $\Delta s > 0$ and a twice continuously differentiable function

$$\phi : (-\Delta s, \Delta s) \to \mathbb{R}, \qquad \phi(0) = \phi'(0) = 0,$$

such that the intersection of a small open neighborhood of $x = (0,0,0)^T$ in $\partial S(t)$ with the $\theta\zeta$-plane can be expressed as the set

$$\left\{ \gamma(\theta) := \begin{pmatrix} \theta \\ \phi(\theta) \\ 0 \end{pmatrix} : \ \theta \in (-\Delta s, \Delta s) \right\}. \tag{2.2.5}$$

Without loss of generality we assume points below the curve γ defined by (2.2.5) (i.e. points $(\theta,\zeta,0)^T \in (-\Delta s, \Delta s) \times \mathbb{R} \times \{0\}$ with $\zeta < \phi(\theta)$) to lie inside of $S(t)$, while the points above γ (i.e. points $(\theta,\zeta,0)^T \in (-\Delta s, \Delta s) \times \mathbb{R} \times \{0\}$ with $\zeta \geqslant \phi(\theta)$) are assumed to lie outside of $S(t)$. Shifting the curve γ upwards as well as downwards along the axis ζ over a small distance $\Delta l > 0$, respectively, we introduce a curved rectangle ΔF, enclosed by the four edges

$$S_1 = \left\{ \gamma_1(\zeta) := (\Delta s, \phi(\Delta s) + \zeta, 0)^T : \ \zeta \in (-\Delta l, \Delta l) \right\}, \tag{2.2.6}$$

$$S_2 = \left\{ \gamma_2(\theta) := (-\theta, \phi(-\theta) + \Delta l, 0)^T : \ \theta \in (-\Delta s, \Delta s) \right\}, \tag{2.2.7}$$

$$S_3 = \left\{ \gamma_3(\zeta) := (-\Delta s, \phi(-\Delta s) - \zeta, 0)^T : \ \zeta \in (-\Delta l, \Delta l) \right\}, \tag{2.2.8}$$

$$S_4 = \left\{ \gamma_4(\theta) := (\theta, \phi(\theta) - \Delta l, 0)^T : \ \theta \in (-\Delta s, \Delta s) \right\}, \tag{2.2.9}$$

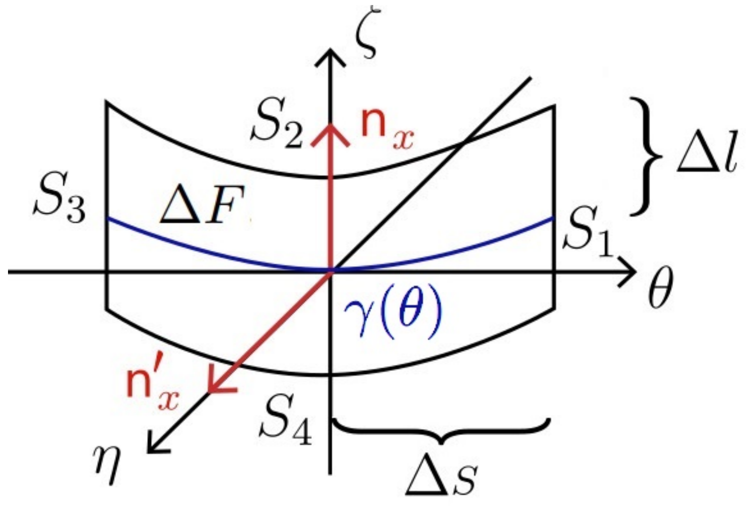

Figure 2.1: The curved rectangle ΔF.

in the $\theta\zeta$-plane, cf. Figure 2.1. We multiply the Maxwell-Faraday equation in the fluid domain (see (2.1.2)) and in the solid domain (see (2.1.21)) by the vector n'_x - which is normal to ΔF - and integrate the result over ΔF. This yields the identity

$$\begin{aligned}&\int_{S_1} E\cdot ds + \int_{S_2} E\cdot ds + \int_{S_3} E\cdot ds + \int_{S_4} E\cdot ds\\ &= \int_{\partial\Delta F} E\cdot ds = \int_{\Delta F} (\mathrm{curl}\, E)\cdot \mathrm{n}'_x \, dA = -\int_{\Delta F} \partial_t B\cdot \mathrm{n}'_x \, dA,\end{aligned} \tag{2.2.10}$$

where, for the sake of readability, we neglect the argument t in the notation of the involved functions. Next, we rewrite the integrals on the left-hand side of this equation. To this end we first exploit the fact that ϕ is twice continuously differentiable with $\phi(0) = \phi'(0) = 0$ and E is continuously differentiable in both the fluid and the solid domain, cf. (2.1.14). This allows us to use the fundamental theorem of calculus and calculate

$$\begin{aligned}\int_{S_1} E\cdot ds &= \int_{-\Delta l}^{\Delta l} E\left(\begin{pmatrix}\Delta s\\ \phi(\Delta s)+\zeta\\ 0\end{pmatrix}\right)\cdot\begin{pmatrix}0\\1\\0\end{pmatrix} d\zeta\\ &= \int_{-\Delta l}^{\Delta l} E\left(\begin{pmatrix}0\\ \zeta\\ 0\end{pmatrix}\right)\cdot\begin{pmatrix}0\\1\\0\end{pmatrix} + \int_0^{\Delta s}\frac{d}{d\xi}E\left(\begin{pmatrix}\xi\\ \phi(\xi)+\zeta\\ 0\end{pmatrix}\right)\cdot\begin{pmatrix}0\\1\\0\end{pmatrix} d\xi d\zeta,\end{aligned} \tag{2.2.11}$$

$$\begin{aligned}\int_{S_2} E\cdot ds &= \int_{-\Delta s}^{\Delta s} E\left(\begin{pmatrix}-\theta\\ \phi(-\theta)+\Delta l\\ 0\end{pmatrix}\right)\cdot\begin{pmatrix}-1\\ -\phi'(-\theta)\\ 0\end{pmatrix} d\theta\\ &= \int_{-\Delta s}^{\Delta s} E\left(\begin{pmatrix}0\\ \Delta l\\ 0\end{pmatrix}\right)\cdot\begin{pmatrix}-1\\0\\0\end{pmatrix} + \int_0^{\theta}\frac{d}{d\xi}\left[E\left(\begin{pmatrix}-\xi\\ \phi(-\xi)+\Delta l\\ 0\end{pmatrix}\right)\cdot\begin{pmatrix}-1\\ -\phi'(-\xi)\\ 0\end{pmatrix}\right] d\xi d\theta,\end{aligned} \tag{2.2.12}$$

$$\int_{S_3} E\cdot ds = \int_{-\Delta l}^{\Delta l} E\left(\begin{pmatrix}0\\ -\zeta\\ 0\end{pmatrix}\right)\cdot\begin{pmatrix}0\\-1\\0\end{pmatrix} + \int_0^{\Delta s}\frac{d}{d\xi}E\left(\begin{pmatrix}-\xi\\ \phi(-\xi)-\zeta\\ 0\end{pmatrix}\right)\cdot\begin{pmatrix}0\\-1\\0\end{pmatrix} d\xi d\zeta, \tag{2.2.13}$$

$$\int_{S_4} E\cdot ds = \int_{-\Delta s}^{\Delta s} E\left(\begin{pmatrix}0\\ -\Delta l\\ 0\end{pmatrix}\right)\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} + \int_0^{\theta}\frac{d}{d\xi}\left[E\left(\begin{pmatrix}\xi\\ \phi(\xi)-\Delta l\\ 0\end{pmatrix}\right)\cdot\begin{pmatrix}1\\ \phi'(\xi)\\ 0\end{pmatrix}\right] d\xi d\theta. \tag{2.2.14}$$

We remark that for the first terms on the right-hand sides of the equations (2.2.11) and (2.2.13), respectively, it holds that

$$\int_{-\Delta l}^{\Delta l} E\left(\begin{pmatrix}0\\ \zeta\\ 0\end{pmatrix}\right)\cdot\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}\,d\zeta + \int_{-\Delta l}^{\Delta l} E\left(\begin{pmatrix}0\\ -\zeta\\ 0\end{pmatrix}\right)\cdot\begin{pmatrix}0\\ -1\\ 0\end{pmatrix}\,d\zeta = 0. \tag{2.2.15}$$

With the identities (2.2.11)–(2.2.15) at hand we rearrange the equation (2.2.10). From the boundedness of E, B and ϕ as well as their derivatives (cf. (2.1.15)) we then infer that

$$\left|\int_{-\Delta s}^{\Delta s} E\left(\begin{pmatrix}0\\ \Delta l\\ 0\end{pmatrix}\right)\cdot\begin{pmatrix}-1\\ 0\\ 0\end{pmatrix}\,d\theta + \int_{-\Delta s}^{\Delta s} E\left(\begin{pmatrix}0\\ -\Delta l\\ 0\end{pmatrix}\right)\cdot\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}\,d\theta\right| \leqslant c\left(\Delta s\Delta l + (\Delta s)^2\right)$$

for some constant $c > 0$ independent of Δs and Δl. Dividing this inequality by $2\Delta s$ we conclude that

$$\left|\tau_x\cdot E\left(\begin{pmatrix}0\\ \Delta l\\ 0\end{pmatrix}\right) - \tau_x\cdot E\left(\begin{pmatrix}0\\ -\Delta l\\ 0\end{pmatrix}\right)\right| \leqslant c(\Delta l + \Delta s), \tag{2.2.16}$$

where $\tau_x := \mathrm{n}_x' \times \mathrm{n}_x = (-1, 0, 0)^T$. We let Δl and Δs tend to zero and infer that

$$\left(E^f\left(\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\right) - E^s\left(\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\right)\right)\cdot\left(\mathrm{n}_x'\times\mathrm{n}_x\right) = \left(E^f\left(\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\right) - E^s\left(\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\right)\right)\cdot\tau_x = 0.$$

Due to the arbitrary choice of the axis η and hence the vector n_x' as well as the arbitrary choice of the origin $x = (0, 0, 0)^T \in \partial S(t)$ of the local coordinate system we infer the second equation for E in (2.2.1). The first identity for E in (2.2.1) follows by the same arguments and under exploitation of the fact that $E = 0$ in Q^{ext} since the exterior domain is assumed to be a perfect conductor, cf. (2.1.16).
In order to deduce the corresponding conditions (2.2.2) for the tangential component of H we again fix an arbitrary point $x \in \partial S(t)$. We consider x as the origin $x = (0, 0, 0)^T$ of a local coordinate system and as the center of a curved rectangle ΔF, determined by its edges (2.2.6)–(2.2.9), with the same notation as in the derivation of the condition (2.2.1) on E. After multiplying Ampère's law in the fluid domain (see (2.1.1)) and in the solid domain (see (2.1.20)) by n_x', we integrate the result over ΔF. This leads to the identity

$$\int_{S_1} H\cdot ds + \int_{S_2} H\cdot ds + \int_{S_3} H\cdot ds + \int_{S_4} H\cdot ds = \int_{\Delta F} (\mathrm{curl}\, H)\cdot \mathrm{n}_x'\; dA = \int_{\Delta F} (\partial_t D + j + J)\cdot\mathrm{n}_x'\; dA.$$

Exactly as in the derivation of the estimate (2.2.16), we exploit the regularity and boundedness assumptions (2.1.14), (2.1.15) on H, D and J and the C^2-regularity of ϕ to deduce from this equation that

$$\left|\tau_x\cdot H\left(\begin{pmatrix}0\\ \Delta l\\ 0\end{pmatrix}\right) - \tau_x\cdot H\left(\begin{pmatrix}0\\ -\Delta l\\ 0\end{pmatrix}\right) - \frac{1}{2\Delta s}K(\Delta F)\right| \leqslant c(\Delta l + \Delta s),$$

where $\tau_x = \mathrm{n}_x' \times \mathrm{n}_x = (-1, 0, 0)^T$, the constant $c > 0$ is independent of Δl and Δs and

$$K(\Delta F) := \int_{\Delta F} j\cdot\mathrm{n}_x'\; dA$$

denotes the current in the curved rectangle ΔF. We let first Δl and subsequently Δs tend to zero. Then $\frac{1}{2\Delta s}K(\Delta F)$ converges to the surface current density (i.e. the current per unit length)

$$k = k_x := \lim_{\Delta s\to 0}\lim_{\Delta l\to 0}\frac{1}{2\Delta s}K(\Delta F) \tag{2.2.17}$$

in the origin $x = (0,0,0)^T$ of the local coordinate system. Since $x \in \partial S(t)$ as well as the vector n'_x in $\tau_x = \mathrm{n}'_x \times \mathrm{n}_x$ can be chosen arbitrarily, we may drop the index x and infer the interface condition

$$\left(H^f - H^S\right) \times \mathrm{n} = k \quad \text{on } \partial S(t). \tag{2.2.18}$$

Here, we assume that $k = 0$ on $\partial S(t)$, i.e. the right-hand side of the identity (2.2.18) is zero and consequently we infer the second equation for H in (2.2.2). This is in accordance with the literature, according to which the surface current density k vanishes on the surfaces of most materials. More specifically, for example in [61, Section 9.4.2], it is stated that the equality (2.2.18) with zero right-hand side holds true on the interfaces between Ohmic conductors, i.e. conductors (with finite conductivity $\sigma < \infty$) in which Ohm's law holds true. Indeed, provided that Ohm's law is also satisfied on the interface itself, the identity $k = 0$ follows directly from the definition of k in (2.2.17) and the boundedness assumptions (2.1.15) on E, u and B. However, we point out that the applicability of Ohm's law on the interface poses an additional assumption, the validity of which does not seem to be generally accepted. It remains an open problem to mathematically justify $k = 0$ in the case that Ohm's law is not satisfied at the interface.

In general, k does not necessarily need to vanish. From the mathematical point of view, k may take values different from zero if j becomes infinite on the considered surface, which can be expressed mathematically via the use of a Dirac delta distribution. Also according to the physical literature there exist materials, such as superconductors, on the surfaces of which k takes non-zero values, cf. [60, Chapter 11]. In the condition for H on $\partial\Omega$ in (2.2.2), which is derived by the same arguments as the identity (2.2.18), we thus refrain from the restrictive assumption $k = 0$. Indeed, while on $\partial S(t)$ we make this assumption in order for the interface condition to match the one used in the models 1.3.1 and 1.3.2 for the analytical work in Chapter 3 and Chapter 4, on $\partial\Omega$ we regard it more appropriate to allow k to take non-zero values.

Our next goal is the deduction of the conditions (2.2.3) for the normal component of B. Again we choose an arbitrary point $x \in \partial S(t)$ as the origin $x = (0,0,0)^T$ of a local coordinate system with axes θ, ζ and η, where ζ is orthogonal to $\partial S(t)$ at x while θ and η are tangential to $\partial S(t)$ at x. We denote by $B_r(0)$ the 2-dimensional open ball with radius $r > 0$ centered at $x = (0,0,0)^T$ in the $\theta\eta$-plane of this local coordinate system. Due to the smoothness assumptions (2.1.13) on $S(t)$ we can choose r sufficiently small and find some twice continuously differentiable function

$$\Phi : B_r(0) \to \mathbb{R}, \qquad \Phi(0,0) = \partial_i \Phi(0,0) = 0 \quad \text{for } i = 1,2,$$

where $\partial_i \Phi$ denotes the derivative of Φ with respect to the i-th variable, such that a small open neighborhood of x in $\partial S(t)$ can be written as the set

$$\left\{ \Gamma(\theta,\eta) := \begin{pmatrix} \theta \\ \Phi(\theta,\eta) \\ \eta \end{pmatrix} : \ (\theta,\eta) \in B_r(0) \right\}. \tag{2.2.19}$$

Without loss of generality we assume points below the surface Γ defined by (2.2.19) (i.e. points $(\theta,\zeta,\eta)^T$ with $(\theta,\eta) \in B_r(0)$ and $\zeta < \Phi(\theta,\eta)$) to lie inside of $S(t)$ while the points above Γ (i.e. points $(\theta,\zeta,\eta)^T$ with $(\theta,\eta) \in B_r(0)$ and $\zeta > \Phi(\theta,\eta)$) are assumed to lie outside of $S(t)$. By shifting the surface Γ upwards and downwards along the axis ζ over a small distance $\Delta l > 0$ we define a cylinder C with the curved bases

$$\Delta_{\text{top}} := \left\{ \Gamma_{\text{top}}(s,\alpha) := \begin{pmatrix} s\cos(\alpha) \\ \Phi(s\cos(\alpha), s\sin(\alpha)) + \Delta l \\ s\sin(\alpha) \end{pmatrix} : \ 0 \leqslant s \leqslant r, \ 0 \leqslant \alpha \leqslant 2\pi \right\}, \tag{2.2.20}$$

$$\Delta_{\text{bottom}} := \left\{ \Gamma_{\text{bottom}}(s,\alpha) := \begin{pmatrix} s\cos(-\alpha) \\ \Phi(s\cos(-\alpha), s\sin(-\alpha)) - \Delta l \\ s\sin(-\alpha) \end{pmatrix} : \ 0 \leqslant s \leqslant r, \ 0 \leqslant \alpha \leqslant 2\pi \right\} \tag{2.2.21}$$

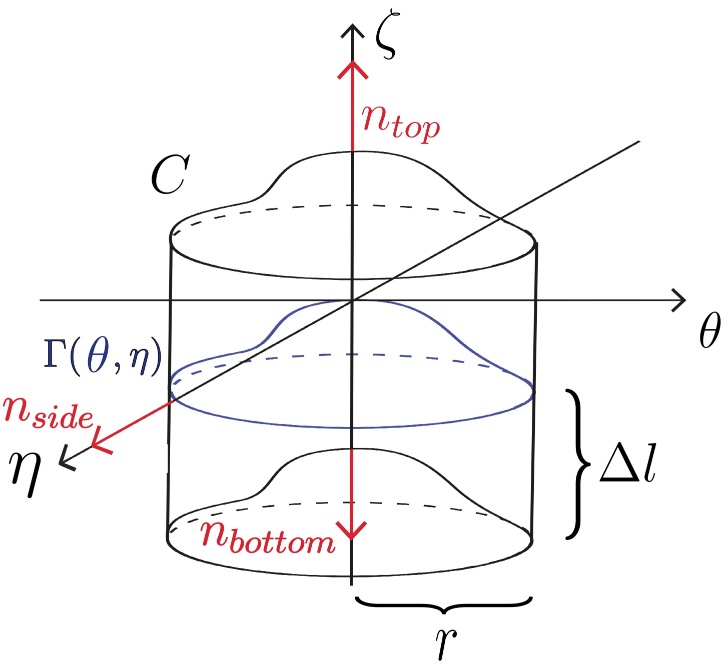

Figure 2.2: The cylinder C with curved bases.

and the lateral area

$$\Delta_{\mathrm{side}} := \left\{ \Gamma_{\mathrm{side}}(\alpha, h) := \begin{pmatrix} r\cos(\alpha) \\ \Phi(r\cos(\alpha), r\sin(\alpha)) + h \\ r\sin(\alpha) \end{pmatrix} : \ 0 \leqslant \alpha < 2\pi, \ -\Delta l \leqslant h \leqslant \Delta l \right\}, \qquad (2.2.22)$$

cf. Figure 2.2. Moreover, we denote by n_C the outer unit normal vector on ∂C, which we split into the outer unit normal vector $\mathrm{n}_{\mathrm{top}}$ on Δ_{top}, the outer unit normal vector $\mathrm{n}_{\mathrm{bottom}}$ on Δ_{bottom} and the outer unit normal vector $\mathrm{n}_{\mathrm{side}}$ on Δ_{side}.

We integrate Gauss's law for magnetism in the fluid domain (cf. (2.1.4)) and in the solid domain (cf. (2.1.23)) over the cylinder C. This leads to the identity

$$\int_{\Delta_{\mathrm{top}}} B\cdot \mathrm{n}_{\mathrm{top}}\, dA + \int_{\Delta_{\mathrm{bottom}}} B\cdot \mathrm{n}_{\mathrm{bottom}}\, dA + \int_{\Delta_{\mathrm{side}}} B\cdot \mathrm{n}_{\mathrm{side}}\, dA = \int_{\partial C} B\cdot \mathrm{n}_C\, dA = \int_C \mathrm{div} B\, dx = 0. \quad (2.2.23)$$

Next, we reformulate the integrals on the left-hand side of this identity under usage of the parametrizations (2.2.20)–(2.2.22). We start by calculating, from the paramtetrization (2.2.20),

$$\left| \frac{\partial \Gamma_{\mathrm{top}}(s,\alpha)}{\partial s} \times \frac{\partial \Gamma_{\mathrm{top}}(s,\alpha)}{\partial \alpha} \right| \mathrm{n}_{\mathrm{top}}(s,\alpha) = s \begin{pmatrix} \partial_1 \Phi(s\cos(\alpha), s\sin(\alpha)) \\ -1 \\ \partial_2 \Phi(s\cos(\alpha), s\sin(\alpha)) \end{pmatrix}.$$

This in combination with the parametrization (2.2.20), the fundamental theorem of calculus - which may be applied since B is continuously differentiable in both the fluid and the solid domain and Φ is twice continuously differentiable - and the identities $\Phi(0,0) = \partial_1\Phi(0,0) = \partial_2\Phi(0,0) = 0$ allows us to calculate the first integral on the left-hand side of the identity (2.2.23) as

$$\begin{aligned}\int_{\Delta_{\mathrm{top}}} B\cdot \mathrm{n}_{\mathrm{top}}\, dA =& \int_0^{2\pi}\int_0^r sB\left(\begin{pmatrix} s\cos(\alpha)\\ \Phi(s\cos(\alpha), s\sin(\alpha))+\Delta l\\ s\sin(\alpha)\end{pmatrix}\right)\cdot\begin{pmatrix}\partial_1\Phi(s\cos(\alpha), s\sin(\alpha))\\ -1\\ \partial_2\Phi(s\cos(\alpha), s\sin(\alpha))\end{pmatrix} dsd\alpha\\ =&\pi r^2 B\left(\begin{pmatrix}0\\ \Delta l\\ 0\end{pmatrix}\right)\cdot\begin{pmatrix}0\\ -1\\ 0\end{pmatrix}\\ &+\int_0^{2\pi}\int_0^r s\int_0^s \frac{d}{d\xi}\left[B\left(\begin{pmatrix}\xi\cos(\alpha)\\ \Phi(\xi\cos(\alpha),\xi\sin(\alpha))+\Delta l\\ \xi\sin(\alpha)\end{pmatrix}\right)\cdot\begin{pmatrix}\partial_1\Phi(\xi\cos(\alpha),\xi\sin(\alpha))\\ -1\\ \partial_2\Phi(\xi\cos(\alpha),\xi\sin(\alpha))\end{pmatrix}\right] d\xi ds d\alpha.\end{aligned} \tag{2.2.24}$$

Analogously, for the second integral on the left-hand side of the identity (2.2.23), we calculate, from the parametrization (2.2.21),

$$\begin{aligned}&\int_{\Delta_{\mathrm{bottom}}} B\cdot \mathrm{n}_{\mathrm{bottom}}\, dA = \pi r^2 B\left(\begin{pmatrix}0\\ -\Delta l\\ 0\end{pmatrix}\right)\cdot\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}\\ &+\int_0^{2\pi}\int_0^r s\int_0^s \frac{d}{d\xi}\left[B\left(\begin{pmatrix}\xi\cos(-\alpha)\\ \Phi(\xi\cos(-\alpha),\xi\sin(-\alpha))-\Delta l\\ \xi\sin(-\alpha)\end{pmatrix}\right)\cdot\begin{pmatrix}-\partial_1\Phi(\xi\cos(-\alpha),\xi\sin(-\alpha))\\ 1\\ -\partial_2\Phi(\xi\cos(-\alpha),\xi\sin(-\alpha))\end{pmatrix}\right] d\xi ds d\alpha.\end{aligned} \tag{2.2.25}$$

Finally, a similar calculation for the third integral on the left-hand side of (2.2.23) under exploitation of the parametrization (2.2.22) leads to

$$\begin{aligned}&\int_{\Delta_{\mathrm{side}}} B\cdot \mathrm{n}_{\mathrm{side}}\, dA\\ =&\int_{-\Delta l}^{\Delta l}\int_0^{2\pi} r\left[B\left(\begin{pmatrix}0\\ h\\ 0\end{pmatrix}\right)\cdot\begin{pmatrix}-\cos(\alpha)\\ 0\\ -\sin(\alpha)\end{pmatrix}\right.\\ &\qquad\left.+\int_0^r\frac{d}{d\xi}B\left(\begin{pmatrix}\xi\cos(\alpha)\\ \Phi(\xi\cos(\alpha),\xi\sin(\alpha))+h\\ \xi\sin(\alpha)\end{pmatrix}\right)\cdot\begin{pmatrix}-\cos(\alpha)\\ 0\\ -\sin(\alpha)\end{pmatrix}d\xi\right] d\alpha dh\\ =&\int_{-\Delta l}^{\Delta l}\int_0^{2\pi} r\int_0^r\frac{d}{d\xi}B\left(\begin{pmatrix}\xi\cos(\alpha)\\ \Phi(\xi\cos(\alpha),\xi\sin(\alpha))+h\\ \xi\sin(\alpha)\end{pmatrix}\right)\cdot\begin{pmatrix}-\cos(\alpha)\\ 0\\ -\sin(\alpha)\end{pmatrix} d\xi d\alpha dh.\end{aligned} \tag{2.2.26}$$

We use the identities (2.2.24), (2.2.25) and (2.2.26) to reformulate the terms on the left-hand side of the equation (2.2.23). Rearranging the resulting equation and using the fact that B and ϕ as well as their derivatives are bounded we estimate

$$\pi r^2\left|B\left(\begin{pmatrix}0\\ \Delta l\\ 0\end{pmatrix}\right)\cdot\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}-B\left(\begin{pmatrix}0\\ -\Delta l\\ 0\end{pmatrix}\right)\cdot\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}\right|\leqslant c\left(r^3+r^2\Delta l\right)$$

for a constant $c > 0$ independent of r and Δl. Dividing this inequality by πr^2 we arrive at the estimate

$$\left|\mathrm{n}_x\cdot\left[B\left(\begin{pmatrix}0\\ -\Delta l\\ 0\end{pmatrix}\right)-B\left(\begin{pmatrix}0\\ \Delta l\\ 0\end{pmatrix}\right)\right]\right|\leqslant c(r+\Delta l)$$

for the outer unit normal vector $\mathrm{n}_x = (0,1,0)^T$ on $\partial S(t)$ in the origin $x = (0,0,0)^T$ of the local coordinate system. We let r and Δl tend to zero. Since $x \in \partial S(t)$ was chosen arbitrarily we infer the second condition for B in (2.2.3). The first condition follows by the same arguments as well as the fact that $B = 0$ in Q^{ext}, cf. (2.1.16).

2.3 Magnetohydrodynamic approximation via nondimensionalization

In this section we carry out a nondimensionalization of various equations from the system introduced in Section 2.1 in order to single out several negligibly small terms. We will then neglect these terms from the system in order to obtain a simplified model. In particular, in the fluid domain this simplification constitutes the classical magnetohydrodynamic approximation, an alternative derivation of which can be found in [74, 75]. For any physical quantity a we denote by $\overline{a} > 0$ an associated characteristic scale. Introducing the dimensionless variables

$$t' := \frac{t}{\overline{t}}, \qquad T' := \frac{T}{\overline{t}}, \qquad x' := \frac{x}{\overline{x}} \tag{2.3.1}$$

we may then define nondimensionalized versions of our several domains,

$$\Omega' := \frac{1}{\overline{x}}\Omega, \qquad S'(t') := \frac{1}{\overline{x}}S(t'\overline{t}) = \frac{1}{\overline{x}}S(t), \qquad F'(t') := \Omega' \backslash \overline{S'(t')} = \frac{1}{\overline{x}}F(t)$$

and

$$Q' := (0,T') \times \Omega', \qquad Q^{s\prime} := \left\{(t',x') \in Q' : \ x' \in S'(t')\right\}, \qquad Q^{f\prime} := \left\{(t',x') \in Q' : \ x' \in F'(t')\right\}.$$

We remark that

$$(t,x) \in Q \Leftrightarrow (t',x') \in Q', \qquad (t,x) \in Q^S \Leftrightarrow (t',x') \in Q^{S\prime}, \qquad (t,x) \in Q^f \Leftrightarrow (t',x') \in Q^{f\prime}.$$

Moreover, for any $(t',x') \in Q'$ we introduce the dimensionless mechanical quantities

$$\rho'(t',x') := \frac{\rho(t,x)}{\overline{\rho}}, \quad u'(t',x') := \frac{u(t,x)}{\overline{u}}, \quad p'(t',x') := \frac{p(t,x)}{\overline{p}}, \quad g'(t',x') := \frac{g(t,x)}{\overline{g}}, \tag{2.3.2}$$

and for all $(t',x') \in (0,T') \times \mathbb{R}^3$ we introduce the dimensionless electromagnetic quantities

$$D'(t',x') := \frac{D(t,x)}{\overline{D}}, \quad E'(t',x') := \frac{E(t,x)}{\overline{E}}, \quad B'(t',x') := \frac{B(t,x)}{\overline{B}}, \quad H'(t',x') := \frac{H(t,x)}{\overline{H}}, \tag{2.3.3}$$

$$\rho_c'(t',x') := \frac{\rho_c(t,x)}{\overline{\rho}_c}, \quad j'(t',x') := \frac{j(t,x)}{\overline{j}}, \quad J'(t',x') := \frac{J(t,x)}{\overline{j}}. \tag{2.3.4}$$

2.3.1 The Maxwell system

In the classical magnetohydrodynamic approximation, the Maxwell system in the fluid domain is simplified by dropping the quantity $\partial_t D$ in Ampère's law. The purpose of this section is to justify this simplification. Furthermore, proceeding similarly in the solid region, we carry out the same reduction in Ampère's law in the solid domain. We start by nondimensionalizing the Maxwell-Faraday equation (2.1.2) in the fluid and the solid domain. Expressing the magnetic induction B and the electric field E through the dimensionless variables introduced in (2.3.1) and (2.3.3) and making use of the chain rule, the equation (2.1.2) becomes

$$\frac{\overline{B}}{\overline{t}}\partial_{t'}B'(t',x') + \frac{\overline{E}}{\overline{x}}\nabla_{x'} \times E'(t',x') = 0 \qquad \text{in } Q^{f\prime} \text{ and } Q^{s\prime}.$$

In this relation we assume the two terms $\partial_{t'}B'(t',x')$ and $\nabla_{x'} \times E'(t',x')$ to be equally significant, meaning that the coefficients in front of these terms have to coincide,

$$\frac{\overline{B}}{\overline{t}} = \frac{\overline{E}}{\overline{x}}. \tag{2.3.5}$$

We first use this relation to simplify Ampère's law (2.1.1) in the fluid domain, which, under exploitation of the linear relations (2.1.26), can be expressed as

$$\nabla_{x'} \times B'(t',x') = \frac{\mu_0\mu_r^f\overline{x}\epsilon_0\epsilon_r^f\overline{E}}{\overline{t}\,\overline{B}}\partial_{t'}E'(t',x') + \frac{\mu^f\overline{x}}{\overline{B}}\overline{j}j'(t',x') + \frac{\mu^f\overline{x}}{\overline{B}}\overline{j}J'(t',x') \qquad \text{in } Q^{f\prime} \tag{2.3.6}$$

by recalling that $\mu^f = \mu_0 \mu_r^f$ and $\epsilon^f = \epsilon_0 \epsilon_r^f$ for the values μ_0 and ϵ_0 of μ and ϵ in vacuum, cf. (2.1.27). In the equation (2.3.6) we assume the quantities $j'(t',x')$ and $\nabla_{x'} \times B'(t',x')$ to be equally significant, i.e.

$$\overline{B} = \mu^f \overline{x}\,\overline{j}, \tag{2.3.7}$$

and we further assume that

$$\overline{u} = \frac{\overline{x}}{\overline{t}}, \qquad \overline{u} \ll c, \qquad \mu_r^f \epsilon_r^f \approx 1, \tag{2.3.8}$$

where $c > 0$ denotes the speed of light. We point out that the latter of the assumptions in (2.3.8) is reasonable for isotropic liquid electrical conductors (i.e. for example electrolytes, molten salts and liquid metals), cf. [88, Section II.1]. The first equation in (2.3.8) together with the relation (2.3.5) implies that

$$\frac{\overline{E}}{\overline{B}} = \overline{u}. \tag{2.3.9}$$

As the magnetic permeability μ_0 and the dielectric permittivity ϵ_0 in vacuum are known to satisfy

$$c = (\mu_0 \epsilon_0)^{-\frac{1}{2}}, \tag{2.3.10}$$

we may use the relations (2.3.7)–(2.3.9) to express the equation (2.3.6) in the form

$$\nabla_{x'} \times B'(t',x') = \frac{\overline{u}^2}{c^2} \partial_{t'} E'(t',x') + j'(t',x') + J'(t',x') \qquad \text{in } Q^{f'}. \tag{2.3.11}$$

Due to the second assumption in (2.3.8) the first term on the left-hand side of this equation is negligibly small. Thus, formally, this term may be neglected, leaving us with the identity

$$\nabla_{x'} \times B'(t',x') = j'(t',x') + J'(t',x') \qquad \text{in } Q^{f'}. \tag{2.3.12}$$

Finally, transforming this relation back into a dimensional form, we obtain the desired simplified version of Ampère's law in the fluid domain

$$\nabla \times H(t,x) = j(t,x) + J(t,x) \qquad \text{in } Q^f. \tag{2.3.13}$$

For the corresponding simplification in the solid domain we recall that $\mu^s = \mu_0$ and $\epsilon^s = \epsilon_0$, cf. (2.1.27). Combining this with the conditions (2.3.5) and (2.3.8), we can treat Ampère's law (2.1.20) in the solid domain similarly as in the fluid domain (cf. (2.3.11)) and obtain

$$\nabla_{x'} \times B'(t',x') = \frac{\overline{u}^2}{c^2} \partial_{t'} E'(t',x') \qquad \text{in } Q^{s'}. \tag{2.3.14}$$

Due to the second assumption in (2.3.8) the term on the left-hand side of (2.3.14) can be neglected. Converting the resulting relation back into a dimensional form we obtain the reduced version of Ampère's law in the solid domain,

$$\nabla \times H(t,x) = 0 \qquad \text{in } Q^s. \tag{2.3.15}$$

2.3.2 The Navier-Stokes system

In the Navier-Stokes system we nondimensionalize the momentum equation (2.1.6) with the aim of simplifying the Lorentz force $\rho_c E + (j + J) \times B$ as it is common in the magnetohydrodynamic approximation. A straight forward calculation under exploitation of the relations (2.3.1), (2.3.2), (2.3.3) and (2.3.4) allows us to express the momentum equation (2.1.6) in the form

$$\begin{aligned} &\frac{\overline{\rho}\,\overline{u}}{\overline{t}} \partial_{t'} \left(\rho'(t',x') u'(t',x')\right) + \frac{\overline{\rho}\,\overline{u}^2}{\overline{x}} \nabla_{x'} \cdot \left(\rho'(t',x') u'(t',x') \otimes u'(t',x')\right) + \frac{\overline{p}}{\overline{x}} \nabla_{x'} p'(t',x') \\ =&\frac{2\nu\overline{u}}{\overline{x}^2} \nabla_{x'} \cdot \mathbb{D}_{x'} \left(u'(t',x')\right) + \overline{\rho}\,\overline{g} \rho'(t',x') g(t,x) + \overline{\rho}_c \overline{E} \rho_c'(t',x') E'(t',x') \\ &+ \overline{j}\,\overline{B} \left(j'(t',x') + J'(t',x')\right) \times B'(t',x') \qquad \text{in } Q^{f'}. \end{aligned} \tag{2.3.16}$$

We proceed by taking the divergence of Ohm's law (2.1.9), which, under exploitation of Gauss's law (2.1.3) and the linear relations (2.1.26), yields the identity

$$\frac{1}{\sigma}\nabla \cdot j(t,x) - \nabla \cdot (u(t,x) \times B(t,x)) = \frac{1}{\epsilon^f}\nabla \cdot D(t,x) = \frac{\rho_c(t,x)}{\epsilon^f} \qquad \text{in } Q^f.$$

Nondimensionalizing this relation we obtain

$$\frac{\overline{j}}{\sigma\overline{x}}\nabla_{x'} \cdot j'(t',x') - \frac{\overline{u}\overline{B}}{\overline{x}}\nabla_{x'} \cdot \left(u'(t',x') \times B'(t',x')\right) = \frac{\overline{\rho}_c}{\epsilon^f}\rho_c'(t',x') \qquad \text{in } Q^{f'}.$$

Here the first term on the left-hand side is equal to zero since $\nabla_{x'} \cdot j' = 0$ due to Ampère's law (2.3.12) and the fact that (by the assumptions on J in (2.1.12)) J' is divergence-free. We infer that

$$\frac{\overline{\rho}_c\overline{x}}{\epsilon^f\overline{u}\overline{B}}\rho_c'(t',x') = -\nabla_{x'} \cdot \left(u'(t',x') \times B'(t',x')\right) \quad \text{in } Q^{f'}.$$

In this identity we assume the quantities ρ_c' and $-\nabla_{x'} \cdot (u' \times B')$ to be equally significant, meaning that

$$\overline{\rho}_c = \frac{\epsilon^f\overline{u}\overline{B}}{\overline{x}}. \tag{2.3.17}$$

This yields the equation

$$\overline{\rho}_c\overline{E} = \frac{\epsilon^f\overline{u}\overline{B}}{\overline{x}}\overline{E} = \frac{\epsilon^f\overline{B}^2\overline{u}^2}{\overline{x}} = \frac{\epsilon_0\epsilon_r^f\mu_0\mu_r^f\overline{x}\overline{j}\ \overline{B}\overline{u}^2}{\overline{x}} \approx \frac{\overline{u}^2}{c^2}\overline{j}\ \overline{B},$$

where we used the relation (2.3.9) for the second identity, the relations (2.1.27) and (2.3.7) for the third identity and the third assumption in (2.3.8) as well as the relation (2.3.10) for the last identity. For the (nondimensionalized) Lorentz force this means that

$$\begin{aligned}&\overline{\rho}_c\ \overline{E}\rho_c'(t',x')E'(t',x') + \overline{j}\ \overline{B}\left(j'(t',x') + J'(t',x')\right) \times B'(t',x')\\ \approx&\frac{\overline{u}^2}{c^2}\overline{j}\ \overline{B}\rho_c'(t',x')E'(t',x') + \overline{j}\ \overline{B}\left(j'(t',x') + J'(t',x')\right) \times B'(t',x') \qquad \text{in } Q^{f'}.\end{aligned}$$

Due to the second assumption in (2.3.8) we see that the first term on the right-hand side of this relation is negligibly small compared to the second one. Hence, formally, we may neglect it and the nondimensionalized momentum equation (2.3.16) reduces to

$$\begin{aligned}&\frac{\overline{\rho}\ \overline{u}}{\overline{t}}\partial_{t'}\left(\rho'(t',x')u'(t',x')\right) + \frac{\overline{\rho}\ \overline{u}^2}{\overline{x}}\nabla_{x'} \cdot \left(\rho'(t',x')u'(t',x') \otimes u'(t',x')\right) + \frac{\overline{p}}{\overline{x}}\nabla_{x'}p'(t',x')\\ =&\frac{2\nu\overline{u}}{\overline{x}^2}\nabla_{x'} \cdot \mathbb{D}_{x'}\left(u'(t',x')\right) + \overline{\rho}\ \overline{g}\rho'(t',x')g(t,x) + \overline{j}\ \overline{B}\left(j'(t',x') + J'(t',x')\right) \times B'(t',x') \qquad \text{in } Q^{f'}.\end{aligned}$$

Transforming this equation back into a dimensional form and exploiting Ampère's law (2.3.13) and the linear relation (2.1.26) to rewrite the remaining part of the Lorentz force, we obtain the desired simplified momentum equation

$$\begin{aligned}&\partial_t\left(\rho(t,x)u(t,x)\right) + \nabla \cdot \left(\rho(t,x)u(t,x) \otimes u(t,x)\right) + \nabla p(t,x)\\ =&2\nu\nabla \cdot \mathbb{D}\left(u(t,x)\right) + \rho(t,x)g(t,x) + \frac{1}{\mu}\left(\nabla \times B(t,x)\right) \times B(t,x) \qquad \text{in } Q^f.\end{aligned} \tag{2.3.18}$$

2.4 Summary of the derived system

In the following we present a summary of the system derived in the previous sections. The mechanical part of this system coincides with the mechanical part (2.1.5)–(2.1.8), (2.1.11) of the original system except for the momentum equation, which was simplified in the course of the magnetohydrodynamic approximation, cf. (2.3.18).

In the electromagnetic part of the derived system, the Maxwell system in the fluid domain consists of the Maxwell-Faraday equation (2.1.2), Gauss's law (2.1.3) and Gauss's law for magnetism (2.1.4) taken from the original model, as well as the simplified version (2.3.13) of Ampère's law derived in the magnetohydrodynamic approximation. In the solid domain the Maxwell system has been adjusted to the assumption of the solid being insulating in (2.1.20)–(2.1.23). In addition, Ampère's law has been further simplified in the same way as in the fluid domain in (2.3.15), leaving us with the equations (2.1.21)–(2.1.23) and (2.3.15) in the solid domain. The Maxwell system in the exterior domain in our derived model consists, in accordance with the assumption of the exterior domain being a perfect conductor, only of Ampère's law (2.1.17) and Gauss's law (2.1.18), since the remaining equations become superfluous due to the trivial relations (2.1.16). Ohm's law in the derived system keeps its form (2.1.9) from the original model, however it can be omitted in the exterior domain because of the trivial relations (2.1.16). The constitutive relations (2.1.10) from the original model reduce according to the trivial relations (2.1.16) in the exterior domain while they take the form (2.1.26) in the fluid and the solid domain.
Finally, our derived model also includes the interface conditions (2.2.1)–(2.2.4) for the electromagnetic fields. Before we present the derived system in its complete form we recall the conditions which we obtained through our scaling assumptions we used for the magnetohydrodynamic approximation in Section 2.3. These conditions consist of the relations

$$\frac{\overline{B}}{\overline{t}} = \frac{\overline{E}}{\overline{x}}, \qquad \overline{B} = \mu^f \overline{x}\overline{j}, \qquad \overline{u} = \frac{\overline{x}}{\overline{t}}, \qquad \overline{u} \ll c, \qquad \mu_r^f \epsilon_r^f \approx 1, \qquad \overline{\rho}_c = \frac{\epsilon^f \overline{B}\overline{u}}{\overline{x}}$$

for the characteristic scales $\overline{a} > 0$ of the physical quantities a in our model, cf. (2.3.5), (2.3.7), (2.3.8) and (2.3.17).
The full system we derived under these conditions for the modeling of the motion of an insulating rigid body through an electrically conducting diamagnetic dielectric viscous non-homogeneous and incompressible fluid surrounded by a perfect conductor reads

$$\operatorname{curl} H = j + J \qquad \text{in } Q^f, \tag{2.4.1}$$

$$\operatorname{curl} H = 0 \qquad \text{in } Q^s, \tag{2.4.2}$$

$$\operatorname{curl} H = \partial_t D \qquad \text{in } Q^{\text{ext}}, \tag{2.4.3}$$

$$\partial_t B + \operatorname{curl} E = 0 \qquad \text{in } Q^f \text{ and } Q^s, \tag{2.4.4}$$

$$\operatorname{div} D = \rho_c \qquad \text{in } Q^f \text{ and } Q^{\text{ext}}, \tag{2.4.5}$$

$$\operatorname{div} D = 0 \qquad \text{in } Q^s, \tag{2.4.6}$$

$$\operatorname{div} B = 0 \qquad \text{in } Q^f \text{ and } Q^s, \tag{2.4.7}$$

$$\operatorname{div} u = 0, \quad \partial_t \rho + u \cdot \nabla \rho = 0 \qquad \text{in } Q^f, \tag{2.4.8}$$

$$\partial_t (\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \operatorname{div} \mathbb{T} + \rho g + \frac{1}{\mu} \operatorname{curl} B \times B \qquad \text{in } Q^f, \tag{2.4.9}$$

$$m \frac{d}{dt} V(t) = \frac{d}{dt} \int_{S(t)} \rho u \, dx = \int_{\partial S(t)} [\mathbb{T} - p \, \mathrm{id}] \mathrm{n} \, dA + \int_{S(t)} \rho g \, dx, \tag{2.4.10}$$

$$\begin{aligned} \frac{d}{dt} (\mathbb{J}(t) w(t)) &= \frac{d}{dt} \int_{S(t)} \rho (x - X) \times u \, dx \\ &= \int_{\partial S(t)} (x - X) \times [\mathbb{T} - p \, \mathrm{id}] \mathrm{n} \, dA + \int_{S(t)} \rho (x - X) \times g \, dx, \end{aligned} \tag{2.4.11}$$

supplemented by the trivial relations

$$B = E = j = 0 \qquad \text{in } Q^{\text{ext}}, \tag{2.4.12}$$

the relations

$$j = \sigma(E + u \times B) \quad \text{in } Q^f \text{ and } Q^s, \qquad \sigma = \begin{cases} \sigma^f > 0 & \text{in } Q^f, \\ \sigma^s = 0 & \text{in } Q^s, \end{cases} \tag{2.4.13}$$

$$B = \mu H \quad \text{in } Q^f \text{ and } Q^s, \qquad D = \epsilon E \quad \text{in } Q^f \text{ and } Q^s, \tag{2.4.14}$$

$$H = -M \quad \text{in } Q^{\text{ext}}, \qquad D = P \quad \text{in } Q^{\text{ext}} \tag{2.4.15}$$

and the boundary and interface conditions

$$E^f(t) \times \mathrm{n} = 0 \quad \text{on } \partial\Omega, \qquad \Big(E^f(t) - E^s(t)\Big) \times \mathrm{n} = 0 \quad \text{on } \partial S(t), \tag{2.4.16}$$

$$\Big(H^{\text{ext}}(t) - H^f(t)\Big) \times \mathrm{n} = k(t) \quad \text{on } \partial\Omega, \qquad \Big(H^f(t) - H^s(t)\Big) \times \mathrm{n} = 0 \quad \text{on } \partial S(t), \tag{2.4.17}$$

$$B^f(t) \cdot \mathrm{n} = 0 \quad \text{on } \partial\Omega, \qquad \Big(B^f(t) - B^s(t)\Big) \cdot \mathrm{n} = 0 \quad \text{on } \partial S(t), \tag{2.4.18}$$

$$\Big(D^{\text{ext}}(t) - D^f(t)\Big) \cdot \mathrm{n} = \omega(t) \quad \text{on } \partial\Omega, \qquad \Big(D^f(t) - D^s(t)\Big) \cdot \mathrm{n} = \omega(t) \quad \text{on } \partial S(t), \tag{2.4.19}$$

$$u^f(t) = 0 \quad \text{on } \partial\Omega, \qquad u^f(t) - u^s(t) = 0 \quad \text{on } \partial S(t). \tag{2.4.20}$$

In this system, the external force $J : (0, T) \times \mathbb{R}^3 \to \mathbb{R}^3$ in Ampère's law (2.4.1) in the fluid domain is assumed to satisfy

$$\nabla \cdot J = 0 \quad \text{in } Q^f, \qquad J = 0 \quad \text{in } Q^s \text{ and } Q^{\text{ext}}. \tag{2.4.21}$$

The stress tensor $\mathbb{T}$ in the momentum equation (2.4.9) is given by

$$\mathbb{T} = \mathbb{T}(u) := 2\nu\mathbb{D}(u), \qquad \mathbb{D}(u) := \frac{1}{2}\nabla u + \frac{1}{2}(\nabla u)^T,$$

the function $g : Q \to \mathbb{R}^3$ denotes another external forcing term. In the balance of linear momentum (2.4.10) and the balance of angular momentum (2.4.11) the quantities V and w constitute the translational velocity and the rotational velocity of the rigid body respectively while

$$m := \int_{S(t)} \rho(t, x) dx, \qquad X(t) := \frac{1}{m} \int_{S(t)} \rho(t, x) x dx,$$

$$\mathbb{J}(t) a \cdot b := \int_{S(t)} \rho(t, x) \left[a \times (x - X(t))\right] \cdot \left[b \times (x - X(t))\right] dx \qquad \forall a, b \in \mathbb{R}^3,$$

represent its total mass m, its center of mass X and its inertia tensor $\mathbb{J}$. The magnetic permeability μ and the dielectric permittivity ϵ in the relations (2.4.14) are given by

$$\mu := \begin{cases} \mu^f := \mu_0 \mu_r^f > 0 & \text{in } Q^f, \\ \mu^s := \mu_0 > 0 & \text{in } Q^s, \end{cases} \qquad \epsilon = \begin{cases} \epsilon^f := \epsilon_0 \epsilon_r^f > 0 & \text{in } Q^f, \\ \epsilon^s := \epsilon_0 > 0 & \text{in } Q^s, \end{cases}$$

wherein μ_0 and ϵ_0 denote the magnetic permeability and the dielectric permittivity in a vacuum and μ_r^f and ϵ_r^f denote the relative permeability and the relative permittivity of the fluid. Finally, in the interface conditions (2.4.17), (2.4.19) the quantities k and ω denote the surface current density and the surface charge density, respectively, on $\partial\Omega$ and $\partial S(t)$.

The derived system (2.4.1)–(2.4.20) finds itself in an intermediate state between the general system (2.1.1)–(2.1.11) and the system (1.3.1)–(1.3.14), which constitutes the basis of our mathematical analysis in Chapter 3 and the derivation of which is the ultimate goal of the present chapter. The system (2.1.1)–(2.1.11) models the interaction between an electrically conducting fluid and an insulating rigid body in full generality. The equations (1.3.1)–(1.3.14), as a reduced version of this system, are less general, however, they bear the advantage that we are able to prove the existence of weak solutions to them, cf. Theorem 3.1.1. The disadvantage of the latter system lies in the fact that some of its modifications in comparison to the system (2.1.1)–(2.1.11) are made for purely mathematical reasons. This, in turn, shows why the system (2.4.1)–(2.4.20) is interesting: It constitutes an intermediate result in the derivation of the system (1.3.1)–(1.3.14) from the system (2.1.1)–(2.1.11) containing only those modifications for which we have been able to provide - under certain assumptions - physical arguments. However, we point out that, so far, we have not been able to prove the existence of weak solutions to the system (2.4.1)–(2.4.20). This is mainly due to the jump of the magnetic permeability μ across the interface between the fluid and the solid in this system, which would require a new variational formulation and, probably, a new methodology.

We end this chapter by discussing the final adjustments which need to be made in order to turn the system (2.4.1)–(2.4.20) into the system (1.3.1)–(1.3.14). In the latter system, as it is common practice in many mathematical works in magnetohydrodynamics (cf. for example [12]), the Maxwell equations are only considered inside of the domain Ω. This is due to the additional assumption of the linear relations (2.4.14) holding true also in the exterior domain,

$$B = \mu H \qquad \text{in } Q^{\text{ext}}, \qquad\qquad D = \epsilon E \qquad \text{in } Q^{\text{ext}} \tag{2.4.22}$$

for the magnetic permeability $\mu > 0$ and the dielectric permittivity $\epsilon > 0$ in the perfect conductor $\mathbb{R}^3 \backslash \overline{\Omega}$. Indeed, under this assumption the trivial relations (2.4.12) and the equation (2.4.5) immediately imply that also the quantities H, D and ρ_c vanish in Q^{ext}, cf. for example [27, Chapter 1: Part A: §4.2.4.3]. In this case the whole electromagnetic subsystem in the exterior domain becomes trivial and it is sufficient to examine the problem in the interior domain. Thus, in the system (1.3.1)–(1.3.14), in which the condition (2.4.22) is assumed implicitly, all electromagnetic equations in the exterior domain are neglected. However, we point out that there does not seem to be a physical argument for the validity of the linear relations (2.4.22) in a perfect conductor. For this reason we here decided to formulate the system (2.4.1)–(2.4.20) in a more general form, without the condition (2.4.22) and with the electromagnetic subsystem in the exterior domain.
Furthermore, Gauss's law (2.4.5) in the fluid domain is not included in the system (1.3.1)–(1.3.14). This is explained as follows: In mathematical works in magnetohydrodynamics the (reduced) Maxwell system is commonly compressed into two equations for the magnetic induction B, Gauss's law for magnetism (2.4.7) and the so-called induction equation,

$$\operatorname{div} B = 0 \quad \text{in } Q^f, \qquad \partial_t B + \nabla \times (B \times u) + \frac{1}{\sigma} \nabla \times \left(\frac{1}{\mu} \nabla \times (B) - J \right) = 0 \quad \text{in } Q^f. \tag{2.4.23}$$

Indeed, the latter equation results directly from a combination of Ohm's law (2.4.13), Ampère's law (2.4.1) and the Maxwell-Faraday equation (2.4.4). The idea behind this further reduction is that the unknown B may be determined independently of all the other unknowns from the Maxwell system in the fluid domain. After determining B, we have the magnetic field H given explicitly from the relation (2.4.14) and consequently also the electric current density j from Ampère's law (2.4.1). Subsequently, the electric field E and the electric induction D can be computed directly from Ohm's law (2.4.13) and the relation (2.4.14) and we can use Gauss's law (2.4.5) to immediately obtain the density of electric charges ρ_c. Therefore, the Maxwell system in the fluid domain can be solved by solving only the system (2.4.23) and in particular Gauss's law (2.4.5) becomes superfluous for the mathematical analysis. For this reason Gauss's law in the fluid domain is neglected in the system (1.3.1)–(1.3.14).

As opposed to in the fluid domain, Gauss's law (2.4.6) in the solid domain cannot be dropped. Indeed, in the solid domain ρ_c is equal to 0 (cf. (2.1.19)), while E cannot be determined via Ohm's law, so (2.4.6) rather constitutes a condition required for determining D and E than for determining ρ_c. We remark that in the system (1.3.1)–(1.3.14) this equation is expressed in terms of E instead of D, cf. (1.3.4). Thus the only remaining equation involving D is the linear relation between D and E in (2.4.14), which therefore becomes redundant for the analysis and consequently does not appear in the system (1.3.1)–(1.3.14).
The linear relation between B and H in (2.4.14) instead constitutes a crucial component of the system (1.3.1)–(1.3.14). In the latter system, however, it is modified in the sense that μ is assumed to take the same value in both Q^f and Q^s, i.e. μ is a constant in the whole domain Q, cf. (1.3.11). This modification, which is only justified if the magnetic permeability in Q^f and Q^s is (almost) the same, is made for purely mathematical reasons. Namely, in combination with the interface conditions for H and B on $\partial S(t)$ in (2.4.17) and (2.4.18) it ensures continuity of B across the fluid-solid interface. This is necessary for the weak formulation of the system in Definition 3.1.1 below, in which the magnetic induction is assumed to be a Sobolev function over the whole domain Q.
Moreover, we remark that the interface conditions for H on $\partial\Omega$ in (2.4.17) as well as the interface conditions (2.4.19) on D do not appear in the system (1.3.1)–(1.3.14) as they do not enter its weak formulation in Chapter 3. Finally, we point out that also the conditions (2.4.21) on J are left out of the system (1.3.1)–(1.3.14) as the analysis in Chapter 3 can also be carried out without them. This concludes the derivation of the system (1.3.1)–(1.3.14).

Chapter 3

Fluid-rigid body interaction in an incompressible electrically conducting fluid

In this chapter we investigate the interaction between a viscous non-homogeneous incompressible and electrically conducting fluid, an insulating rigid body traveling through the fluid as well as the electromagnetic fields present in both materials. More specifically, the electrically conducting fluid interacts directly with both the electromagnetic fields and the solid body, while the body, being insulating, only interacts indirectly with the electromagnetic fields via the fluid. Mathematically this situation is described by the system (1.3.1)–(1.3.14) of partial differential equations presented in Section 1.3.1. The main result in this chapter, which is joint work with Barbora Benešová, Šárka Nečasová and Anja Schlömerkemper and has been published in the article [8], guarantees the existence of weak solutions to this model up to the first time at which there occurs a contact between the body and the boundary of the domain. It constitutes one of the first results of its kind in the combination of the - in themselves well studied - research fields of fluid-structure interaction and magnetohydrodynamics.

The proof we provide for this result in the present chapter can be found in almost the same way in [8]. As an additional value, however, we complement the proof given here by some supplementary technical details. The main difficulty in this proof is the high coupling of the system, caused by the test functions in the variational formulation of the induction equation, which depend on the moving solid domain and therefore on the overall solution to the problem. We evade this inconvenience by implementing a time discretization via the Rothe method, in which the equations are decoupled by the use of time-lagging functions. In this way, at each fixed discrete time, we may first determine the solid domain and with it the test functions for the induction equation at this time. Subsequently, the induction equation can be solved by a standard procedure. The time discretization, however, generates an additional problem in the transport equation for the characteristic function of the solid body. Indeed, if the transport equation is discretized, this function cannot be guaranteed to take only the values 0 and 1, making it impossible for us to determine the position of the body. We therefore do not discretize the whole system but instead solve this transport equation as a continuous equation on the small intervals between all consecutive discrete time points. This idea of a hybrid approximation scheme - consisting of both discrete and continuous equations - later turns out to also play an important role in the compressible case in Chapter 4 below. The problem of test functions depending on the moving solid domain moreover arises in the momentum equation. As this situation, however, is classical in fluid-structure interaction we may take advantage of the well-studied Brinkman penalization to handle it in that case.

Applications of our results can be found predominantly in the area of biomechanics. More specifically, the fluid-structure interaction problem with an electrically conducting fluid finds use for example in capsule endoscopy (cf. [59]) or remote drug delivery (cf. [58, Section 4.4]), whereby microscopically small robots are steered via the application of electromagnetic forces through the human blood stream for medical purposes. We point out that blood is widely assumed to be an incompressible fluid (see e.g. [108]), which makes our results in the present chapter particularly suitable for these applications. Further applications include the description of the interplay between cell membranes and extracellular and intercellular fluids in organisms. For more details we refer to Section 1.1.

3.1 Weak solutions and main result

3.1.1 Notation

We study a bounded domain $\Omega \subset \mathbb{R}^3$ occupied by a viscous non-homogeneous incompressible and electrically conducting fluid as well as an insulating rigid body, moving through and interacting with the fluid over some time interval $[0,T]$, $T>0$. For the motion of the solid body we introduce the following notation: The initial position of the body is characterized through a bounded domain $S_0 \subset \Omega$. Due to the rigidity of the body, its position at any time $s \in [0,T]$ can be mapped to its position at any time $t \in [0,T]$ using only a translation and a rotation. Mathematically speaking, this means that there exists an orientation preserving isometry $\eta(s;t,\cdot) : \mathbb{R}^3 \to \mathbb{R}^3$, $s,t \in [0,T]$, such that the position of the body at arbitrary times is expressed through a set-valued function

$$S: \ [0,T] \to 2^{\mathbb{R}^3}, \quad S(t) = \eta(s;t,S(s)) \qquad \forall s,t \in [0,T].$$

More directly, the position $S(t)$ of the body at the time $t \in [0,T]$ is then described by

$$S(t) = \eta(0;t,S_0).$$

The codomain of S being $2^{\mathbb{R}^3}$ instead of only 2^{Ω} hints that the solid body might leave the domain Ω. While this is not foreseen in our model and cannot happen for the solution we construct in our main result, Theorem 3.1.1 below, such a behavior cannot be excluded in the early stages of the proof of this result, which is why it needs to be included into our notation. Denoting by $\chi(t) = \chi(t,\cdot) : \mathbb{R}^3 \to \{0,1\}$ the characteristic function of the solid at time t we also write alternatively

$$S(t) = \left\{x \in \mathbb{R}^3 : \ \chi(t,x) = 1\right\}.$$

Next, for each time $T' \in (0,T]$, we introduce the time-space domain $Q(T') := (0,T') \times \Omega$, which we split into the solid time-space domain $Q^s(S,T')$ as well as its fluid counterpart $Q^f(S,T')$,

$$Q^s\left(S,T'\right) := \left\{(t,x) \in \left(0,T'\right) \times \mathbb{R}^3 : \ x \in S(t)\right\}, \qquad Q^f\left(S,T'\right) := \left\{(t,x) \in Q\left(T'\right) : \ x \in F(t)\right\},$$

where $F(t) := \Omega \backslash \overline{S(t)}$. In the case that $T' = T$ we shorten the notation to

$$Q(T) = Q, \qquad Q^s(S,T) = Q^s(S), \qquad Q^f(S,T) = Q^f(S).$$

With this notation at hand we further introduce the space $Z(S,T')$ of test function for the momentum equation in our weak formulation of the problem (cf. Definition 3.1.1 below),

$$Z\left(S,T'\right) := \left\{\phi \in \mathcal{D}\left(\left[0,T'\right) \times \Omega\right) : \ \mathrm{div}\phi = 0, \ \mathbb{D}(\phi) = 0 \quad \text{in an open neighborhood of } \overline{Q^s\left(S,T'\right)}\right\}, \tag{3.1.1}$$

meaning that for any $\phi \in Z(S,T')$ there is $\kappa > 0$ such that

$$\mathbb{D}(\phi) = 0 \quad \text{in } \left\{(t,x) \in Q\left(T'\right) : \ \mathrm{dist}\left((t,x), \overline{Q^s\left(S,T'\right)}\right) < \kappa\right\}. \tag{3.1.2}$$

Similarly, we define the test function space $Y(S,T')$ for the induction equation in our weak formulation,

$$Y\left(S,T'\right) := \left\{b \in \mathcal{D}\left(\left[0,T'\right) \times \Omega\right) : \mathrm{curl}\ b = 0 \quad \text{in an open neighborhood of } \overline{Q^s\left(S,T'\right)}\right\}. \tag{3.1.3}$$

In the case $T' = T$ we use the shortened notation

$$Z(S,T) = Z(S), \qquad Y(S,T) = Y(S).$$

Furthermore, in order to be able to characterize neighborhoods of sets $S \subset \mathbb{R}^3$ more specifically, we denote by S^κ, $\kappa > 0$, the κ-neighborhood of S

$$S^\kappa := \left\{ x \in \mathbb{R}^3 : \ \mathrm{dist}\,(x, S) < \kappa \right\}. \tag{3.1.4}$$

Similarly, we denote by S_κ the κ-kernel of S, i.e.

$$S_\kappa := \{ x \in S : \ \mathrm{dist}\,(x, \partial S) > \kappa \}. \tag{3.1.5}$$

Moreover, in addition to the standard notation for the Lebesgue-, Sobolev- and Bochner spaces we use the notation

$$\begin{aligned} H^r_{\mathrm{div}}(\Omega) &:= \left\{ v \in H^r(\Omega) : \ \mathrm{div}\, v = 0 \text{ in } \mathcal{D}'(\Omega) \right\} && \text{for } r \geqslant 0, \\ H^r_{0,\mathrm{div}}(\Omega) &:= \{ v \in H^r_{\mathrm{div}}(\Omega) : \ v|_{\partial\Omega} = 0 \} && \text{for } r > \frac{1}{2}, \end{aligned}$$

for the (potentially fractional) Sobolev spaces of functions which are in addition divergence-free.

Finally, an important tool in the proof of our existence result Theorem 3.1.1 is the Brinkman penalization, in which the deviation of the velocity field from its orthogonal projection onto velocity fields which are rigid in a (given) solid body is penalized, cf. Section 3.2. For the definition of this projection we introduce the following notation: If $\chi(t) \in L^\infty(\mathbb{R}^3; \{0, 1\})$ denotes the characteristic function of a bounded domain $S(t) \subset \mathbb{R}^3$, if $\rho(t) \in L^\infty(\Omega; \mathbb{R})$ satisfies $\rho(t) \geqslant \underline{\rho}$ almost everywhere in $S(t)$ for some constant $\underline{\rho} > 0$ and if $u(t) \in L^2(\mathbb{R}^3; \mathbb{R}^3)$, we define a rigid velocity field $\Pi_{[\chi,\rho,u]}(t, \cdot) : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\Pi_{[\chi,\rho,u]}(t, x) := (u_G)_{[\chi,\rho,u]}(t) + \omega_{[\chi,\rho,u]}(t) \times \left(x - a_{[\chi,\rho]}(t) \right) \qquad \forall t \in [0, T], \ x \in \mathbb{R}^3, \tag{3.1.6}$$

with

$$\begin{aligned} (u_G)_{[\chi,\rho,u]}(t) &:= \frac{\int_{\mathbb{R}^3} \rho(t,x)\chi(t,x)u(t,x)\, dx}{\int_{\mathbb{R}^3} \rho(t,x)\chi(t,x)\, dx}, \\ \omega_{[\chi,\rho,u]}(t) &:= \left(I_{[\chi,\rho]}(t) \right)^{-1} \int_{\mathbb{R}^3} \rho(t,x)\chi(t,x) \left(x - a_{[\chi,\rho]}(t) \right) \times u(t,x)\, dx, \\ I_{[\chi,\rho]}(t) &:= \int_{\mathbb{R}^3} \rho(t,x)\chi(t,x) \left(\left| x - a_{[\chi,\rho]}(t) \right|^2 \mathrm{id} - \left(x - a_{[\chi,\rho]}(t) \right) \otimes \left(x - a_{[\chi,\rho]}(t) \right) \right) dx, \\ a_{[\chi,\rho]}(t) &:= \frac{\int_{\mathbb{R}^3} \rho(t,x)\chi(t,x)x\, dx}{\int_{\mathbb{R}^3} \rho(t,x)\chi(t,x)\, dx}. \end{aligned}$$

The quantity $\Pi_{[\chi,\rho,u]}(t)$ can be understood as orthogonal projection of $u(t) \in L^2(\mathbb{R}^3)$ onto a rigid velocity field in $S(t)$ in the sense that

$$\int_{\mathbb{R}^3} \rho(t,x)\chi(t,x) \left(u(t,x) - \Pi_{[\chi,\rho,u]}(t,x) \right) \cdot \Pi(t,x)\, dx = 0 \tag{3.1.7}$$

for any rigid velocity field

$$\Pi(t, x) := v(t) + w(t) \times x, \qquad v(t), w(t) \in \mathbb{R}^3,$$

cf. Lemma A.5.1 in the appendix.

3.1.2 Weak solutions

We introduce our definition of weak solutions to the system (1.3.1)–(1.3.14), describing the interaction between an insulating rigid body and an incompressible electrically conducting fluid surrounding it. As, thanks to the non-conductivity of the rigid body, the expressions containing the electrical conductivity $\sigma^s = 0$ of the solid region are not visible in our weak formulation, we slightly abuse the notation to write $\sigma = \sigma^f > 0$ in the following.

Definition 3.1.1. *Let $T > 0$, let $\Omega \subset \mathbb{R}^3$ and $S_0 \subset \Omega$ be bounded domains such that*

$$\varnothing \neq S_0 \text{ is open and connected, } |\partial S_0| = 0 \text{ and } \operatorname{dist}(S_0, \partial\Omega) > 0. \tag{3.1.8}$$

Consider $\underline{\rho}, \overline{\rho}, \nu, \sigma, \mu > 0$, consider some external data $g, J \in L^\infty(Q)$ and consider some initial data $\chi_0, \rho_0 \in L^\infty(\Omega)$ and $u_0, B_0 \in L^2_{\operatorname{div}}(\Omega)$ satisfying

$$\chi_0(x) = \begin{cases} 1 & \text{if } x \in S_0 \\ 0 & \text{if } x \notin S_0 \end{cases}, \quad 0 < \underline{\rho} \leqslant \rho_0 \leqslant \overline{\rho} < \infty \quad \text{a.e. in } \Omega, \quad B_0 \cdot \mathrm{n} = 0 \text{ on } \partial\Omega \tag{3.1.9}$$

Then the system (1.3.1)–(1.3.14) is said to admit a weak solution on $[0, T']$, $T' \in (0, T]$, if there exists an orientation preserving isometry

$$\eta(s; t, \cdot) : \mathbb{R}^3 \to \mathbb{R}^3, \quad s, t \in [0, T'], \tag{3.1.10}$$

and if there exist functions

$$\chi \in C\left([0, T']; L^p(\Omega; \{0, 1\})\right) \quad \forall 1 \leqslant p < \infty, \tag{3.1.11}$$

$$\rho \in C\left([0, T']; L^p(\Omega; \mathbb{R})\right) \quad \forall 1 \leqslant p < \infty, \tag{3.1.12}$$

$$u \in \left\{\phi \in L^\infty\left(0, T'; L^2\left(\Omega; \mathbb{R}^3\right)\right) \bigcap L^2\left(0, T'; H^1_{0,\operatorname{div}}(\Omega)\right) : \mathbb{D}(\phi) = 0 \text{ in } Q^s\left(S, T'\right)\right\}, \tag{3.1.13}$$

$$B \in \left\{b \in L^\infty\left(0, T'; L^2\left(\Omega; \mathbb{R}^3\right)\right) \bigcap L^2\left(0, T'; H^1_{\operatorname{div}}(\Omega)\right) : \operatorname{curl} b = 0 \text{ in } Q^s\left(S, T'\right),\ b \cdot \mathrm{n} = 0 \text{ on } \partial\Omega\right\}, \tag{3.1.14}$$

where $S = S(\cdot) = \eta(0; \cdot, S_0)$, which satisfy

$$-\int_0^{T'} \int_\Omega \chi \partial_t \Theta dxdt - \int_\Omega \chi_0 \Theta(0, x)\ dx = \int_0^{T'} \int_\Omega (\chi u) \cdot \nabla\Theta\ dxdt, \tag{3.1.15}$$

$$-\int_0^{T'} \int_\Omega \rho \partial_t \psi dxdt - \int_\Omega \rho_0 \psi(0, x)\ dx = \int_0^{T'} \int_\Omega (\rho u) \cdot \nabla\psi\ dxdt, \tag{3.1.16}$$

$$\begin{aligned} -\int_0^{T'} \int_\Omega \rho u \cdot \partial_t \phi\ dxdt - \int_\Omega \rho_0 u_0 \cdot \phi(0, x)\ dx = \int_0^{T'} \int_\Omega & (\rho u \otimes u) : \nabla\phi - 2\nu\mathbb{D}(u) : \nabla\phi \\ & + \rho g \cdot \phi + \frac{1}{\mu}(\operatorname{curl} B \times B) \cdot \phi\ dxdt, \end{aligned} \tag{3.1.17}$$

$$-\int_0^{T'} \int_\Omega B \cdot \partial_t b\ dxdt - \int_\Omega B_0 \cdot b(0, x)\ dx = \int_0^{T'} \int_\Omega \left[-\frac{1}{\sigma\mu} \operatorname{curl} B + u \times B + \frac{1}{\sigma} J\right] \cdot \operatorname{curl} b\ dxdt \tag{3.1.18}$$

for all Θ, $\psi \in \mathcal{D}([0, T') \times \Omega)$, $\phi \in Z(S, T')$ and $b \in Y(S, T')$ as well as the relation

$$S(t) = \eta(s; t, S(s)) \quad \forall s, t \in [0, T']. \tag{3.1.19}$$

3.1.3 Main result

The main result of this chapter proves the existence of weak solutions to the system (1.3.1)–(1.3.14) as introduced in Definition 3.1.1.

Theorem 3.1.1. *[8, Theorem 1.1] Let $T > 0$, assume $\Omega \subset \mathbb{R}^3$ to be a simply connected bounded domain of class C^2 and assume $S_0 \subset \Omega$ to be a bounded domain of class C^2 which satisfies the conditions (3.1.8). Let moreover $\underline{\rho}, \overline{\rho}, \nu, \sigma, \mu > 0$ be some positive coefficients and assume the data $g, J \in L^\infty(Q)$, $\chi_0, \rho_0 \in L^\infty(\Omega)$ and $u_0, B_0 \in L^2_{\operatorname{div}}(\Omega)$ to satisfy the conditions (3.1.9). Then there exists $T' > 0$ such that the system (1.3.1)–(1.3.14) admits a weak solution (η, χ, ρ, u, B) on $[0, T']$ in the sense of Definition 3.1.1 which in addition satisfies the energy inequality*

$$\begin{aligned} &\int_\Omega \frac{1}{2}\rho(\tau)|u(\tau)|^2 + \frac{1}{2\mu}|B(\tau)|^2\ dx + \int_0^\tau \int_\Omega 2\nu\,|\nabla u|^2 + \frac{1}{\sigma\mu^2}\,|\operatorname{curl} B|^2\ dxdt \\ &\leqslant \int_\Omega \frac{1}{2}\rho_0|u_0|^2 + \frac{1}{2}|B_0|^2\ dx + \int_0^\tau \int_\Omega \rho g \cdot u + \frac{1}{\sigma\mu} J \cdot \operatorname{curl} B\ dxdt \end{aligned} \tag{3.1.20}$$

for almost all $\tau \in [0, T']$. *Moreover, the time* T' *can be chosen such that*

$$T' = \sup \Big\{ \tau \in [0, T] : \ \operatorname{dist}(S(t), \partial\Omega) > 0 \quad \forall t \leqslant \tau \Big\}. \tag{3.1.21}$$

Remark 3.1.1. *As a slight improvement of the above result, it might be desirable to consider test functions*

$$b \in \mathcal{D}\left([0, T') \times \overline{\Omega}\right) : \ \operatorname{curl} b = 0 \quad \textit{in an open neighborhood of } \overline{Q^s(S, T')}$$

with (potentially) non-compact support in Ω *in the induction equation* (3.1.18) *instead of the test functions* (3.1.3). *This is possible via a slight modification of our proof of Theorem 3.1.1 in Sections 3.3–3.6. Indeed, fully utilizing the Helmholtz-decomposition Lemma A.2.2 in the construction of a solution to our time discrete approximation of the induction equation in Section 3.3.4, we do not need to impose a boundary condition on the test functions on the discrete level. In the subsequent limit passages throughout Sections 3.4–3.6, a non-compact support of the test functions in* Ω *poses no difficulties.*
For future research, it might further be of interest to consider test functions the curl of which vanishes only in the solid domain $Q^s(S, T')$ *itself instead of in a neighbourhood thereof. This modification, however, is not trivial as the vanishing curl of the test functions is crucial in our limit passages carried out in Sections 3.4–3.6.*

Remark 3.1.2. *Due to the transport theorem by DiPerna and Lions [35], the solution* $\rho \in L^2(Q)$ *to the continuity equation (3.1.16), given by Theorem 3.1.1, also solves the renormalized continuity equation*

$$\partial_t \beta(\rho) + u \cdot \nabla \beta(\rho) = 0 \quad \textit{in } \mathcal{D}'(Q) \tag{3.1.22}$$

for any bounded $\beta \in C^1(\mathbb{R})$ *such that*

$$\beta \textit{ is bounded, } \beta \textit{ vanishes near 0 and } (\beta'(1 + |\cdot|))^{-1} \textit{ is bounded.} \tag{3.1.23}$$

The proof of Theorem 3.1.1 will be accomplished via an approximation method in Sections 3.3–3.6. Large parts of this proof are taken directly from the article [8], wherein Theorem 3.1.1 was published by the author of the present thesis in joint work with Barbora Benešová, Šárka Nečasová and Anja Schlömerkemper. Nonetheless, we here supplement individual steps of the proof given in the article by further details with the aim of improving the readability for the convenience of the reader. An outline of the proof is given in the following section.

3.2 Approximate system

We introduce the approximation to the original system. This approximate system is chosen such that it is easily solvable; a solution to the original system is obtained subsequently by passing to the limit in all levels of the approximation. Our approximation consists of three different levels, characterized by three parameters $\Delta t, \epsilon > 0$, $m \in \mathbb{N}$:

- On the Δt-level, we employ a time discretization by the Rothe method, cf. [99, Section 8.2]. To this end, $\Delta t > 0$ is chosen in such a way that $\frac{T}{\Delta t} \in \mathbb{N}$ and we split up the interval $[0, T]$ into the discrete times $k\Delta t$, $k = 1, ..., \frac{T}{\Delta t}$.

- On the ϵ-level, we add several regularization terms to the system, which help us to solve the approximate system and pass to the limit as $\Delta t \to 0$.

- On the m-level we add a penalization term to the momentum equation, which guarantees us that after passing to the limit in $m \to \infty$, the limit velocity will coincide - on the solid part of the domain - with the rigid velocity of the body.

We now present the full approximate system, containing all three approximation levels, and give a more detailed explanation afterwards: Assuming for some discrete time $k\Delta t$, $k \in \{1, ..., \frac{T}{\Delta t}\}$, the solution at time $(k-1)\Delta t$, indexed by $k-1$, to be given we seek functions

$$\begin{aligned}
\chi_{\Delta t,k} \in& C\left([(k-1)\Delta t, k\Delta t]; L^p\left(\mathbb{R}^3\right)\right) \quad \forall 1 \leqslant p < \infty, && (3.2.1)\\
\rho_{\Delta t}^k \in& \left\{\psi \in H^1(\Omega) : \ \underline{\rho} \leqslant \psi \leqslant \overline{\rho} \text{ in } \Omega\right\}, && (3.2.2)\\
u_{\Delta t}^k \in& H^2_{0,\mathrm{div}}(\Omega), && (3.2.3)\\
B_{\Delta t}^k \in& Y^k\left(S_{\Delta t,k}\right) := \left\{b \in H^1_{\mathrm{div}}\left(\Omega\right) : \ \operatorname{curl} b \in H^1(\Omega), \ \operatorname{curl} b = 0 \text{ in } S_{\Delta t,k}(k\Delta t) \bigcap \Omega, \ b \cdot \mathrm{n}|_{\partial\Omega} = 0\right\} && (3.2.4)
\end{aligned}$$

satisfying the discrete system at time $k\Delta t$,

$$\begin{aligned}
-\int_{(k-1)\Delta t}^{k\Delta t}\int_{\mathbb{R}^3} \chi_{\Delta t,k}\partial_t\Theta \ dxdt =& \int_{\mathbb{R}^3} \chi_{\Delta t}^{k-1}\Theta((k-1)\Delta t, x) \ dx - \int_{\mathbb{R}^3} \chi_{\Delta t}^k\Theta(k\Delta t, x) \ dx\\
&+ \int_{(k-1)\Delta t}^{k\Delta t}\int_{\mathbb{R}^3} \left(\chi_{\Delta t,k}\Pi_{\Delta t}^{k-1}\right) \cdot \nabla\Theta \ dxdt, && (3.2.5)\\
-\int_\Omega \frac{\rho_{\Delta t}^k - \rho_{\Delta t}^{k-1}}{\Delta t}\psi dx =& \int_\Omega u_{\Delta t}^{k-1} \cdot \nabla\rho_{\Delta t}^k\psi + \epsilon\nabla\rho_{\Delta t}^k \cdot \nabla\psi \ dx, && (3.2.6)\\
-\int_\Omega \frac{\rho_{\Delta t}^k u_{\Delta t}^k - \rho_{\Delta t}^{k-1}u_{\Delta t}^{k-1}}{\Delta t} \cdot \phi \ dx =& \int_\Omega \left[\operatorname{div}\left(\rho_{\Delta t}^k u_{\Delta t}^{k-1} \otimes u_{\Delta t}^k\right) - 2\nu \operatorname{div}\left(\mathbb{D}u_{\Delta t}^k\right) + \epsilon\nabla u_{\Delta t}^k\nabla\rho_{\Delta t}^k\right] \cdot \phi \ dx\\
&+ \epsilon\Delta u_{\Delta t}^k\Delta\phi + \left[m\rho_{\Delta t}^{k-1}\chi_{\Delta t}^k\left(u_{\Delta t}^{k-1} - \Pi_{\Delta t}^{k-1}\right) - \rho_{\Delta t}^{k-1}g_{\Delta t}^k\right.\\
&\left.- \frac{1}{\mu}\left(\operatorname{curl} B_{\Delta t}^{k-1} \times B_{\Delta t}^{k-1}\right)\right] \cdot \phi \ dx, && (3.2.7)\\
-\int_\Omega \frac{B_{\Delta t}^k - B_{\Delta t}^{k-1}}{\Delta t} \cdot b \ dx =& \int_\Omega \left[\frac{1}{\sigma\mu}\operatorname{curl} B_{\Delta t}^k - u_{\Delta t}^k \times B_{\Delta t}^{k-1} + \frac{\epsilon}{\mu^2}\left|\operatorname{curl} B_{\Delta t}^k\right|^2 \operatorname{curl} B_{\Delta t}^k\right.\\
&\left.- \frac{1}{\sigma}J_{\Delta t}^k\right] \cdot \operatorname{curl} b + \epsilon\left(\nabla \operatorname{curl} B_{\Delta t}^k\right) : \left(\nabla \operatorname{curl} b\right) \ dx && (3.2.8)
\end{aligned}$$

for all $\Theta \in \mathcal{D}([(k-1)\Delta t, k\Delta t] \times \mathbb{R}^3)$, $\psi \in H^1(\Omega)$, $\phi \in H^2_{0,\mathrm{div}}(\Omega)$ and

$$b \in W^k(S_{\Delta t,k}) := \left\{b \in H^1\left(\Omega\right) : \ \operatorname{curl} b \in H^1(\Omega), \ \operatorname{curl}\, b = 0 \text{ in } S_{\Delta t,k}(k\Delta t) \bigcap \Omega, \ b \cdot \mathrm{n}|_{\partial\Omega} = 0\right\} \quad (3.2.9)$$

as well as the (discrete) initial conditions

$$\begin{aligned}
\chi_{\Delta t,k}((k-1)\Delta t, x) = \chi_{\Delta t,k-1}((k-1)\Delta t, x), \quad \chi_{\Delta t,1}(0, x) = \chi_0(x), \qquad \forall x \in \Omega, && (3.2.10)\\
\rho_{\Delta t}^0(x) = \rho_{0,m}(x), \quad u_{\Delta t}^0(x) = u_{0,m}(x), \quad B_{\Delta t}^0(x) = B_{0,m}(x) \qquad \forall x \in \Omega. && (3.2.11)
\end{aligned}$$

Before we proceed with explaining the different approximation levels in (3.2.1)–(3.2.11), let us clarify the notation introduced in this system: The spaces $Y^k(S_{\Delta t,k})$ and $W^k(S_{\Delta t,k})$ in (3.2.4) and (3.2.9) are equipped with the norm

$$\|\cdot\|_{Y^k(S_{\Delta t,k})} := \|\cdot\|_{W^k(S_{\Delta t,k})} := \|\cdot\|_{H^1(\Omega)} + \|\operatorname{curl}\left(\cdot\right)\|_{H^1(\Omega)}. \quad (3.2.12)$$

The functions $\chi_{\Delta t}^k$ and $\Pi_{\Delta t}^{k-1}$, introduced in the equations (3.2.5) and (3.2.7), are defined by:

$$\chi_{\Delta t}^k := \chi_{\Delta t,k}(k\Delta t), \quad \Pi_{\Delta t}^{k-1} = (u_G)_{\Delta t}^{k-1} + \omega_{\Delta t}^{k-1} \times \left(x - a_{\Delta t}^{k-1}\right) \quad (3.2.13)$$

and

$$
\begin{aligned}
(u_G)_{\Delta t}^{k-1} &:= \frac{\int_{\mathbb{R}^3} \rho_{\Delta t}^k \chi_{\Delta t}^{k-1} u_{\Delta t}^{k-1}\, dx}{\int_{\mathbb{R}^3} \rho_{\Delta t}^k \chi_{\Delta t}^{k-1}\, dx},\\
\omega_{\Delta t}^{k-1} &:= \left(I_{\Delta t}^{k-1}\right)^{-1} \int_{\mathbb{R}^3} \rho_{\Delta t}^k \chi_{\Delta t}^{k-1} \left(x - a_{\Delta t}^{k-1}\right) \times u_{\Delta t}^{k-1}\, dx,\\
I_{\Delta t}^{k-1} &:= \int_{\mathbb{R}^3} \rho_{\Delta t}^k \chi_{\Delta t}^{k-1} \left(|x - a_{\Delta t}^{k-1}|^2 \mathrm{id} - \left(x - a_{\Delta t}^{k-1}\right) \otimes \left(x - a_{\Delta t}^{k-1}\right)\right)\, dx,\\
a_{\Delta t}^{k-1} &:= \frac{\int_{\mathbb{R}^3} \rho_{\Delta t}^k \chi_{\Delta t}^{k-1} x\, dx}{\int_{\mathbb{R}^3} \rho_{\Delta t}^k \chi_{\Delta t}^{k-1}\, dx}.
\end{aligned}
$$

In order to keep the latter terms well-defined, we extend the functions $\rho_{\Delta t}^l$ by $\underline{\rho}$ and $u_{\Delta t}^l$ by 0 outside of Ω for any $l = 0, ..., k$. For the definition of the set $S_{\Delta t,k}(k\Delta t)$ in (3.2.4) and (3.2.9), we first denote by $\eta_{\Delta t}^{\Pi_{\Delta t}^{k-1}} : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ the solution to the initial value problem

$$
\frac{d\eta_{\Delta t}^{\Pi_{\Delta t}^{k-1}}(s;t,x)}{dt} = \Pi_{\Delta t}^{k-1}\left(\eta_{\Delta t}^{\Pi_{\Delta t}^{k-1}}(s;t,x)\right), \qquad \eta_{\Delta t}^{\Pi_{\Delta t}^{k-1}}(s;s,x) = x, \quad x \in \mathbb{R}^3, \quad s,t \in \mathbb{R}, \tag{3.2.14}
$$

where t represents the time variable and s the initial time. By the transport theory (cf. [35, Theorem III.2]) this solution is related to the solution $\chi_{\Delta t,k}$ to the transport equation (3.2.5) via the formula

$$
\chi_{\Delta t,k}(t,x) = \chi_{\Delta t,k-1}\left((k-1)\Delta t, \eta_{\Delta t}^{\Pi_{\Delta t}^{k-1}}(t;(k-1)\Delta t, x)\right) \quad \text{for } t \in [(k-1)\Delta t, k\Delta t]. \tag{3.2.15}
$$

Setting $S_{\Delta t,1}(0) = S_0$ we can now define the set $S_{\Delta t,k}(k\Delta t)$ recursively via the formula

$$
S_{\Delta t,k}(t) := \eta_{\Delta t}^{\Pi_{\Delta t}^{k-1}}((k-1)\Delta t; t, S_{\Delta t,k-1}((k-1)\Delta t)) = \left\{x \in \mathbb{R}^3 : \ \chi_{\Delta t,k}(t,x) = 1\right\}
$$

for $t \in [(k-1)\Delta t, k\Delta t]$. Moreover, since the given L^∞-functions g and J are not necessarily defined in the discrete times, we regularize them as in [99, (7.10)],

$$
g_\gamma(t) := \int_0^T \theta_\gamma\left(t + \xi_\gamma(t) - s\right) g(s)\, ds, \quad J_\gamma(t) := \int_0^T \theta_\gamma\left(t + \xi_\gamma(t) - s\right) J(s)\, ds, \quad \xi_\gamma(t) := \gamma\frac{T-2t}{T}.
$$

Here $\theta_\gamma : \mathbb{R} \to \mathbb{R}$ is a mollifier with support in $[-\gamma,\gamma]$ and ξ_γ has the purpose of shifting the support of $s \mapsto \theta_\gamma(t-s)$ into $[0,T]$ for any fixed $t \in [0,T]$. Then we choose $\gamma = \gamma(\Delta t)$, $\gamma(\Delta t) \to 0$ for $\Delta t \to 0$ and define the discrete approximations $g_{\Delta t}^k$ and $J_{\Delta t}^k$ of g and J from (3.2.7) and (3.2.8) by

$$
g_{\Delta t}^k := g_{\gamma(\Delta t)}(k\Delta t), \quad J_{\Delta t}^k := J_{\gamma(\Delta t)}(k\Delta t). \tag{3.2.16}
$$

Finally, the functions $\rho_{0,m}$, $u_{0,m}$, $B_{0,m}$ in the initial conditions (3.2.11) denote regularizations of the initial data in Theorem 3.1.1 chosen such that

$$
\rho_{0,m} \in H^1(\Omega), \quad u_{0,m},\ B_{0,m} \in H^2_{\mathrm{div}}(\Omega). \tag{3.2.17}
$$

We are now in the position to shed light on the ideas behind the various levels of the approximate system, beginning with the highest level.

The usage of a time discretization on the Δt-level constitutes the greatest novelty in our proof. Its purpose is explained as follows: The fact that the test functions in both the momentum equation and the induction equation depend on the solution of the system (cf. (3.1.1), (3.1.3)) prevents us from solving all of the equations in the original system simultaneously. In the case of the momentum equation this problem is circumvented via the application of a penalization method (see the m-level below). In the case of the induction equation, however, no similar method appears to be available. Instead we decouple the system via a time discretization, which allows us to solve the equations one after another by using time-lagging functions in the coupling terms. More precisely, in our discretization we are

able to first determine the position of the solid up to a certain discrete time and subsequently choose the test functions for the induction equation at this specific time accordingly. The existence of the magnetic induction on the discrete level then follows by standard methods.
We also want to point out that the function $\chi_{\Delta t,k}$ represents an exception in this system: It is the only function which is immediately constructed as a time-dependent function. The reason for this is that we want it to take only the values 0 and 1 so that we are able to determine the position of the rigid body at any time. Inspired by [56], we can guarantee this by constructing $\chi_{\Delta t,k}$ by solving a classical transport equation on the small interval $[(k-1)\Delta t, k\Delta t]$, in case of a discrete transport equation we might lose the property. As a consequence we remark that, strictly speaking, the approximate problem (3.2.1)–(3.2.9) does not constitute a discrete but rather a hybrid system, consisting of both discrete and continuous (on the small intervals between the discrete time points) equations. This realization is important for the idea of our proof in the compressible setting in Section 4.2 below. Indeed, in that proof we also make use of a hybrid approximate system, however, the weighting between discrete and continuous equations is reversed. While the induction equation is again approximated discretely, the whole mechanical part of the system needs to be studied as a time-dependent problem already on approximation level in this case.
Next we note that the mapping $\Pi_{\Delta t}^{k-1}$ is, by definition, a rigid velocity field with the translational velocity $(u_G)_{\Delta t}^{k-1}$ and the rotational velocity $w_{\Delta t}^{k-1}$. The constant terms $I_{\Delta t}^{k-1}$ and $a_{\Delta t}^{k-1}$ can be considered as discrete versions of the inertia tensor and the center of mass of the rigid body described by the characteristic function $\chi_{\Delta t}^{k}$. In fact, $\Pi_{\Delta t}^{k-1}$ constitutes a discretization of the projection (3.1.6) of the velocity onto a rigid velocity field. This comes into play in the penalization term from the m-level of the approximation mentioned above, namely the term

$$m\rho_{\Delta t}^{k-1}\chi_{\Delta t}^{k}\left(u_{\Delta t}^{k-1}-\Pi_{\Delta t}^{k-1}\right),$$

from (3.2.7). We can use this term to infer that after letting $m\to\infty$ the limit velocity coincides, in the solid area, with the velocity of the rigid body, which is what we require to obtain (3.1.15). This penalization method, which is known as Brinkman penalization, is discussed rigorously in [15]. Physically speaking, it describes an extension of the fluid into the solid region, i.e. the approximate body, while still moving via a rigid velocity field, is now permeable and the limit passage $m\to\infty$ represents the process of letting the permeability vanish. This technique can be considered as an extension of the penalty method used in [2] for a fluid-structure interaction problem in which the movement of the solid is prescribed. It further finds use in [92], where the examined solid is additionally deformable and self-propelled and it is moreover of interest for finite element approaches to the problem, cf. [26, 69]. There are also other penalization methods available as for example in [43, 103], where an approach is used in which the solids are approximated by a fluid with viscosity rising to infinity. The latter penalization method is especially useful for the case of a compressible fluid, in which the density is not necessarily bounded away from zero. In particular we will choose it as our approach in the proof of our main result in the compressible case in Section 4 below.
Finally, it remains to discuss the various regularization terms from the ϵ-level. In the continuity equation, the Laplacian of the density is added to the right-hand side, which allows us to show an upper bound for ρ as well as some bound away from 0. This is needed because such bounds cannot be guaranteed from the discrete version of the transport equation. The parabolic regularization via the Laplacian of the density is well known from the theory for the compressible Navier-Stokes equations (cf. [94, Section 7.6.5]) and turns out to still guarantee upper and lower bounds of the density on the discrete level in the incompressible case. In order to compensate for this term in the energy inequality, the quantity $\epsilon\nabla u_{\Delta t}^{k}\nabla\rho_{\Delta t}^{k}$ is added to the momentum equation. The second new term in this equation, $\epsilon\Delta^2 u_{\Delta t}^{k}$, is needed for controlling the latter quantity when passing to the limit in $\Delta t\to 0$. Moreover, we have two regularization terms in the induction equation, the term $\mathrm{curl}(\Delta(\mathrm{curl}\, B_{\Delta t}^{k}))$ and the term $\mathrm{curl}(|\mathrm{curl} B_{\Delta t}^{k}|^2\mathrm{curl} B_{\Delta t}^{k})$, which is also known as the 4-double-curl. The first one is used to express the induction equation through a weakly continuous operator on $Y^k(S_{\Delta t,k})$. Showing that this operator is also coercive, we then conclude the existence of $B_{\Delta t}^{k}$. The latter one is required in the energy inequality: In the time-dependent version of the system, the mixed terms from the momentum and the induction equation cancel each other. On the discrete level this is not the case, as the involved functions are

chosen from distinct discrete times. However, the 4-double-curl enables us to absorb the problematic terms into the positive left-hand side, so that we can get the uniform bounds needed for the limit passage as $\Delta t \to 0$. We also remark, that the 4-double-curl was chosen instead of the 4-Laplacian $\nabla \cdot (|\nabla B_{\Delta t}^k|^2 \nabla B_{\Delta t}^k)$ in order to allow us to apply the Helmholtz-decomposition [107, Theorem 4.2]. This is why the test functions $b \in W^k(S_{\Delta t,k})$, which are non-solenoidal, can be used in the induction equation (3.2.8).

3.3 Existence of the approximate solution

We consider some discrete time index $k \in \left\{1, ..., \frac{T}{\Delta t}\right\}$ and prove the existence of a solution to the approximate system (3.2.5)–(3.2.8) under the assumption that a solution has already been determined for all time indices $l = 1, ..., k-1$.

3.3.1 Existence of the density

We introduce the bilinear form

$$a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}, \qquad a(\rho, \psi) := \int_\Omega \frac{\rho}{\Delta t}\psi + u_{\Delta t}^{k-1} \cdot \nabla\rho\psi + \epsilon\nabla\rho \cdot \nabla\psi \, dx \quad \forall \rho, \psi \in H^1(\Omega),$$

which allows us to write the continuity equation (3.2.6) in the form

$$a\left(\rho_{\Delta t}^k, \psi\right) = \int_\Omega \frac{\rho_{\Delta t}^{k-1}}{\Delta t}\psi \, dx \quad \forall \psi \in H^1(\Omega). \tag{3.3.1}$$

Clearly, the bilinear form a is bounded. Moreover, due to the identity

$$\int_\Omega u_{\Delta t}^{k-1} \cdot \nabla\rho\rho \, dx = \int_\Omega \frac{1}{2} u_{\Delta t}^{k-1} \cdot \nabla |\rho|^2 \, dx = -\int_\Omega \frac{1}{2} |\rho|^2 \operatorname{div} u_{\Delta t}^{k-1} \, dx = 0 \qquad \forall \rho \in H^1(\Omega), \tag{3.3.2}$$

it holds that

$$a(\rho, \rho) \geqslant \frac{1}{\Delta t} \|\rho\|_{L^2(\Omega)}^2 + \epsilon \|\nabla\rho\|_{L^2(\Omega)}^2 \qquad \forall \rho \in H^1(\Omega),$$

so a is also coercive. Consequently the Lax-Milgram Lemma implies the existence of a unique solution $\rho_{\Delta t}^k \in H^1(\Omega)$ to the equation (3.3.1) and hence to the continuity equation (3.2.6). For the proof of the upper and lower bounds for $\rho_{\Delta t}^k$ in (3.2.2) we transfer the arguments from the time-dependent and compressible case in [94, Section 7.6.5] to our setting: Since $\overline{\rho}$ is a constant, the function $\overline{r}_{\Delta t}^k := \rho_{\Delta t}^k - \overline{\rho}$ also satisfies a corresponding version of the discrete continuity equation,

$$-\int_\Omega \frac{\overline{r}_{\Delta t}^k - \overline{r}_{\Delta t}^{k-1}}{\Delta t}\psi \, dx = \int_\Omega u_{\Delta t}^{k-1} \cdot \nabla\overline{r}_{\Delta t}^k\psi \, dx + \int_\Omega \epsilon\nabla\overline{r}_{\Delta t}^k \cdot \nabla\psi \, dx \quad \forall \psi \in H^1(\Omega).$$

We use the function $\max\{\overline{r}_{\Delta t}^k, 0\} \in H^1(\Omega)$ as a test function in this identity. The first integral on the right-hand side vanishes since $u_{\Delta t}^{k-1}$ is divergence-free (cf. the identity (3.3.2)) and we are left with the estimate

$$\int_\Omega \frac{\left|\max\{\overline{r}_{\Delta t}^k, 0\}\right|^2}{\Delta t} \, dx \leqslant \int_\Omega \frac{\left|\max\{\overline{r}_{\Delta t}^k, 0\}\right|^2}{\Delta t} + \epsilon \left|\nabla \max\{\overline{r}_{\Delta t}^k, 0\}\right|^2 \, dx = \int_\Omega \frac{\overline{r}_{\Delta t}^{k-1} \max\{\overline{r}_{\Delta t}^k, 0\}}{\Delta t} \, dx \leqslant 0,$$

wherein the last inequality follows from the fact that $\overline{r}_{\Delta t}^{k-1} = \rho_{\Delta t}^{k-1} - \overline{\rho} \leqslant 0$. Consequently it holds that

$$\max\left\{\rho_{\Delta t}^k - \overline{\rho}, 0\right\} = \max\left\{\overline{r}_{\Delta t}^k, 0\right\} = 0 \quad \text{a.e. in } \Omega,$$

which proves the upper bound of the desired estimates in (3.2.2). The corresponding lower bound can be shown via a similar procedure in which $\overline{r}_{\Delta t}^k$ is replaced by $\underline{r}_{\Delta t}^k := \rho_{\Delta t}^k - \underline{\rho}$.

3.3.2 Existence of the characteristic function

The function $\Pi_{\Delta t}^{k-1}$, being constant with respect to the time variable and, according to (3.2.13), a rigid velocity field and thus affine with respect to the spatial variable, is Lipschitz-continuous. Consequently, by the theory of ordinary differential equations, the initial value problem (3.2.14) defines a unique mapping

$$\eta_{\Delta t}^{\Pi_{\Delta t}^{k-1}} : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3. \tag{3.3.3}$$

From the transport theory (cf. [35, Theorem III.2]) it follows that the function $\chi_{\Delta t,k}$, defined by the formula (3.2.15), is the (unique renormalized) solution to the transport equation (3.2.5).

3.3.3 Existence of the velocity field

For the construction of the velocity field we define the bilinear form

$$\tilde{a} : H^2_{0,\mathrm{div}}(\Omega) \times H^2_{0,\mathrm{div}}(\Omega) \to \mathbb{R}, \qquad \tilde{a}(u,\phi) := \int_\Omega \left(\frac{\rho_{\Delta t}^k u}{\Delta t}\right) \cdot \phi + \operatorname{div}\left(\rho_{\Delta t}^k u_{\Delta t}^{k-1} \otimes u\right) \cdot \phi$$
$$+ 2\nu \mathbb{D}(u) : \nabla\phi + \epsilon\left(\nabla u \nabla \rho_{\Delta t}^k\right) \cdot \phi + \epsilon(\Delta u)\cdot(\Delta\phi)\, dx$$

for all $u, \phi \in H^2_{0,\mathrm{div}}(\Omega)$. This allows us to express the momentum equation (3.2.7) in the form

$$\tilde{a}\left(u_{\Delta t}^k, \phi\right)$$
$$= \int_\Omega \frac{\rho_{\Delta t}^{k-1} u_{\Delta t}^{k-1}}{\Delta t} \cdot \phi - \left[m\rho_{\Delta t}^{k-1} \chi_{\Delta t}^k \left(u_{\Delta t}^{k-1} - \Pi_{\Delta t}^{k-1}\right) - \rho_{\Delta t}^{k-1} g_{\Delta t}^k - \frac{1}{\mu}\left(\operatorname{curl} B_{\Delta t}^{k-1} \times B_{\Delta t}^{k-1}\right)\right] \cdot \phi\, dx \tag{3.3.4}$$

for all $\phi \in H^2_{0,\mathrm{div}}(\Omega)$. Clearly, $\tilde{a}$ is bounded. In order to show that $\tilde{a}$ is also coercive, we first test the continuity equation (3.2.6) by $\frac{1}{2}|u|^2 \in H^1(\Omega)$ for some arbitrary $u \in H^2_{0,\mathrm{div}}(\Omega)$,

$$0 = \int_\Omega \frac{\rho_{\Delta t}^k - \rho_{\Delta t}^{k-1}}{\Delta t} \frac{1}{2}|u|^2 + u_{\Delta t}^{k-1} \cdot \nabla \rho_{\Delta t}^k \frac{1}{2}|u|^2 + \epsilon\left(\nabla u \nabla \rho_{\Delta t}^k\right) \cdot u\, dx. \tag{3.3.5}$$

Due to the divergence theorem and the no-slip boundary condition satisfied by u we see that

$$\int_\Omega \operatorname{div}\left(\rho_{\Delta t}^k u_{\Delta t}^{k-1} \otimes u\right) \cdot u\, dx - \int_\Omega u_{\Delta t}^{k-1} \cdot \nabla \rho_{\Delta t}^k \frac{1}{2}|u|^2\, dx$$
$$= \int_\Omega \operatorname{div}\left(\rho_{\Delta t}^k u_{\Delta t}^{k-1}\right)|u|^2 + \frac{1}{2}\rho_{\Delta t}^k u_{\Delta t}^{k-1} \cdot \left(\nabla |u|^2\right)\, dx - \int_\Omega u_{\Delta t}^{k-1} \cdot \nabla \rho_{\Delta t}^k \frac{1}{2}|u|^2\, dx$$
$$= \int_\Omega \frac{1}{2}\operatorname{div}\left(\rho_{\Delta t}^k u_{\Delta t}^{k-1} |u|^2\right)\, dx = 0. \tag{3.3.6}$$

Hence, subtracting the right-hand side of the identity (3.3.5) from $\tilde{a}(u,u)$ we find that

$$\tilde{a}(u,u) \geqslant \int_\Omega \frac{\rho}{\Delta t}|u|^2 + 2\nu|\nabla u|^2 + \epsilon|\Delta u|^2\, dx \geqslant c\|u\|^2_{H^2(\Omega)} \qquad \forall u \in H^2_{0,\mathrm{div}}(\Omega)$$

for a constant $c > 0$ independent of u, where the last inequality follows from the estimate (A.2.1) in Lemma A.2.1 in the appendix, holding true on the C^2-domain Ω. Consequently, $\tilde{a}$ is indeed coercive and hence the Lax-Milgram Lemma implies the existence of a unique solution $u_{\Delta t}^k \in H^2_{0,\mathrm{div}}(\Omega)$ to the problem (3.3.4) and therefore to the discrete momentum equation (3.2.7).

3.3.4 Existence of the magnetic induction

In order to solve the discrete induction equation (3.2.8), we introduce the operator

$$A : Y^k(S_{\Delta t,k}) \to \left(Y^k(S_{\Delta t,k})\right)^*,$$
$$\langle A(B), b\rangle_{(Y^k(S_{\Delta t,k}))^* \times Y^k(S_{\Delta t,k})} := \int_\Omega \frac{B}{\Delta t} \cdot b + \epsilon(\nabla \operatorname{curl} B) : (\nabla \operatorname{curl} b)$$
$$+ \left[\frac{1}{\sigma\mu}\operatorname{curl} B + \frac{\epsilon}{\mu^2}|\operatorname{curl} B|^2 \operatorname{curl} B\right] \cdot \operatorname{curl} b\, dx. \tag{3.3.7}$$

and first study the problem

$$\left\langle A\left(B_{\Delta t}^{k}\right), b\right\rangle_{\left(Y^{k}(S_{\Delta t,k})\right)^{*}\times Y^{k}(S_{\Delta t,k})} = \int_{\Omega} \frac{B_{\Delta t}^{k-1}}{\Delta t}\cdot b + \left[u_{\Delta t}^{k}\times B_{\Delta t}^{k-1} + \frac{1}{\sigma}J_{\Delta t}^{k}\right]\cdot \operatorname{curl} b \, dx \qquad \forall b \in Y^{k}\left(S_{\Delta t,k}\right). \tag{3.3.8}$$

Since for the divergence-free functions $B \in Y^k(S_{\Delta t,k})$, satisfying $B\cdot \mathrm{n}\,|_{\partial\Omega} = 0$, it is well known (cf. for example [107, Theorem 3.1]) that there exists a constant $c > 0$ such that

$$\|B\|_{H^1(\Omega)}^2 \leqslant c\left(\|B\|_{L^2(\Omega)}^2 + \|\operatorname{curl} B\|_{L^2(\Omega)}^2\right) \qquad \forall B \in Y^k\left(S_{\Delta t,k}\right),$$

we find another constant $c > 0$ such that

$$\langle A(B), B\rangle_{\left(Y^{k}(S_{\Delta t,k})\right)^{*}\times Y^{k}(S_{\Delta t,k})} \geqslant \frac{1}{\Delta t}\|B\|_{L^2(\Omega)}^2 + \frac{1}{\sigma\mu}\|\operatorname{curl} B\|_{L^2(\Omega)}^2 + \epsilon\|\nabla \operatorname{curl} B\|_{L^2(\Omega)}^2 \geqslant c\|B\|_{Y^k(S_{\Delta t,k})}^2$$

for all $B \in Y^k(S_{\Delta t,k})$, where the norm in $Y^k(S_{\Delta t,k})$ on the right-hand side is defined in (3.2.12). Consequently, the operator A is coercive on $Y^k(S_{\Delta t,k})$. Moreover, it is weakly continuous: Indeed, let $(B_l)_{l\in\mathbb{N}} \subset Y^k(S_{\Delta t,k})$ be an arbitrary sequence satisfying $B_l \rightharpoonup B$ in $Y^k(S_{\Delta t,k})$. Then the compactness of the embedding $H^1(\Omega) \subset L^4(\Omega)$ implies that $\operatorname{curl} B_l \to \operatorname{curl} B$ in $L^4(\Omega)$ and hence

$$\int_{\Omega} \frac{\epsilon}{\mu^2}\left|\operatorname{curl} B_l\right|^2 \operatorname{curl} B_l \cdot \operatorname{curl} b \, dx \to \int_{\Omega} \frac{\epsilon}{\mu^2}\left|\operatorname{curl} B\right|^2 \operatorname{curl} B \cdot \operatorname{curl} b \, dx \qquad \forall b \in Y^k\left(S_{\Delta t,k}\right).$$

It follows that

$$\langle A\left(B_l\right), b\rangle_{\left(Y^{k}(S_{\Delta t,k})\right)^{*}\times Y^{k}(S_{\Delta t,k})} \to \langle A(B), b\rangle_{\left(Y^{k}(S_{\Delta t,k})\right)^{*}\times Y^{k}(S_{\Delta t,k})} \qquad \forall b \in Y^k\left(S_{\Delta t,k}\right)$$

or, in other words, weak continuity of A on $Y^k(S_{\Delta t,k})$. This, together with the coercivity shown above, is sufficient to infer that A is also surjective, cf. [49, Theorem 1.2]. Thus there exists a solution $B_{\Delta t}^k \in Y^k(S_{\Delta t,k})$ to the problem (3.3.8). Finally, in order to prove that $B_{\Delta t}^k$ is the desired solution to the induction equation (3.2.8), it remains to show that it in fact satisfies the relation (3.3.8) also for all non-solenoidal test functions $b \in W^k(S_{\Delta t,k})$. This is achieved via the Helmholtz-decomposition in Lemma A.2.2 in the appendix, according to which any function $b \in W^k(S_{\Delta t,k})$ can be split up into

$$b = \nabla q + \operatorname{curl} w,$$

where the functions on the right-hand side satisfy

$$q \in H^1(\Omega), \quad w \in L^2(\Omega), \quad \operatorname{curl} w \in H^1(\Omega), \quad \operatorname{div} w = 0 \quad \text{in } \Omega, \quad (\operatorname{curl} w)\cdot \mathrm{n}|_{\partial\Omega} = 0. \tag{3.3.9}$$

In fact, the properties of w even show that

$$\operatorname{curl} w \in Y^k(S_{\Delta t,k}). \tag{3.3.10}$$

Indeed, since $\operatorname{curl}(\operatorname{curl} w) = \operatorname{curl}(b - \nabla q) = \operatorname{curl} b$ and $b \in W^k(S_{\Delta t,k})$ it immediately follows that

$$\operatorname{curl}(\operatorname{curl} w) \in H^1(\Omega), \qquad \operatorname{curl}(\operatorname{curl} w) = 0 \quad \text{in } S_{\Delta t,k}(k\Delta t)\bigcap \Omega.$$

This, together with the relations (3.3.9) and the fact that $\operatorname{div}(\operatorname{curl} w) = 0$, implies the desired inclusion (3.3.10). As a consequence of the inclusion (3.3.10), $\operatorname{curl} w$ constitutes an admissible test function in the equation (3.3.8). Using it as such we infer from the identities $\operatorname{curl}(\nabla q) = 0$ and $\operatorname{div} B_{\Delta t}^k = \operatorname{div} B_{\Delta t}^{k-1} = 0$ that $B_{\Delta t}^k$ indeed satisfies the relation (3.3.8) for any $b \in W^k(S_{\Delta t,k})$ and thus, as desired, the discrete induction equation (3.2.8).

Proposition 3.3.1. *Let all the assumptions of Theorem 3.1.1 be satisfied and let $\Delta t, \epsilon > 0$, $m \in \mathbb{N}$. Let further $g_{\Delta t}^k$ and $J_{\Delta t}^k$ be given by (3.2.16) for any $k = 0, ..., \frac{T}{\Delta t}$ and assume the regularized initial data $\rho_{0,m}$, $u_{0,m}$, $B_{0,m}$ to satisfy the conditions* (3.2.11). *Then, for all $k = 1, ..., \frac{T}{\Delta t}$, there exist functions*

$$\chi_{\Delta t,k} \in C\left([(k-1)\Delta t, k\Delta t]; L^p\left(\mathbb{R}^3\right)\right) \quad \forall 1 \leqslant p < \infty,$$
$$\rho_{\Delta t}^k \in \left\{\psi \in H^1(\Omega) : \ \underline{\rho} \leqslant \psi \leqslant \overline{\rho} \text{ in } \Omega\right\}, \quad u_{\Delta t}^k \in H_{0,\text{div}}^2(\Omega), \quad B_{\Delta t}^k \in Y^k\left(S_{\Delta t,k}\right)$$

which satisfy the variational equations (3.2.5)–(3.2.8) for all test functions $\Theta \in \mathcal{D}([(k-1)\Delta t, k\Delta t] \times \mathbb{R}^3)$, $\psi \in H^1(\Omega)$, $\phi \in H_{0,\text{div}}^2(\Omega)$ and $b \in W^k(S_{\Delta t,k})$ as well as the initial conditions (3.2.10), (3.2.11).

Remark 3.3.1. *For any fixed $s, t \in \mathbb{R}$ the mapping (3.3.3) is an isometry. Indeed, from $\Pi_{\Delta t}^{k-1}$ being a rigid velocity field and the ordinary differential equation (3.2.14), it follows that*

$$\frac{dl}{dt}\left|\eta_{\Delta t}^{\Pi_{\Delta t}^{k-1}}(s;t,x) - \eta_{\Delta t}^{\Pi_{\Delta t}^{k-1}}(s;t,y)\right|^2 = 0$$

for any $x, y \in \mathbb{R}^3$.

3.4 Limit passage with respect to $\Delta t \to 0$

Our next step is the return from the discrete in time system to a continuous system, i.e. the limit passage with respect to $\Delta t \to 0$. To this end we construct piecewise constant and piecewise affine interpolants of the discrete quantities, defined on the whole time interval $[0, T]$ instead of only in the discrete time points. More precisely, for all time independent functions $h_{\Delta t}^k$, defined on Ω for $k = 0, ..., \frac{T}{\Delta t}$, we define

$$h_{\Delta t}(t) := \left(\frac{t}{\Delta t} - (k-1)\right) h_{\Delta t}^k + \left(k - \frac{t}{\Delta t}\right) h_{\Delta t}^{k-1} \qquad \forall t \in ((k-1)\Delta t, k\Delta t], \quad k = 1, ..., \frac{T}{\Delta t}, \tag{3.4.1}$$
$$\overline{h}_{\Delta t}(t) := h_{\Delta t}^k \qquad \forall t \in ((k-1)\Delta t, k\Delta t], \quad k = 0, ..., \frac{T}{\Delta t}, \tag{3.4.2}$$
$$\overline{h}'_{\Delta t}(t) := h_{\Delta t}^{k-1} \qquad \forall t \in ((k-1)\Delta t, k\Delta t], \quad k = 1, ..., \frac{T}{\Delta t}. \tag{3.4.3}$$

We note that the piecewise affine interpolants are piecewise differentiable with respect to the time variable and their derivatives satisfy the relation

$$\partial_t h_{\Delta t}(t) = \frac{h_{\Delta t}^k - h_{\Delta t}^{k-1}}{\Delta t} \qquad \forall t \in ((k-1)\Delta t, k\Delta t), \quad k = 1, ..., \frac{T}{\Delta t}. \tag{3.4.4}$$

Regarding the solution to the transport equation on $[0, T]$, we glue together the already time-dependent functions $\chi_{\Delta t,k}$, defined on $[(k-1)\Delta t, k\Delta t] \times \Omega$ for $k = 1, ..., \frac{T}{\Delta t}$. More specifically, we set

$$\chi_{\Delta t}(t) := \chi_{\Delta t,k}(t) \qquad \forall t \in ((k-1)\Delta t, k\Delta t], \quad k = 1, ..., \frac{T}{\Delta t}.$$

Similarly, for the description of the position of the rigid body throughout the whole time interval $[0, T]$ we set

$$S_{\Delta t}(t) := S_{\Delta t,k}(t) = \left\{x \in \mathbb{R}^3 : \ \chi_{\Delta t}(t,x) = 1\right\} \qquad \forall t \in ((k-1)\Delta t, k\Delta t], \quad k = 1, ..., \frac{T}{\Delta t},$$
$$\overline{\overline{S}}_{\Delta t}(t) := S_{\Delta t,k}(k\Delta t) = \left\{x \in \mathbb{R}^3 : \ \overline{\chi}_{\Delta t}(t,x) = 1\right\} \qquad \forall t \in ((k-1)\Delta t, k\Delta t], \quad k = 0, ..., \frac{T}{\Delta t},$$
$$\overline{\overline{S}}'_{\Delta t}(t) := S_{\Delta t,k}((k-1)\Delta t) = \left\{x \in \mathbb{R}^3 : \ \overline{\chi}'_{\Delta t}(t,x) = 1\right\} \qquad \forall t \in ((k-1)\Delta t, k\Delta t], \quad k = 1, ..., \frac{T}{\Delta t}.$$

We point out that here we use the notation $\overline{\overline{S}}_{\Delta t}(t)$ instead of $\overline{S}_{\Delta t}(t)$ for the piecewise constant interpolants in order to avoid confusion with the notation for the closure of sets. The interpolations (3.4.1)–(3.4.3) allow us to express the time-independent equations on the Δt-level as continuous equations on the interval $[0,T]$. Indeed, for any function $\psi \in L^2(0,T;H^1(\Omega))$ and almost all $t \in [(k-1)\Delta t, k\Delta t]$, $k = 1, ..., \frac{T}{\Delta t}$, the discrete continuity equation (3.2.6) may be tested by $\psi(t)$. Integrating the resulting identity over $[(k-1)\Delta t, k\Delta t]$ and summing over all k we then infer, under exploitation of the identity (3.4.4), the relation

$$-\int_0^T \int_\Omega \partial_t \rho_{\Delta t} \psi \, dxdt = \int_0^T \int_\Omega \overline{u}'_{\Delta t} \cdot \nabla \overline{\rho}_{\Delta t} \psi + \epsilon \nabla \overline{\rho}_{\Delta t} \cdot \nabla \psi \, dxdt \tag{3.4.5}$$

for all $\psi \in L^2(0,T;H^1(\Omega))$. Arguing in the same way for the discrete momentum equation (3.2.7) we obtain the relation

$$\begin{aligned}\int_0^T \int_\Omega \partial_t \left(\rho_{\Delta t} u_{\Delta t}\right) \cdot \phi \, dxdt = \int_0^T \int_\Omega &\left[\operatorname{div}\left(\overline{\rho}_{\Delta t} \overline{u}'_{\Delta t} \otimes \overline{u}_{\Delta t}\right) - 2\nu \operatorname{div}\left(\mathbb{D}(\overline{u}_{\Delta t})\right) + \epsilon \left(\nabla \overline{u}_{\Delta t} \nabla \overline{\rho}_{\Delta t}\right)\right] \cdot \phi \\ &+ \epsilon \Delta \overline{u}_{\Delta t} \cdot \Delta \phi + \left[m \overline{\rho}'_{\Delta t} \overline{\chi}_{\Delta t} \left(\overline{u}'_{\Delta t} - \overline{\Pi}'_{\Delta t}\right) - \overline{\rho}'_{\Delta t} \overline{g}_{\Delta t}\right] \cdot \phi \\ &- \frac{1}{\mu} \left(\operatorname{curl} \overline{B}'_{\Delta t} \times \overline{B}'_{\Delta t}\right) \cdot \phi \, dxdt\end{aligned} \tag{3.4.6}$$

for all $\phi \in L^4(0,T;H^2_{0,\mathrm{div}}(\Omega))$. For the continuous version of the induction equation, we choose test functions

$$b \in L^4\left(0,T;H_0^{2,2}(\Omega)\right) \quad \text{such that} \quad b(t) \in W^k\left(S_{\Delta t,k}\right) \quad \text{for a.a. } t \in [(k-1)\Delta t, k\Delta t],\ k = 1, ..., \frac{T}{\Delta t}. \tag{3.4.7}$$

For any such b and almost all $t \in [(k-1)\Delta t, k\Delta t]$, $k = 1, ..., \frac{T}{\Delta t}$ the function $b(t)$ is an admissible test function in the discrete induction equation (3.2.8) at the time $k\Delta t$. Using it as such, integrating the resulting identity over $[(k-1)\Delta t, k\Delta t]$ and summing over the time indices we thus infer the identity

$$\begin{aligned}-\int_0^T \int_\Omega \partial_t B_{\Delta t} \cdot b \, dxdt = \int_0^T \int_\Omega &\left[\frac{1}{\sigma\mu} \operatorname{curl} \overline{B}_{\Delta t} - \overline{u}_{\Delta t} \times \overline{B}'_{\Delta t} + \frac{\epsilon}{\mu^2} \left|\operatorname{curl} \overline{B}_{\Delta t}\right|^2 \operatorname{curl} \overline{B}_{\Delta t} - \frac{1}{\sigma} \overline{J}_{\Delta t}\right] \cdot \operatorname{curl} b \\ &+ \epsilon \left(\nabla \operatorname{curl} \overline{B}_{\Delta t}\right) \cdot \left(\nabla \operatorname{curl} b\right) \, dxdt\end{aligned} \tag{3.4.8}$$

for any b as in (3.4.7). Finally, by the construction of $\chi_{\Delta t,k}$ in Proposition 3.3.1 it holds that $\chi_{\Delta t} \in C([0,T];L^p_{\mathrm{loc}}(\mathbb{R}^3))$, $1 \leqslant p < \infty$, and $\chi_{\Delta t}$ is the solution to

$$-\int_0^T \int_{\mathbb{R}^3} \chi_{\Delta t} \partial_t \Theta dxdt - \int_{\mathbb{R}^3} \chi_0 \Theta(0,x) \, dx = \int_0^T \int_{\mathbb{R}^3} \left(\chi_{\Delta t} \overline{\Pi}'_{\Delta t}\right) \cdot \nabla \Theta \, dxdt \tag{3.4.9}$$

for any $\Theta \in \mathcal{D}([0,T) \times \mathbb{R}^3)$. According to the transport theory by DiPerna and Lions, cf. [35, Thoerem III.2], this solution is unique and can be represented in the form

$$\chi_{\Delta t}(t,x) := \chi_0\left(\eta_{\Delta t}^{\overline{\Pi}'_{\Delta t}}(t;0,x)\right) \quad \text{for } t \in [0,T], \tag{3.4.10}$$

where $\eta_{\Delta t}^{\overline{\Pi}'_{\Delta t}}$ denotes the unique Carathéodory solution (cf. Theorem A.1.1 in the appendix) to the initial value problem

$$\frac{d\eta_{\Delta t}^{\overline{\Pi}'_{\Delta t}}(s;t,x)}{dt} = \overline{\Pi}'_{\Delta t}\left(t, \eta_{\Delta t}^{\overline{\Pi}'_{\Delta t}}(s;t,x)\right), \qquad \eta_{\Delta t}^{\overline{\Pi}'_{\Delta t}}(s;s,x) = x, \quad x \in \mathbb{R}^3, \quad s,t \in [0,T]. \tag{3.4.11}$$

By the uniqueness of this solution, the function $\eta_{\Delta t}^{\overline{\Pi}'_{\Delta t}}$ can also be written as a composition of the mappings (3.3.3). In particular, by the corresponding property of those functions (cf. Remark 3.3.1), the mapping

$$x \mapsto \eta_{\Delta t}^{\overline{\Pi}'_{\Delta t}}(s;t,x), \quad s,t \in [0,T]$$

is an isometry from $\mathbb{R}^3$ to $\mathbb{R}^3$.

3.4.1 Energy inequality on the Δt-level

In order to extract convergent subsequences for the limit passage with respect to $\Delta t \to 0$ we need to derive an energy estimate and therewith uniform bounds for the solution on the Δt-level. We begin by showing that the rigid velocity field $\Pi_{\Delta t}^{l-1}$ can be controlled in terms of the velocity field $u_{\Delta t}^{l-1}$ for any $l = 1, ..., \frac{T}{\Delta t}$. More precisely, writing

$$v_{\Delta t}^{l-1} := (u_G)_{\Delta t}^{l-1} - \omega_{\Delta t}^{l-1} \times a_{\Delta t}^{l-1} \qquad \text{and} \qquad w_{\Delta t}^{l-1} := \omega_{\Delta t}^{l-1}, \tag{3.4.12}$$

we aim at proving the estimate

$$\left|v_{\Delta t}^{l-1}\right|, \ \left|w_{\Delta t}^{l-1}\right| \leqslant c \left\|u_{\Delta t}^{l-1}\right\|_{L^2(\Omega)} \qquad \forall \Delta t > 0, \ l = 1, ..., \frac{T}{\Delta t} \tag{3.4.13}$$

with a constant $c > 0$ independent of Δt and l and consequently

$$\left\|\Pi_{\Delta t}^{l-1}\right\|_{L^2(\Omega)} = \left\|v_{\Delta t}^{l-1} + w_{\Delta t}^{l-1} \times (\cdot)\right\|_{L^2(\Omega)} \leqslant c\|u_{\Delta t}^{l-1}\|_{L^2(\Omega)} \qquad \forall \Delta t > 0, \ l = 1, ..., \frac{T}{\Delta t}. \tag{3.4.14}$$

Since in the final system we do not care about the behavior of the density and the velocity outside of Ω, we can extend these functions without loss of generality in an arbitrary way to the exterior domain and set

$$u_{\Delta t}^{l} = 0 \quad \text{in } \mathbb{R}^3 \backslash \Omega, \quad \rho_{\Delta t}^{l} = \underline{\rho} \quad \text{in } \mathbb{R}^3 \backslash \Omega \qquad \text{for all } l = 0, ..., \frac{T}{\Delta t}. \tag{3.4.15}$$

We consider some arbitrary $l = 1, ..., \frac{T}{\Delta t}$ and distinguish between two cases, the first one being

$$\operatorname{supp}\chi_{\Delta t}^{l-1} \bigcap \Omega \neq \emptyset. \tag{3.4.16}$$

Since $\chi_{\Delta t}^{l-1}$ is the characteristic function of the solid body, the motion of which is described via an isometry (cf. Remark 3.3.1), we can find a compact set $K \subset \mathbb{R}^3$, independent of Δt and l, which satisfies

$$\operatorname{supp}\chi_{\Delta t}^{l-1} \subset K \tag{3.4.17}$$

in this case. Due to the lower bound of the density in (3.2.2) and the extension (3.4.15) to the exterior domain, it further holds that

$$\int_{\mathbb{R}^3} \rho_{\Delta t}^{l} \chi_{\Delta t}^{l-1} \, dx \geqslant \underline{\rho} \, |S_0| > 0. \tag{3.4.18}$$

Combining this bound with the inclusion (3.4.17), we find a constant $c > 0$ independent of Δt and l such that

$$\left|a_{\Delta t}^{l-1}\right| = \left|\frac{\int_{\mathbb{R}^3} \rho_{\Delta t}^{l} \chi_{\Delta t}^{l-1} x \, dx}{\int_{\mathbb{R}^3} \rho_{\Delta t}^{l} \chi_{\Delta t}^{l-1} \, dx}\right| \leqslant \frac{1}{\underline{\rho} \, |S_0| |} \left|\int_K \rho_{\Delta t}^{l} x \, dx\right| \leqslant c \tag{3.4.19}$$

as well as

$$\left|(u_G)_{\Delta t}^{l-1}\right| = \left|\frac{\int_{\mathbb{R}^3} \rho_{\Delta t}^{l} \chi_{\Delta t}^{l-1} u_{\Delta t}^{l-1} \, dx}{\int_{\mathbb{R}^3} \rho_{\Delta t}^{l} \chi_{\Delta t}^{l-1} \, dx}\right| \leqslant \frac{1}{\underline{\rho} \, |S_0|} \left|\int_\Omega \rho_{\Delta t}^{l} u_{\Delta t}^{l-1}\right| \leqslant c \left\|u_{\Delta t}^{l-1}\right\|_{L^2(\Omega)}. \tag{3.4.20}$$

Further, we recall that the initial domain S_0 of the solid body is open and non-empty by the assumptions (3.1.8). Thus the fact that the motion of the body is characterized via an isometry guarantees the existence of a ball $B_r(\Delta t, l-1) \subset S_{\Delta t, l-1}((l-1)\Delta t)$ with radius $r > 0$ independent of Δt and l. Hence for any arbitrary $v \in \mathbb{R}^3$ we may estimate

$$\begin{aligned} v^T I_{\Delta t}^{l-1} v &= v^T \left[\int_{\mathbb{R}^3} \rho_{\Delta t}^{l} \chi_{\Delta t}^{l-1} \left(|x - a_{\Delta t}^{l-1}|^2 \mathrm{id} - \left(x - a_{\Delta t}^{l-1}\right) \otimes \left(x - a_{\Delta t}^{l-1}\right)\right) \, dx\right] v \\ &= \int_{\mathbb{R}^3} \rho_{\Delta t}^{l} \chi_{\Delta t}^{l-1} \left|(x - a_{\Delta t}^{l-1}) \times v\right|^2 dx \\ &\geqslant \underline{\rho} \int_{B_r(\Delta t, l-1)} \left|(x - a_{\Delta t}^{l-1}) \times v\right|^2 dx \\ &\geqslant \underline{\rho} \int_{B_r(0)} |y \times v|^2 dy \geqslant c|v|^2, \end{aligned} \tag{3.4.21}$$

for another constant $c > 0$ independent of Δt, l and v. This estimate implies invertibility of the matrix $I_{\Delta t}^{l-1}$ with an inverse $(I_{\Delta t}^{l-1})^{-1}$ bounded uniformly with respect to Δt and l. Therefore, taking into account the bound (3.4.19), we may estimate

$$\begin{aligned}\left|\omega_{\Delta t}^{l-1}\right| &= \left|\left(I_{\Delta t}^{l-1}\right)^{-1}\int_{\mathbb{R}^3}\rho_{\Delta t}^l\chi_{\Delta t}^{l-1}\left(x-a_{\Delta t}^{l-1}\right)\times u_{\Delta t}^{l-1}\,dx\right|\\ &\leqslant c\left|\int_K \rho_{\Delta t}^l\left(x-a_{\Delta t}^{l-1}\right)\times u_{\Delta t}^{l-1}\,dx\right| \leqslant c\left\|u_{\Delta t}^{l-1}\right\|_{L^2(\Omega)}. \end{aligned} \tag{3.4.22}$$

Consequently, combining the estimates (3.4.19), (3.4.20) and (3.4.22), we have shown the estimate (3.4.13) provided that the inequality (3.4.16) holds true. In the second case, i.e. if $\operatorname{supp}\chi_{\Delta t}^{l-1}\bigcap\Omega = \varnothing$, the extension (3.4.15) of $u_{\Delta t}^{l-1}$ by 0 in $\mathbb{R}^3\backslash\Omega$ implies that $(u_G)_{\Delta t}^{l-1} = 0$ and $\omega_{\Delta t}^{l-1} = 0$, so the estimate (3.4.13) is satisfied trivially. It follows that the desired estimates (3.4.13) and (3.4.14) indeed hold true for any $\Delta t > 0$ and any $l = 1, ..., \frac{T}{\Delta t}$. With the estimate (3.4.14) at hand we now proceed with the proof of the desired energy estimate. We fix some $k \in \left\{1, ..., \frac{T}{\Delta t}\right\}$ and test, for any $l \leqslant k$, the discrete induction equation (3.2.8) at the time $l\Delta t$ by $\frac{1}{\mu}B_{\Delta t}^l$. Under exploitation of the estimate

$$\begin{aligned}\int_\Omega \frac{B_{\Delta t}^l - B_{\Delta t}^{l-1}}{\Delta t}\cdot B_{\Delta t}^l\,dx &= \int_\Omega \frac{1}{2\Delta t}\left(\left|B_{\Delta t}^l\right|^2 + \left|B_{\Delta t}^l - B_{\Delta t}^{l-1}\right|^2 - \left|B_{\Delta t}^{l-1}\right|^2\right)\,dx\\ &\geqslant \int_\Omega \frac{1}{2\Delta t}\left(\left|B_{\Delta t}^l\right|^2 - \left|B_{\Delta t}^{l-1}\right|^2\right)\,dx \end{aligned} \tag{3.4.23}$$

this leads us to the inequality

$$\begin{aligned}&\int_\Omega \frac{1}{2\mu\Delta t}\left|B_{\Delta t}^l\right|^2 - \frac{1}{2\mu\Delta t}\left|B_{\Delta t}^{l-1}\right|^2 + \frac{1}{\sigma\mu^2}\left|\operatorname{curl}B_{\Delta t}^l\right|^2\,dx\\ \leqslant &-\int_\Omega \frac{\epsilon}{\mu^3}\left|\operatorname{curl}B_{\Delta t}^l\right|^4 + \frac{\epsilon}{\mu}\left|\nabla\operatorname{curl}B_{\Delta t}^l\right|^2 - \frac{1}{\mu}(u_{\Delta t}^l\times B_{\Delta t}^{l-1})\cdot\operatorname{curl}B_{\Delta t}^l - \frac{1}{\sigma\mu}J_{\Delta t}^l\cdot\operatorname{curl}B_{\Delta t}^l\,dx. \end{aligned} \tag{3.4.24}$$

We further test the discrete momentum equation (3.2.7) at the time $l\Delta t$ by $u_{\Delta t}^l$ and subtract the discrete continuity equation (3.2.6) at the time $l\Delta t$, tested by $\frac{1}{2}|u_{\Delta t}^l|^2$. Making use of the relations

$$\begin{aligned}&\int_\Omega \frac{\rho_{\Delta t}^l u_{\Delta t}^l - \rho_{\Delta t}^{l-1}u_{\Delta t}^{l-1}}{\Delta t}\cdot u_{\Delta t}^l dx - \int_\Omega \frac{\rho_{\Delta t}^l - \rho_{\Delta t}^{l-1}}{\Delta t}\frac{1}{2}\left|u_{\Delta t}^l\right|^2 dx\\ \geqslant &\int_\Omega \frac{1}{\Delta t}\rho_{\Delta t}^l\left|u_{\Delta t}^l\right|^2 - \frac{1}{2\Delta t}\rho_{\Delta t}^{l-1}\left|u_{\Delta t}^l\right|^2 - \frac{1}{2\Delta t}\rho_{\Delta t}^{l-1}\left|u_{\Delta t}^{l-1}\right|^2 dx - \int_\Omega \frac{\rho_{\Delta t}^l - \rho_{\Delta t}^{l-1}}{\Delta t}\frac{1}{2}\left|u_{\Delta t}^l\right|^2 dx\\ = &\int_\Omega \frac{1}{2\Delta t}\rho_{\Delta t}^l\left|u_{\Delta t}^l\right|^2 - \frac{1}{2\Delta t}\rho_{\Delta t}^{l-1}\left|u_{\Delta t}^{l-1}\right|^2 dx\end{aligned}$$

and

$$\int_\Omega \operatorname{div}\left(\rho_{\Delta t}^l u_{\Delta t}^{l-1}\otimes u_{\Delta t}^l\right)\cdot u_{\Delta t}^l\,dx - \int_\Omega u_{\Delta t}^{l-1}\cdot\nabla\rho_{\Delta t}^l\frac{1}{2}\left|u_{\Delta t}^l\right|^2\,dx = 0,$$

cf. (3.3.6), we then obtain the inequality

$$\begin{aligned}&\int_\Omega \frac{1}{2\Delta t}\rho_{\Delta t}^l\left|u_{\Delta t}^l\right|^2 - \frac{1}{2\Delta t}\rho_{\Delta t}^{l-1}\left|u_{\Delta t}^{l-1}\right|^2 + 2\nu\left|\nabla u_{\Delta t}^l\right|^2 + m\rho_{\Delta t}^{l-1}\chi_{\Delta t}^l\left(u_{\Delta t}^{l-1} - \Pi_{\Delta t}^{l-1}\right)\cdot u_{\Delta t}^l + \epsilon\left|\Delta u_{\Delta t}^l\right|^2\,dx\\ \leqslant &\int_\Omega \rho_{\Delta t}^{l-1}g_{\Delta t}^l\cdot u_{\Delta t}^l + \frac{1}{\mu}\left(\operatorname{curl}B_{\Delta t}^{l-1}\times B_{\Delta t}^{l-1}\right)\cdot u_{\Delta t}^l\,dx. \end{aligned} \tag{3.4.25}$$

We add up the inequalities (3.4.24) and (3.4.25), multiply the result by Δt and sum over all indices $l = 1, ..., k$. Under exploitation of the bounds for the density in (3.2.2), the bound (3.4.14) for the rigid velocity field $\Pi_{\Delta t}^{l-1}$ as well as Hölder's and Young's inequalities this yields the inequality

$$
\begin{aligned}
&\frac{1}{2}\left\|\sqrt{\rho_{\Delta t}^{k}}u_{\Delta t}^{k}\right\|_{L^2(\Omega)}^2+\frac{1}{2\mu}\left\|B_{\Delta t}^{k}\right\|_{L^2(\Omega)}^2+\Delta t\sum_{l=1}^{k}\Big[2\nu\left\|\nabla u_{\Delta t}^{l}\right\|_{L^2(\Omega)}^2+\epsilon\left\|\Delta u_{\Delta t}^{l}\right\|_{L^2(\Omega)}^2\\
&+\frac{1}{\sigma\mu^2}\left\|\mathrm{curl}B_{\Delta t}^{l}\right\|_{L^2(\Omega)}^2+\frac{\epsilon}{\mu^3}\left\|\mathrm{curl}B_{\Delta t}^{l}\right\|_{L^4(\Omega)}^4+\frac{\epsilon}{\mu}\left\|\nabla\,\mathrm{curl}\,B_{\Delta t}^{l}\right\|_{L^2(\Omega)}^2\Big]\\
\leqslant&\frac{1}{2}\left\|\sqrt{\rho_{0,m}}u_{0,m}\right\|_{L^2(\Omega)}^2+\frac{1}{2\mu}\left\|B_{0,m}\right\|_{L^2(\Omega)}^2+\Delta t\sum_{l=1}^{k}\int_\Omega -m\rho_{\Delta t}^{l-1}\chi_{\Delta t}^{l}(u_{\Delta t}^{l-1}-\Pi_{\Delta t}^{l-1})\cdot u_{\Delta t}^{l}+\rho_{\Delta t}^{l-1}g_{\Delta t}^{l}\cdot u_{\Delta t}^{l}\\
&+\frac{1}{\mu}\left(\mathrm{curl}B_{\Delta t}^{l-1}\times B_{\Delta t}^{l-1}\right)\cdot u_{\Delta t}^{l}+\frac{1}{\mu}\left(u_{\Delta t}^{l}\times B_{\Delta t}^{l-1}\right)\cdot\mathrm{curl}B_{\Delta t}^{l}+\frac{1}{\sigma\mu}J_{\Delta t}^{l}\cdot\mathrm{curl}\Delta t^{l}\,dx\\
\leqslant&\frac{1}{2}\overline{\rho}\left\|u_{0,m}\right\|_{L^2(\Omega)}^2+\frac{1}{2\mu}\left\|B_{0,m}\right\|_{L^2(\Omega)}^2+\Delta t\sum_{l=1}^{k}\Big[\frac{\overline{\rho}m}{2}\left\|u_{\Delta t}^{l-1}\right\|_{L^2(\Omega)}^2+\frac{\overline{\rho}m}{2}\left\|u_{\Delta t}^{l}\right\|_{L^2(\Omega)}^2+\frac{\overline{\rho}}{2}\left\|g_{\Delta t}^{l}\right\|_{L^2(\Omega)}^2\\
&+\frac{\overline{\rho}}{2}\left\|u_{\Delta t}^{l}\right\|_{L^2(\Omega)}^2+\frac{1}{\mu}\left\|\mathrm{curl}B_{\Delta t}^{l-1}\right\|_{L^4(\Omega)}\left\|B_{\Delta t}^{l-1}\right\|_{L^4(\Omega)}\left\|u_{\Delta t}^{l}\right\|_{L^2(\Omega)}\\
&+\frac{1}{\mu}\left\|\mathrm{curl}B_{\Delta t}^{l}\right\|_{L^4(\Omega)}\left\|B_{\Delta t}^{l-1}\right\|_{L^4(\Omega)}\left\|u_{\Delta t}^{l}\right\|_{L^2(\Omega)}+\frac{1}{2\sigma}\left\|J_{\Delta t}^{l}\right\|_{L^2(\Omega)}^2+\frac{1}{2\sigma\mu^2}\left\|\mathrm{curl}B_{\Delta t}^{l}\right\|_{L^2(\Omega)}^2\Big].
\end{aligned}
\tag{3.4.26}
$$

On the right-hand side of this inequality we estimate, due to the Gagliardo-Nirenberg inequality and the Young inequality,

$$
\begin{aligned}
\left\|B_{\Delta t}^{l-1}\right\|_{L^4(\Omega)}&\leqslant c\left\|B_{\Delta t}^{l-1}\right\|_{L^2(\Omega)}^{\frac{1}{4}}\left\|\nabla B_{\Delta t}^{l-1}\right\|_{L^2(\Omega)}^{\frac{3}{4}}+c\left\|B_{\Delta t}^{l-1}\right\|_{L^2(\Omega)}\\
&\leqslant\frac{c}{4}\left\|B_{\Delta t}^{l-1}\right\|_{L^2(\Omega)}+\frac{3c}{4}\left\|\nabla B_{\Delta t}^{l-1}\right\|_{L^2(\Omega)}+c\left\|B_{\Delta t}^{l-1}\right\|_{L^2(\Omega)}\leqslant c\left\|B_{\Delta t}^{l-1}\right\|_{H^1(\Omega)}.
\end{aligned}
$$

This allows us to control the $L^4(\Omega)$-norm of the divergence-free vector field $B_{\Delta t}^{l-1}$, satisfying $B_{\Delta t}^{l-1}\cdot \mathrm{n}|_{\partial\Omega}=0$, via the Poincaré-type estimate

$$
\left\|B_{\Delta t}^{l-1}\right\|_{L^4(\Omega)}\leqslant c\left\|B_{\Delta t}^{l-1}\right\|_{H^1(\Omega)}\leqslant c\left\|\mathrm{curl}B_{\Delta t}^{l-1}\right\|_{L^2(\Omega)}\leqslant c\left\|\mathrm{curl}B_{\Delta t}^{l-1}\right\|_{L^4(\Omega)},
\tag{3.4.27}
$$

see [107, Corollary 3.2]. Consequently we may further estimate

$$
\begin{aligned}
\frac{1}{\mu}\left\|\mathrm{curl}B_{\Delta t}^{l}\right\|_{L^4(\Omega)}\left\|B_{\Delta t}^{l-1}\right\|_{L^4(\Omega)}\left\|u_{\Delta t}^{l}\right\|_{L^2(\Omega)}&\leqslant\frac{1}{\mu}\left[c\frac{\sqrt{2}\sqrt{\epsilon}\mu}{\sqrt{2}\sqrt{\epsilon}\mu}\left\|\mathrm{curl}B_{\Delta t}^{l}\right\|_{L^4(\Omega)}\left\|\mathrm{curl}B_{\Delta t}^{l-1}\right\|_{L^4(\Omega)}\left\|u_{\Delta t}^{l}\right\|_{L^2(\Omega)}\right]\\
&\leqslant\frac{c^2\mu}{\epsilon}\left\|u_{\Delta t}^{l}\right\|_{L^2(\Omega)}^2+\frac{\epsilon}{4\mu^3}\left\|\mathrm{curl}B_{\Delta t}^{l-1}\right\|_{L^4(\Omega)}^2\left\|\mathrm{curl}B_{\Delta t}^{l}\right\|_{L^4(\Omega)}^2\\
&\leqslant\frac{c^2\mu}{\epsilon}\left\|u_{\Delta t}^{l}\right\|_{L^2(\Omega)}^2+\frac{\epsilon}{8\mu^3}\left\|\mathrm{curl}B_{\Delta t}^{l-1}\right\|_{L^4(\Omega)}^4+\frac{\epsilon}{8\mu^3}\left\|\mathrm{curl}B_{\Delta t}^{l}\right\|_{L^4(\Omega)}^4
\end{aligned}
$$

and in the same fashion

$$
\frac{1}{\mu}\left\|\mathrm{curl}B_{\Delta t}^{l-1}\right\|_{L^4(\Omega)}\left\|B_{\Delta t}^{l-1}\right\|_{L^4(\Omega)}\left\|u_{\Delta t}^{l}\right\|_{L^2(\Omega)}\leqslant\frac{c^2\mu}{\epsilon}\left\|u_{\Delta t}^{l}\right\|_{L^2(\Omega)}^2+\frac{\epsilon}{4\mu^3}\left\|\mathrm{curl}B_{\Delta t}^{l-1}\right\|_{L^4(\Omega)}^4.
$$

The latter two estimates allow us to absorb the terms depending on $B_{\Delta t}^{l}$, $l=1,...,k$, on the right-hand side of the inequality (3.4.26) into the left-hand side. For Δt sufficiently small the same is possible for the $L^2(\Omega)$-norm of $u_{\Delta t}^{k}$, since the density is bounded from below by $\underline{\rho}$ according to (3.2.2). Hence we infer the inequality

$$\begin{aligned}
&\frac{\rho}{4}\left\|u^k_{\Delta t}\right\|^2_{L^2(\Omega)} + \frac{1}{2\mu}\left\|B^k_{\Delta t}\right\|^2_{L^2(\Omega)} + \Delta t\sum_{l=1}^{k}\Bigg[2\nu\left\|\nabla u^l_{\Delta t}\right\|^2_{L^2(\Omega)} + \epsilon\left\|\Delta u^l_{\Delta t}\right\|^2_{L^2(\Omega)} \\
&+\frac{1}{2\sigma\mu^2}\left\|\mathrm{curl}B^l_{\Delta t}\right\|^2_{L^2(\Omega)} + \frac{\epsilon}{2\mu^3}\left\|\mathrm{curl}B^l_{\Delta t}\right\|^4_{L^4(\Omega)} + \frac{\epsilon}{\mu}\left\|\nabla\,\mathrm{curl}\,B^l_{\Delta t}\right\|^2_{L^2(\Omega)}\Bigg] \\
\leqslant&\frac{\overline{\rho}}{2}\left\|u_{0,m}\right\|^2_{L^2(\Omega)} + \frac{1}{2\mu}\left\|B_{0,m}\right\|^2_{L^2(\Omega)} + \frac{3\epsilon\Delta t}{8\mu^3}\left\|\mathrm{curl}B_{0,m}\right\|^4_{L^4(\Omega)} + \frac{\overline{\rho}m\Delta t}{2}\left\|u_{0,m}\right\|^2_{L^2(\Omega)} \\
&+\Delta t\sum_{l=1}^{k}\left[\frac{\overline{\rho}}{2}\left\|g^l_{\Delta t}\right\|^2_{L^2(\Omega)} + \frac{1}{2\sigma}\left\|J^l_{\Delta t}\right\|^2_{L^2(\Omega)}\right] \\
&+\Delta t\sum_{l=1}^{k-1}\left[\overline{\rho}m\left\|u^l_{\Delta t}\right\|^2_{L^2(\Omega)} + \frac{\overline{\rho}}{2}\left\|u^l_{\Delta t}\right\|^2_{L^2(\Omega)} + \frac{2c^2\mu}{\epsilon}\left\|u^l_{\Delta t}\right\|^2_{L^2(\Omega)}\right].
\end{aligned} \tag{3.4.28}$$

From the definition of the discretized external forcing terms $g^l_{\Delta t}$ and $J^l_{\Delta t}$ in (3.2.16) it immediately follows that the sum of their $L^2(\Omega)$-norms on the right-hand side of the inequality (3.4.28) is bounded uniformly with respect to Δt, cf. [99, Lemma 8.7]. Hence we may apply the discrete Gronwall estimate, cf. [99, (1.67)], to infer the desired energy estimate

$$\begin{aligned}
&\left\|u^k_{\Delta t}\right\|^2_{L^2(\Omega)} + \left\|B^k_{\Delta t}\right\|^2_{L^2(\Omega)} + \Delta t\sum_{l=1}^{k}\Bigg[\left\|\nabla u^l_{\Delta t}\right\|^2_{L^2(\Omega)} + \epsilon\left\|\Delta u^l_{\Delta t}\right\|^2_{L^2(\Omega)} \\
&+\left\|\mathrm{curl}\,B^l_{\Delta t}\right\|^2_{L^2(\Omega)} + \epsilon\left\|\mathrm{curl}\,B^l_{\Delta t}\right\|^4_{L^4(\Omega)} + \epsilon\left\|\nabla\,\mathrm{curl}\,B^l_{\Delta t}\right\|^2_{L^2(\Omega)}\Bigg] \leqslant c
\end{aligned} \tag{3.4.29}$$

for all $k = 1, ..., \frac{T}{\Delta t}$ and a constant $c = c(u_0, B_0, \overline{\rho}, \underline{\rho}, g, J, \sigma, \mu, \nu, m, T, \Omega) > 0$ independent of Δt and k. We supplement this energy estimate by a uniform bound for the density gradient: We choose an arbitrary time index $k = 1, ..., \frac{T}{\Delta t}$ and test the discrete continuity equation (3.2.6) at the time $l\Delta t$ by $\rho^l_{\Delta t}$ for any $l = 1, ..., k$. Because of the identity (3.3.2) (for $\rho = \rho^l_{\Delta t}$) and the estimate

$$\int_\Omega \frac{\rho^l_{\Delta t} - \rho^{l-1}_{\Delta t}}{\Delta t}\cdot\rho^l_{\Delta t}\,dx \geqslant \int_\Omega \frac{1}{2\Delta t}\left(\left|\rho^l_{\Delta t}\right|^2 - \left|\rho^{l-1}_{\Delta t}\right|^2\right)\,dx,$$

which holds true by the same argument as the corresponding estimate (3.4.23) for the magnetic induction, this yields the inequality

$$\int_\Omega \frac{1}{2\Delta t}\left(\left|\rho^l_{\Delta t}\right|^2 - \left|\rho^{l-1}_{\Delta t}\right|^2\right) + \epsilon\left|\nabla\rho^l_{\Delta t}\right|^2\,dx \leqslant 0.$$

We sum over all indices $l = 1, ..., k$ and infer the desired bound

$$\frac{1}{2}\left\|\rho^k_{\Delta t}\right\|^2_{L^2(\Omega)} + \Delta t\sum_{l=1}^{k}\epsilon\left\|\nabla\rho^l_{\Delta}\right\|^2_{L^2(\Omega)} \leqslant \frac{1}{2}\left\|\rho_{0,m}\right\|^2_{L^2(\Omega)}. \tag{3.4.30}$$

The uniform estimates (3.4.29) and (3.4.30) for our discrete solution translate to uniform bounds for the interpolated functions defined via the formulas (3.4.1)–(3.4.3). More precisely, these estimates imply the existence of a constant $c > 0$ independent of Δt such that

$$\epsilon^{\frac{1}{2}} \|\rho_{\Delta t}\|_{L^2(0,T;H^1(\Omega))} + \epsilon^{\frac{1}{2}} \|\overline{\rho}_{\Delta t}\|_{L^2(0,T;H^1(\Omega))} + \epsilon^{\frac{1}{2}} \|\overline{\rho}'_{\Delta t}\|_{L^2(0,T;H^1(\Omega))} \leqslant c, \tag{3.4.31}$$

$$\|u_{\Delta t}\|_{L^\infty(0,T;L^2(\Omega))} + \|\overline{u}_{\Delta t}\|_{L^\infty(0,T;L^2(\Omega))} + \|\overline{u}'_{\Delta t}\|_{L^\infty(0,T;L^2(\Omega))} \leqslant c, \tag{3.4.32}$$

$$\epsilon^{\frac{1}{2}} \|u_{\Delta t}\|_{L^2(0,T;H^2(\Omega))} + \epsilon^{\frac{1}{2}} \|\overline{u}_{\Delta t}\|_{L^2(0,T;H^2(\Omega))} + \epsilon^{\frac{1}{2}} \|\overline{u}'_{\Delta t}\|_{L^2(0,T;H^2(\Omega))} \leqslant c, \tag{3.4.33}$$

$$\|B_{\Delta t}\|_{L^\infty(0,T;L^2(\Omega))} + \|\overline{B}_{\Delta t}\|_{L^\infty(0,T;L^2(\Omega))} + \left\|\overline{B}'_{\Delta t}\right\|_{L^\infty(0,T;L^2(\Omega))} \leqslant c, \tag{3.4.34}$$

$$\|B_{\Delta t}\|_{L^2(0,T;H^1(\Omega))} + \|\overline{B}_{\Delta t}\|_{L^2(0,T;H^1(\Omega))} + \left\|\overline{B}'_{\Delta t}\right\|_{L^2(0,T;H^1(\Omega))} \leqslant c, \tag{3.4.35}$$

$$\epsilon^{\frac{1}{2}} \|\operatorname{curl} B_{\Delta t}\|_{L^2(0,T;H^1(\Omega))} + \epsilon^{\frac{1}{2}} \|\operatorname{curl} \overline{B}_{\Delta t}\|_{L^2(0,T;H^1(\Omega))} + \epsilon^{\frac{1}{2}} \left\|\operatorname{curl} \overline{B}'_{\Delta t}\right\|_{L^2(0,T;H^1(\Omega))} \leqslant c, \tag{3.4.36}$$

$$\epsilon^{\frac{1}{4}} \|\operatorname{curl} B_{\Delta t}\|_{L^4(Q)} + \epsilon^{\frac{1}{4}} \|\operatorname{curl} \overline{B}_{\Delta t}\|_{L^4(Q)} + \epsilon^{\frac{1}{4}} \left\|\operatorname{curl} \overline{B}'_{\Delta t}\right\|_{L^4(Q)} \leqslant c. \tag{3.4.37}$$

These estimates, together with the upper and lower bounds for the density in (3.2.2), allow us to apply the Banach-Alaoglu theorem and infer the existence of functions

$$\begin{aligned}
&\rho \in \left\{\psi \in L^2\left(0,T;H^1(\Omega)\right) \bigcap L^\infty(Q) :\ \underline{\rho} \leqslant \psi \leqslant \overline{\rho} \quad \text{a.e. in } Q\right\}, \\
&u \in L^\infty\left(0,T;L^2(\Omega)\right) \bigcap L^2\left(0,T;H^2_{0,\mathrm{div}}(\Omega)\right), && (3.4.38)\\
&B \in \left\{b \in L^\infty\left(0,T;L^2(\Omega)\right) \bigcap L^2\left(0,T;H^1_{\mathrm{div}}(\Omega)\right) :\ \operatorname{curl} B \in L^2\left(0,T;H^1(\Omega)\right),\ b \cdot \mathrm{n}|_{\partial\Omega} = 0\right\} && (3.4.39)
\end{aligned}$$

such that, possibly after the extraction of non-relabeled subsequences, it holds that

$$\begin{aligned}
&\overline{\rho}'_{\Delta t},\ \overline{\rho}_{\Delta t},\ \rho_{\Delta t} \overset{*}{\rightharpoonup} \rho \quad \text{in } L^\infty(Q), && \overline{\rho}'_{\Delta t},\ \overline{\rho}_{\Delta t},\ \rho_{\Delta t} \rightharpoonup \rho \quad \text{in } L^2\left(0,T;H^1(\Omega)\right),\\
&\overline{u}'_{\Delta t},\ \overline{u}_{\Delta t},\ u_{\Delta t} \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty\left(0,T;L^2(\Omega)\right), && \overline{u}'_{\Delta t},\ \overline{u}_{\Delta t},\ u_{\Delta t} \rightharpoonup u \quad \text{in } L^2\left(0,T;H^2(\Omega)\right), \quad (3.4.40)\\
&\overline{B}'_{\Delta t},\ \overline{B}_{\Delta t},\ B_{\Delta t} \overset{*}{\rightharpoonup} B \quad \text{in } L^\infty\left(0,T;L^2(\Omega)\right), && \overline{B}'_{\Delta t},\ \overline{B}_{\Delta t},\ B_{\Delta t} \rightharpoonup B \quad \text{in } L^2\left(0,T;H^1(\Omega)\right)
\end{aligned}$$

and

$$\operatorname{curl} \overline{B}'_{\Delta t},\ \operatorname{curl} \overline{B}_{\Delta t},\ \operatorname{curl} B_{\Delta t} \rightharpoonup \operatorname{curl} B \quad \text{in } L^2\left(0,T;H^1(\Omega)\right) \tag{3.4.41}$$

The fact that the weak limits of the different interpolants of the same discrete functions here coincide, respectively, is shown in Lemma A.3.1 in the appendix. The properties of the functions u and B stated in the inclusions (3.4.38) and (3.4.39) follow from the inclusions $\overline{u}_{\Delta t}(t) \in H^2_{0,\mathrm{div}}(\Omega)$ for all $t \in [0,T]$ and $\overline{B}_{\Delta t}(t) \in Y^k(S_{\Delta t,k})$ for all $t \in ((k-1)\Delta t, k\Delta t]$, $k = 1, ..., \frac{T}{\Delta t}$. Moreover, by their definition in (3.2.16), the discretized forcing terms $\overline{J}_{\Delta t}$ and $\overline{g}_{\Delta t}$ converge back to the original external forcing terms J and g (see Lemma A.3.2 (i) in the appendix),

$$\overline{J}_{\Delta t} \to J \quad \text{in } L^p(Q), \qquad \overline{g}_{\Delta t} \to g \quad \text{in } L^p(Q) \qquad \forall 1 \leqslant p < \infty. \tag{3.4.42}$$

3.4.2 Continuity equation

In order to pass to the limit in the continuity equation we first deduce strong convergence of the density. Testing the continuity equation (3.4.5) on the Δt-level by an arbitrary function $\psi \in L^2(0,T;H^1(\Omega))$ we estimate

$$\begin{aligned}
\left|\int_0^T \int_\Omega \partial_t \rho_{\Delta t} \psi \, dxdt\right| &= \left|\int_0^T \int_\Omega -\overline{\rho}_{\Delta t} \overline{u}'_{\Delta t} \cdot \nabla \psi + \epsilon \nabla \overline{\rho}_{\Delta t} \cdot \nabla \psi \, dxdt\right| \\
&\leqslant \overline{\rho} \left\|\overline{u}'_{\Delta t}\right\|_{L^2(Q)} \|\nabla \psi\|_{L^2(Q)} + \epsilon \|\nabla \overline{\rho}_{\Delta t}\|_{L^2(Q)} \|\nabla \psi\|_{L^2(Q)} \leqslant c \|\psi\|_{L^2(0,T;H^1(\Omega))}
\end{aligned}$$

for a constant $c > 0$ independent of Δt due to the uniform bounds (3.4.31) and (3.4.32). Exploiting the identity (3.4.4) for the discrete time derivative we have thus shown the dual estimate

$$\left\|\frac{\overline{\rho}_{\Delta t}(\cdot) - \overline{\rho}_{\Delta t}(\cdot - \Delta t)}{\Delta t}\right\|_{L^2(0,T;(H^1(\Omega))^*)} = \|\partial_t \rho_{\Delta t}\|_{L^2(0,T;(H^1(\Omega))^*)} \leqslant c. \tag{3.4.43}$$

For any function $\psi \in L^2(\Delta t, T; H^1(\Omega))$ this further yields the estimate

$$\int_{\Delta t}^{T}\int_\Omega \frac{\overline{\rho}'_{\Delta t}(t)-\overline{\rho}'_{\Delta t}(t-\Delta t)}{\Delta t}\psi(t)\ dxdt$$
$$=\int_0^{T-\Delta t}\int_\Omega \frac{\overline{\rho}_{\Delta t}(t)-\overline{\rho}_{\Delta t}(t-\Delta t)}{\Delta t}\psi(t+\Delta t)\ dxdt \leqslant c\|\psi\|_{L^2(\Delta t,T;H^1(\Omega))},$$

so a corresponding version of the dual estimate (3.4.43) also holds true for the time-lagging interpolant $\overline{\rho}'_{\Delta t}$,

$$\left\|\frac{\overline{\rho}'_{\Delta t}(\cdot)-\overline{\rho}'_{\Delta t}(\cdot-\Delta t)}{\Delta t}\right\|_{L^2(\Delta t,T;(H^1(\Omega))^*)} \leqslant c. \tag{3.4.44}$$

The dual estimates (3.4.43) and (3.4.44) together with the bounds (3.4.31) of both $\overline{\rho}_{\Delta t}$ and $\overline{\rho}'_{\Delta t}$ in $L^2(0,T;H^1(\Omega))$ give us the conditions for the discrete Aubin-Lions Lemma A.3.3 in the appendix and so we infer that

$$\overline{\rho}'_{\Delta t},\ \overline{\rho}_{\Delta t} \to \rho \quad \text{in } L^q\left((0,T)\times\mathbb{R}^3\right) \quad \forall 1\leqslant q<\infty, \qquad \underline{\rho}\leqslant\rho\leqslant\overline{\rho} \quad \text{a.e. in } [0,T]\times\mathbb{R}^3, \tag{3.4.45}$$

where the limit function ρ is extended by $\underline{\rho}$ in $\mathbb{R}^3\backslash\Omega$. As another consequence of the dual estimate (3.4.43) we may further assume that

$$\partial_t\rho_{\Delta t} \overset{*}{\rightharpoonup} \partial_t\rho \quad \text{in } L^2\left(0,T;\left(H^1(\Omega)\right)^*\right). \tag{3.4.46}$$

The convergences (3.4.45), (3.4.46) allow us to pass to the limit in the continuity equation (3.4.5) in two ways, producing the two identities

$$-\int_0^T\int_\Omega \rho\partial_t\psi\ dxdt - \int_\Omega \rho_{0,m}\psi(0,x)\ dx = \int_0^T\int_\Omega (\rho u)\cdot\nabla\psi + \epsilon\rho\Delta\psi\ dxdt \qquad \forall\psi\in\mathcal{D}([0,T)\times\Omega). \tag{3.4.47}$$

and

$$\int_0^\tau\int_\Omega \partial_t\rho\psi - (\rho u)\cdot\nabla\psi + \epsilon\nabla\rho\cdot\nabla\psi\ dxdt = 0 \qquad \forall\psi\in L^2(0,T;H^1(\Omega)),\ \tau\in[0,T]. \tag{3.4.48}$$

While during the limit passage with respect to $\epsilon\to 0$ we will proceed working with the continuity equation in the form (3.4.47), we can use its form (3.4.48) to show strong convergence of the density gradient in the present limit passage, which will be required for passing to the limit in the momentum equation in Section 3.4.5 below. More specifically, we test (3.4.48) by ρ, which yields the identity

$$\|\rho(\tau)\|^2_{L^2(\Omega)} + 2\epsilon\int_0^\tau\int_\Omega |\nabla\rho|^2\ dxdt = \|\rho_{0,m}\|^2_{L^2(\Omega)} \tag{3.4.49}$$

for any $\tau\in[0,T]$. We want to compare this identity to the corresponding relation (3.4.30) on the Δt-level. To this end we first note that this relation can be expressed in the form

$$\left\|\rho^k_{\Delta t}\right\|^2_{L^2(\Omega)} + 2\epsilon\Delta t\sum_{l=1}^{k-1}\left\|\nabla\rho^l_{\Delta t}\right\|^2_{L^2(\Omega)} + 2\epsilon(\Delta t - s)\left\|\nabla\rho^k_{\Delta t}\right\|^2_{L^2(\Omega)} \leqslant \|\rho_{0,m}\|^2_{L^2(\Omega)} \quad \forall k=1,...,\frac{T}{\Delta t},\ s\in[0,\Delta t).$$

As $\overline{\rho}_{\Delta t}=\rho^k_{\Delta t}$ on $((k-1)\Delta t, k\Delta t]$, we may write $\|\rho^k_{\Delta t}\|^2_{L^2(\Omega)} = \|\overline{\rho}_{\Delta t}(k\Delta t - s)\|^2_{L^2(\Omega)}$ and so this estimate reads

$$\|\overline{\rho}_{\Delta t}(k\Delta t-s)\|^2_{L^2(\Omega)} + 2\epsilon\int_0^{k\Delta t-s}\int_\Omega |\nabla\overline{\rho}_{\Delta t}|^2 dxdt \leqslant \|\rho_{0,m}\|^2_{L^2(\Omega)} \qquad \forall k=1,...,\frac{T}{\Delta t},\ s\in[0,\Delta t).$$

Since each $\tau\in(0,T]$ can be written as $\tau = k\Delta t - s$ for certain $k=1,...,\frac{T}{\Delta t}$ and $s\in[0,\Delta t)$ we infer that

$$\|\overline{\rho}_{\Delta t}(\tau)\|^2_{L^2(\Omega)} + 2\epsilon\int_0^\tau\int_\Omega |\nabla\overline{\rho}_{\Delta t}|^2\ dxdt \leqslant \|\rho_{0,m}\|^2_{L^2(\Omega)} \qquad \forall\tau\in[0,T]. \tag{3.4.50}$$

We subtract the identity (3.4.49) from (3.4.50) and infer that, since the strong convergence (3.4.45) of $\overline{\rho}_{\Delta t}$ implies convergence pointwise almost everywhere for a subsequence,

$$\int_0^\tau \int_\Omega |\nabla \rho|^2 \, dxdt \geqslant \lim_{\Delta t \to 0} \int_0^\tau \int_\Omega |\nabla \overline{\rho}_{\Delta t}|^2 \, dxdt \qquad \text{for a.a. } \tau \in [0,T]. \tag{3.4.51}$$

On the other hand, for each τ for which the inequality (3.4.51) holds true we can exploit the weak lower semicontinuity of the norm in $L^2(Q(\tau)) = L^2((0,\tau)\times\Omega)$ to find another subsequence and some value $z_1 = z_1(\tau) \geqslant 0$ satisfying

$$\|\nabla \overline{\rho}_{\Delta t}\|^2_{L^2(Q(\tau))} \to z_1 \geqslant \|\nabla \rho\|^2_{L^2(Q(\tau))} \,. \tag{3.4.52}$$

A comparison between the inequalities (3.4.51) and (3.4.52) implies, for almost all $\tau \in [0,T]$, the existence of a subsequence satisfying

$$\|\nabla \overline{\rho}_{\Delta t}\|_{L^2(Q(\tau))} \to \|\nabla \rho\|_{L^2(Q(\tau))} \,.$$

Since weak convergence in $L^2(Q(\tau))$ combined with convergence of the $L^2(Q(\tau))$-norm implies strong $L^2(Q(\tau))$-convergence, we thus infer from a diagonal argument that

$$\nabla \overline{\rho}_{\Delta t} \to \nabla \rho \quad \text{in } L^2\left(Q(\tau)\right) \qquad \text{for a.a. } \tau \in [0,T]. \tag{3.4.53}$$

Finally, for the later use in the limit passage with respect to $\epsilon \to 0$ in Section 3.5, we show that the density ρ on the ϵ-level satisfies a regularized and integrated version of the renormalized continuity equation (3.1.22). To this end we test the continuity equation (3.4.5) on the Δt-level by $\chi_{[0,\tau]}\beta'(\overline{\rho}_{\Delta t})$ for an arbitrary convex function $\beta \in C^\infty([\underline{\rho},\overline{\rho}])$ and the characteristic function $\chi_{[0,\tau]}$ of the interval $[0,\tau]$, $\tau \in [0,T]$. Under exploitation of the convexity of β and the fact that $\overline{u}'_{\Delta t}$ is divergence-free, this procedure leads to the inequality

$$\begin{aligned}\int_0^\tau \int_\Omega \partial_t \rho_{\Delta t} \beta'(\overline{\rho}_{\Delta t}) \, dx &= -\int_0^\tau \int_\Omega \overline{u}'_{\Delta t} \cdot \nabla \overline{\rho}_{\Delta t} \beta'(\overline{\rho}_{\Delta t}) \, dxdt - \int_0^\tau \int_\Omega \epsilon \nabla \overline{\rho}_{\Delta t} \nabla \beta'(\overline{\rho}_{\Delta t}) \, dxdt \\ &= -\int_0^\tau \int_\Omega \epsilon \left|\nabla \overline{\rho}_{\Delta t}\right|^2 \beta''(\overline{\rho}_{\Delta t}) \, dxdt \leqslant 0 \end{aligned} \tag{3.4.54}$$

for any $\tau \in [0,T]$. Since the smooth function β has bounded derivatives on the compact interval $[\underline{\rho},\overline{\rho}]$, the strong convergence of $\overline{\rho}_{\Delta t}$ in $L^2(0,\tau;H^1(\Omega))$ (cf. (3.4.45), (3.4.53)) shows that

$$\begin{aligned} \beta''(\overline{\rho}_{\Delta t}) &\overset{*}{\rightharpoonup} \beta''(\rho) \quad \text{in } L^\infty(Q), \\ \beta'(\overline{\rho}_{\Delta t}) &\to \beta'(\rho) \quad \text{in } L^2\left(0,\tau;H^1(\Omega)\right) \qquad \text{for a.a. } \tau \in [0,T]. \end{aligned}$$

Combined with the weak-* convergence (3.4.46) of $\partial_t \rho_{\Delta t}$ these convergences are sufficient to pass to the limit in the relation (3.4.54) and infer that

$$\int_\Omega \beta(\rho(\tau)) \, dx - \int_\Omega \beta(\rho_{0,m}) \, dx = \int_0^\tau \int_\Omega \partial_t \beta(\rho) \, dxdt = -\int_0^\tau \int_\Omega \epsilon \beta''(\rho)|\nabla \rho|^2 \, dxdt \leqslant 0 \tag{3.4.55}$$

for almost all $\tau \in [0,T]$ and any convex function $\beta \in C^\infty([\underline{\rho},\overline{\rho}])$.

3.4.3 Transport equation

Recalling the relations (3.4.12) and the estimates (3.4.13) for the individual components of $\overline{\Pi}'_{\Delta t}$ as well as the $L^\infty(0,T;L^2(\Omega))$-bound (3.4.32) of $\overline{u}'_{\Delta t}$ we can write

$$\overline{\Pi}'_{\Delta t}(t,x) = \overline{v}'_{\Delta t}(t) + \overline{w}'_{\Delta t}(t) \times x, \quad \left|\overline{v}'_{\Delta t}(t)\right|, \ \left|\overline{w}'_{\Delta t}(t)\right| \leqslant c \left\|\overline{u}'_{\Delta t}(t)\right\|_{L^2(\Omega)} \leqslant c \qquad \forall t \in [0,T], \ x \in \Omega$$

for a constant $c > 0$ independent of Δt and t. This, in combination with the fact that $\chi_{\Delta t}$ and $\eta_{\Delta t}^{\overline{\Pi}'_{\Delta t}}$ solve the transport equation (3.4.9) and the initial value problem (3.4.11) respectively guarantees us the conditions (A.4.11)–(A.4.13) of Lemma A.4.2 in the appendix. From this and Remark A.4.2 we infer the existence of a function $\Pi \in L^\infty(0,T;W^{1,\infty}_{\text{loc}}(\mathbb{R}^3))$ satisfying

$$\overline{\Pi}'_{\Delta t} \overset{*}{\rightharpoonup} \Pi \quad \text{in } L^\infty\left(0,T;W^{1,\infty}_{\text{loc}}(\mathbb{R}^3)\right), \tag{3.4.56}$$

$$\eta^{\overline{\Pi}'_{\Delta t}}_{\Delta t} \to \eta^{\Pi} \quad \text{in } C\left([0,T]\times[0,T];C_{\text{loc}}\left(\mathbb{R}^3\right)\right), \tag{3.4.57}$$

$$\chi_{\Delta t} \to \chi \quad \text{in } C\left([0,T];L^p\left(\mathbb{R}^3\right)\right) \quad \forall 1 \leqslant p < \infty, \qquad \chi(t,x) = \chi_0\left(\eta^{\Pi}(t;0,x)\right), \tag{3.4.58}$$

where η^Π and χ denote the unique solutions to the initial value problem

$$\frac{d\eta^\Pi(s;t,x)}{dt} = \Pi\left(t,\eta^\Pi(s;t,x)\right), \qquad \eta^\Pi(s;s,x) = x$$

for all $x \in \mathbb{R}^3$, $s,t \in [0,T]$ and the transport equation

$$-\int_0^T\int_{\mathbb{R}^3} \chi\partial_t\Theta dxdt - \int_{\mathbb{R}^3}\chi_0\Theta(0,x)\;dx = \int_0^T\int_{\mathbb{R}^3}(\chi\Pi)\cdot\nabla\Theta\;dxdt$$

for all $\Theta \in \mathcal{D}([0,T)\times\mathbb{R}^3)$ respectively. In particular, from the strong convergence (3.4.58) of the continuous in time function $\chi_{\Delta t}$ we further conclude convergence of the associated piecewise constant functions $\overline{\chi}_{\Delta t}$ and $\overline{\chi}'_{\Delta t}$,

$$\overline{\chi}_{\Delta t},\;\overline{\chi}'_{\Delta t} \to \chi \quad \text{in } C\left([0,T];L^p\left(\mathbb{R}^3\right)\right) \quad \forall 1 \leqslant p < \infty, \tag{3.4.59}$$

cf. Lemma A.3.2 (ii) in the appendix. As a consequence of the strong convergences (3.4.45) of the density and (3.4.59) of the characteristic function, the limit function Π in the convergence (3.4.56) can be identified as the rigid projection $\Pi_{[\chi,\rho,u]}$ (defined in (3.1.6)) of the velocity field u, which can be seen in the following way: We begin by noting that, since the density ρ is extended by $\underline{\rho}$ in $\mathbb{R}^3\backslash\Omega$,

$$\int_{\mathbb{R}^3}\rho(t)\chi(t)dx \geqslant \underline{\rho}\,|S_0| > 0 \qquad \text{for a.a. } t \in [0,T]. \tag{3.4.60}$$

From the weak-$*$ convergence (3.4.40) of the velocity field and the strong convergences (3.4.45) and (3.4.59) of the density and the characteristic function, respectively, we conclude that

$$\int_{\mathbb{R}^3}\overline{\rho}_{\Delta t}\overline{\chi}'_{\Delta t}\;\overline{u}'_{\Delta t}\;dx \overset{*}{\rightharpoonup} \int_{\mathbb{R}^3}\rho\chi\;u\;dx \quad \text{in } L^\infty(0,T),$$

$$\int_{\mathbb{R}^3}\overline{\rho}_{\Delta t}\overline{\chi}'_{\Delta t}\;x\;dx \to \int_{\mathbb{R}^3}\rho\chi\;x\;dx \quad \text{in } L^p(0,T) \qquad \forall 1 \leqslant p < \infty,$$

$$\int_{\mathbb{R}^3}\overline{\rho}_{\Delta t}\overline{\chi}'_{\Delta t}\;dx \to \int_{\mathbb{R}^3}\rho\chi\;dx \quad \text{in } L^p(0,T) \qquad \forall 1 \leqslant p < \infty.$$

These convergences, together with the bounds (3.4.18) and (3.4.60) away from 0 moreover imply that

$$\overline{(u_G)}'_{\Delta t} = \frac{\int_{\mathbb{R}^3}\overline{\rho}_{\Delta t}\overline{\chi}'_{\Delta t}\overline{u}'_{\Delta t}\;dx}{\int_{\mathbb{R}^3}\overline{\rho}_{\Delta t}\overline{\chi}'_{\Delta t}\;dx} \overset{*}{\rightharpoonup} \frac{\int_{\mathbb{R}^3}\rho\chi u\;dx}{\int_{\mathbb{R}^3}\rho\chi\;dx} = (u_G)_{[\chi,\rho,u]} \quad \text{in } L^\infty(0,T) \tag{3.4.61}$$

$$\overline{a}'_{\Delta t} = \frac{\int_{\mathbb{R}^3}\overline{\rho}_{\Delta t}\overline{\chi}'_{\Delta t}x\;dx}{\int_{\mathbb{R}^3}\overline{\rho}_{\Delta t}\overline{\chi}'_{\Delta t}\;dx} \to \frac{\int_{\mathbb{R}^3}\rho\chi x\;dx}{\int_{\mathbb{R}^3}\rho\chi\;dx} = a_{[\chi,\rho]} \quad \text{in } L^p(0,T) \qquad \forall 1 \leqslant p < \infty. \tag{3.4.62}$$

For the matrix

$$\overline{I}'_{\Delta t} = \int_{\mathbb{R}^3}\overline{\rho}_{\Delta t}\overline{\chi}'_{\Delta t}\left(|x-\overline{a}'_{\Delta t}|^2\text{id} - \left(x-\overline{a}'_{\Delta t}\right)\otimes\left(x-\overline{a}'_{\Delta t}\right)\right)\;dx,$$

the convergence (3.4.62) then further implies that

$$\overline{I}'_{\Delta t} \to I_{[\chi,\rho]} = \int_{\mathbb{R}^3}\rho\chi\left(|x-a_{[\chi,\rho]}|^2\text{id} - \left(x-a_{[\chi,\rho]}\right)\otimes\left(x-a_{[\chi,\rho]}\right)\right)\;dx \quad \text{in } L^p\left(0,T\right) \tag{3.4.63}$$

for any $1 \leqslant p < \infty$. This convergence allows us to infer also convergence of the inverse $(\overline{I}'_{\Delta t})^{-1}$ of $\overline{I}'_{\Delta t}$. Indeed, we recall the bound (3.4.21), which implies boundedness of the eigenvalues of $\overline{I}'_{\Delta t}(t)$ away from zero, uniformly with respect to Δt and t. We thus find a constant $c > 0$, independent of Δt and t, such that

$$\left| \left(\overline{I}'_{\Delta t}(t)\right)^{-1} - \left(I_{[\chi,\rho]}(t)\right)^{-1} \right| \leqslant c \left| \overline{I}'_{\Delta t}(t) - I_{[\chi,\rho]}(t) \right| \qquad \forall t \in [0,T].$$

Consequently, the convergence (3.4.63) shows that

$$\left(\overline{I}'_{\Delta t}\right)^{-1} \to \left(I_{[\chi,\rho]}\right)^{-1} \quad \text{in } L^p(0,T) \qquad \forall 1 \leqslant p < \infty.$$

In combination with the convergence (3.4.62) this yields

$$\begin{aligned} \overline{\omega}'_{\Delta t} &= \left(\overline{I}'_{\Delta t}\right)^{-1} \int_{\mathbb{R}^3} \overline{\rho}_{\Delta t} \overline{\chi}'_{\Delta t} \left(x - \overline{a}'_{\Delta t}\right) \times \overline{u}'_{\Delta t} \, dx \\ &\to I^{-1}_{[\chi,\rho]} \int_{\mathbb{R}^3} \rho \chi \left(\left(x - a_{[\chi,\rho]}\right) \times u \right) \, dx = \omega_{[\chi,\rho,u]} \quad \text{in } L^p(0,T) \qquad \forall 1 \leqslant p < \infty. \end{aligned} \tag{3.4.64}$$

Now the convergences (3.4.61), (3.4.62) and (3.4.64) of the individual components of the rigid velocity field $\overline{\Pi}'_{\Delta t}$ indeed allow us to identify its limit, defined by the convergence (3.4.56), as

$$\Pi = (u_G)_{[\chi,\rho,u]} + \omega_{[\chi,\rho,u]} \times \left(x - a_{[\chi,\rho]}\right) = \Pi_{[\chi,\rho,u]}. \tag{3.4.65}$$

We conclude this section by exploiting the uniform convergence (3.4.57) to show, for any $\kappa > 0$, the existence of some value $\delta(\kappa) > 0$ such that

$$(S(t))_\kappa \subset \overline{\overline{S}}_{\Delta t}(t) \subset (S\,(t))^\kappa \qquad \forall t \in [0,T],\ \Delta t < \delta(\kappa), \tag{3.4.66}$$

where

$$S(t) := \eta^{\Pi_{[\chi,\rho,u]}}\,(0;t,S_0) = \left\{x \in \mathbb{R}^3 :\ \chi(t,x) = 1\right\}$$

and the κ-kernel $(S(t))_\kappa$ as well as the κ-neighborhood $(S(t))^\kappa$ of $S(t)$ are defined according to the formulas (3.1.5) and (3.1.4), respectively. Indeed, any point of $S_{\Delta t}(\tau)$, $\tau \in [0,T]$, can be represented in the form $\eta_{\Delta t}^{\overline{\Pi}'_{\Delta t}}(0;\tau,x)$ for some point $x \in S_0$ due to the formula (3.4.10) for the characteristic function of the solid body. The uniform convergence (3.4.57) implies the existence of $\delta(\kappa) > 0$ such that

$$\left| \eta_{\Delta t}^{\overline{\Pi}'_{\Delta t}}\,(0;\tau,x) - \eta^{\Pi_{[\chi,\rho,u]}}\,(0;t,x) \right| < \kappa \quad \text{for all } \Delta t < \delta(\kappa),\ \tau,t \in [0,T] \text{ with } |\tau - t| < \delta(\kappa) \text{ and } x \in S.$$

Since $\eta^{\Pi_{[\chi,\rho,u]}}(0;t,x) \in S(t)$ this shows that

$$S_{\Delta t}\,(\tau) \subset (S\,(t))^\kappa \qquad \text{for all } \Delta t < \delta(\kappa) \text{ and } \tau,t \in [0,T] \text{ with } |\tau - t_0| < \delta(\kappa).$$

Hence, the second inclusion in the relation (3.4.66) follows from the fact that for any $t \in [0,T]$ there exists some $\tau \in [0,T]$ with $|\tau - t| \leqslant \Delta t < \delta(\kappa)$ such that $\overline{\overline{S}}_{\Delta t}(t) = S_{\Delta t}(\tau)$. The first inclusion in (3.4.66) follows from a similar argument.

3.4.4 Induction equation

Before passing to the limit in the induction equation, we show that the limit B of the discrete magnetic induction is again curl-free in the solid domain. Indeed, for any interval $I \subset (0,T)$ and any ball $U \subset \mathbb{R}^3$ such that $\overline{I \times U} \subset Q^s(S) \bigcap Q$, it holds that $\overline{I \times U} \subset Q^s(\overline{\overline{S}}'_{\Delta t}) \bigcap Q$ for all sufficiently small $\Delta t > 0$ due to the first inclusion in (3.4.66). It follows that

$$\operatorname{curl} B = \lim_{\Delta t \to 0} \operatorname{curl} \overline{B}_{\Delta t} = 0 \qquad \text{a.e. in } \overline{I \times U} \text{ and hence in } Q^s(S) \bigcap Q. \tag{3.4.67}$$

Next, in order to obtain weak convergence of all quantities involved in the induction equation, we improve the uniform bounds available for $\overline{u}_{\Delta t}$ via an interpolation between its bounds (3.4.32), (3.4.33) in $L^\infty(0,T;L^2(\Omega))$ and $L^2(0,T;H^2(\Omega))$,

$$\begin{aligned} \epsilon^{\frac{1}{4}} \left\| \overline{u}_{\Delta t} \right\|_{L^4(Q)} \leqslant & \epsilon^{\frac{1}{4}} \left[\int_0^T \left\| \overline{u}_{\Delta t}(t) \right\|_{L^\infty(\Omega)}^{\frac{4}{2}} \left\| \overline{u}_{\Delta t}(t) \right\|_{L^2(\Omega)}^{\frac{4}{2}} dt \right]^{\frac{1}{4}} \\ \leqslant & \epsilon^{\frac{1}{4}} \left\| \overline{u}_{\Delta t} \right\|_{L^2(0,T;L^\infty(\Omega))}^{\frac{1}{2}} \left\| \overline{u}_{\Delta t} \right\|_{L^\infty(0,T;L^2(\Omega))}^{\frac{1}{2}} \\ \leqslant & c\epsilon^{\frac{1}{4}} \left\| \overline{u}_{\Delta t} \right\|_{L^2(0,T;H^2(\Omega))}^{\frac{1}{2}} \left\| \overline{u}_{\Delta t} \right\|_{L^\infty(0,T;L^2(\Omega))}^{\frac{1}{2}} \leqslant c \end{aligned} \tag{3.4.68}$$

for a constant $c > 0$ independent of Δt. A corresponding bound for the magnetic induction is obtained from the Poincaré-type estimate (3.4.27) and the uniform bound (3.4.37),

$$\epsilon^{\frac{1}{4}} \left\| \overline{B}'_{\Delta t} \right\|_{L^4(Q)} \leqslant \epsilon^{\frac{1}{4}} c \left\| \operatorname{curl} \overline{B}'_{\Delta t} \right\|_{L^4(Q)} \leqslant c. \tag{3.4.69}$$

The latter two bounds in combination with the bound (3.4.37) and the Hölder inequality allow us to find functions $z \in L^{\frac{4}{3}}(Q)$, and $z_2, z_3 \in L^2(Q)$ such that, possibly after the extraction of a subsequence,

$$\epsilon \left| \operatorname{curl}\overline{B}_{\Delta t} \right|^2 \operatorname{curl}\overline{B}_{\Delta t} \rightharpoonup \epsilon z \quad \text{in } L^{\frac{4}{3}}(Q), \tag{3.4.70}$$

$$\overline{u}_{\Delta t} \times \overline{B}'_{\Delta t} \rightharpoonup z_2 \quad \text{in } L^2(Q), \tag{3.4.71}$$

$$\operatorname{curl}\overline{B}'_{\Delta t} \times \overline{B}'_{\Delta t} \rightharpoonup z_3 \quad \text{in } L^2(Q), \tag{3.4.72}$$

$$\operatorname{curl}\overline{B}_{\Delta t} \times \overline{B}'_{\Delta t} \rightharpoonup z_4 \quad \text{in } L^2(Q). \tag{3.4.73}$$

While the limit function z in these convergences does not need to be further specified as it will vanish from the system after the limit passage with respect to $\epsilon \to 0$, we need to investigate the identities of the limit functions z_2 and z_3 more precisely. We do so in the solid domain and the fluid domain separately, beginning with the solid domain. Here it holds that, exactly as in the derivation of the curl-free condition (3.4.67) of B,

$$z_3 = z_4 = 0 = \operatorname{curl}B \times B \quad \text{a.e. in } Q^s(S) \bigcap Q. \tag{3.4.74}$$

For the function z_2, which only appears in a product with the curl of the test functions $b \in Y(S)$ in the induction equation, it suffices to remark that

$$z_2 \cdot \operatorname{curl}b = 0 = (u \times B) \cdot \operatorname{curl}b \quad \text{a.e. in } Q^s(S) \bigcap Q \qquad \forall b \in Y(S). \tag{3.4.75}$$

In the fluid region the identification of z_2 and z_3 is more involved and can only be achieved by establishing strong convergence of $\overline{B}'_{\Delta t}$ in a certain sense. To this end we fix some value $\kappa > 0$ and choose an arbitrary function $b \in L^4(0,T;H_0^2(\Omega))$ satisfying

$$\operatorname{curl} b(t) = 0 \quad \text{in } (S(t))^\kappa \qquad \text{for a.a. } t \in [0,T].$$

Choosing $\delta(\kappa) > 0$ as in the inclusion (3.4.66) it follows that

$$\operatorname{curl} b(t) = 0 \quad \text{in } \overline{\overline{S}}_{\Delta t}(t) \qquad \text{for a.a. } t \in [0,T] \text{ and all } \Delta t < \delta(\kappa)$$

and consequently

$$b(t) \in W^k(S_{\Delta t,k}) \qquad \text{for a.a. } t \in ((k-1)\Delta t, k\Delta t] \text{ and all } k = 1, ..., \frac{T}{\Delta t}, \; \Delta t < \delta(\kappa).$$

Hence we have shown the implication

$$b \in L^4\left(0,T;H_0^2(\Omega)\right), \; \operatorname{curl} b(t) = 0 \text{ in } (S(t))^\kappa \text{ for a.a. } t \in [0,T] \quad \Rightarrow \quad b \text{ satisfies (3.4.7) } \forall \Delta t < \delta(\kappa). \tag{3.4.76}$$

We proceed by fixing an arbitrary interval $I := (t_1, t_2)$, $t_1, t_2 \in [0, T]$, as well as an arbitrary ball $U \subset \Omega$ such that $\overline{I \times U} \subset Q^f(S)$ and consider test functions of the form

$$b = \begin{cases} b^f \in L^4\left(I; H_0^2(U)\right) & \text{in } I \times U \\ 0 & \text{in } Q \backslash (I \times U) \end{cases}. \tag{3.4.77}$$

Since $\operatorname{dist}(\overline{I \times U}, Q^s(S)) > \kappa$ for some value $\kappa > 0$ it holds that $\operatorname{curl} b(t) = 0$ in $(S(t))^\kappa$ for any b of the form (3.4.77) and any $t \in [0, T]$. Consequently, according to the implication (3.4.76), for any $\Delta t < \delta(\kappa)$ any test function b of the form (3.4.77) satisfies the conditions (3.4.7), which render b an admissible test function for the induction equation (3.4.8) on the Δt-level. Using it as such, we estimate, under exploitation of the Hölder inequality and the uniform bounds (3.4.35)–(3.4.37) and (3.4.68),

$$\begin{aligned}
&\left| \int_I \int_U \partial_t B_{\Delta t} \cdot b^f \; dxdt \right| \\
=&\left| \int_I \int_U \left[\frac{1}{\sigma\mu} \operatorname{curl} \overline{B}_{\Delta t} - \overline{u}_{\Delta t} \times \overline{B}'_{\Delta t} + \frac{\epsilon}{\mu^2} \left|\operatorname{curl} \overline{B}_{\Delta t}\right|^2 \operatorname{curl} \overline{B}_{\Delta t} - \frac{1}{\sigma} \overline{J}_{\Delta t} \right] \cdot \operatorname{curl} b^f \right. \\
&\left. + \epsilon \left(\nabla \operatorname{curl} \overline{B}_{\Delta t}\right) \cdot \left(\nabla \operatorname{curl} b^f\right) \; dxdt \right| \\
\leqslant&\left[\frac{1}{\sigma\mu} \left\|\operatorname{curl} \overline{B}_{\Delta t}\right\|_{L^{\frac{4}{3}}(Q)} + \left\|\overline{u}_{\Delta t}\right\|_{L^4(Q)} \left\|\overline{B}'_{\Delta t}\right\|_{L^2(Q)} + \frac{\epsilon}{\mu^2} \left\|\operatorname{curl} \overline{B}_{\Delta t}\right\|^3_{L^4(Q)} \right. \\
&\left. + \frac{1}{\sigma} \left\|\overline{J}_{\Delta t}\right\|_{L^{\frac{4}{3}}(Q)} \right] \left\|\operatorname{curl} b^f\right\|_{L^4(I \times U)} + \epsilon \left\|\nabla \operatorname{curl} \overline{B}_{\Delta t}\right\|_{L^2(Q)} \left\|\nabla \operatorname{curl} b^f\right\|_{L^2(I \times U)} \leqslant c \left\|b^f\right\|_{L^4(I; H_0^2(U))}.
\end{aligned}$$

From this inequality we infer the dual estimate

$$\left\| \frac{\overline{B}'_{\Delta t}(\cdot) - \overline{B}'_{\Delta t}(\cdot - \Delta t)}{\Delta t} \right\|_{L^{\frac{4}{3}}(t_1 + \Delta t, t_2; H^{-2}(U))} \leqslant \left\| \frac{\overline{B}_{\Delta t}(\cdot) - \overline{B}_{\Delta t}(\cdot - \Delta t)}{\Delta t} \right\|_{L^{\frac{4}{3}}(t_1, t_2; H^{-2}(U))} \leqslant c.$$

Since moreover $\overline{B}'_{\Delta t}$ is bounded uniformly in $L^2(Q)$ (cf. (3.4.34)) we have thus shown the conditions for the discrete Aubin-Lions Lemma, cf. Lemma A.3.3 (ii) and Remark A.3.1 in the appendix. This yields the desired strong convergence

$$\overline{B}'_{\Delta t} \to B \quad \text{in } L^2\left(I_C; H^{-1}(U)\right)$$

for all compact subintervals $I_C \subset I$. Since $\overline{I \times U} \subset Q^f(S)$ was chosen arbitrarily, this, in combination with the weak $L^2(0, T; H^2(\Omega))$-convergence (3.4.40) of $\overline{u}_{\Delta t}$ and the weak $L^2(0, T; H^1(\Omega))$-convergence (3.4.41) of $\operatorname{curl} \overline{B}_{\Delta t}$ and $\operatorname{curl} \overline{B}'_{\Delta t}$, suffices to identify

$$z_2 = u \times B \quad \text{a.e. in } Q^f(S), \qquad z_3 = z_4 = \operatorname{curl} B \times B \quad \text{a.e. in } Q^f(S). \tag{3.4.78}$$

The first one of these identities gives us the final ingredient for the limit passage in the induction equation. Indeed, since functions $b \in Y(S)$ are curl-free in a neighborhood of the solid body, the implication (3.4.76) shows that any such function can be used as a test function in the induction equation (3.4.8) on the Δt-level for any sufficiently small $\Delta t > 0$. Then, making use of the convergences (3.4.70) and (3.4.71) as well as of the identities (3.4.75) and (3.4.78), we can let Δt tend to 0 and obtain the equation

$$\begin{aligned}
&- \int_0^T \int_\Omega B \cdot \partial_t b \; dxdt - \int_\Omega B_{0,m} \cdot b(0, x) \; dx \\
=& \int_0^T \int_\Omega \left[-\frac{1}{\sigma\mu} \operatorname{curl} B + u \times B - \frac{\epsilon}{\mu^2} z + \frac{1}{\sigma} J \right] \cdot \operatorname{curl} b - \epsilon \left(\nabla \operatorname{curl} B\right) \cdot \left(\nabla \operatorname{curl} b\right) \; dxdt
\end{aligned}$$

for all $b \in Y(S)$.

3.4.5 Momentum equation

In order to pass to the limit in the momentum equation we first show strong convergence of the velocity field. To this end we estimate, under exploitation of the Gagliardo-Nirenberg interpolation inequality,

$$\|\nabla \overline{u}_{\Delta t}(t)\|_{L^2(\Omega)} \leqslant c \left\|\nabla^2 \overline{u}_{\Delta t}(t)\right\|_{L^2(\Omega)}^{\frac{1}{2}} \|\overline{u}_{\Delta t}(t)\|_{L^2(\Omega)}^{\frac{1}{2}} + c \|\overline{u}_{\Delta t}(t)\|_{L^2(\Omega)} \qquad \forall t \in [0,T]. \tag{3.4.79}$$

This, together with the Hölder inequality, allows us to estimate, for any function $\phi \in L^4(0,T;H^2_{0,\mathrm{div}}(\Omega))$,

$$\begin{aligned}
&\int_0^T \int_\Omega \epsilon \left(\nabla \overline{u}_{\Delta t} \nabla \overline{\rho}_{\Delta t}\right) \cdot \phi \, dxdt \\
\leqslant&\epsilon \left(\int_0^T \|\nabla \overline{u}_{\Delta t}(t)\|_{L^2(\Omega)}^4 \, dt\right)^{\frac{1}{4}} \|\nabla \overline{\rho}_{\Delta t}\|_{L^2(Q)} \|\phi\|_{L^4(0,T;L^\infty(\Omega))} \\
\leqslant&c\epsilon \left(\int_0^T \left\|\nabla^2 \overline{u}_{\Delta t}(t)\right\|_{L^2(\Omega)}^2 \|\overline{u}_{\Delta t}(t)\|_{L^2(\Omega)}^2 + \|\overline{u}_{\Delta t}(t)\|_{L^2(\Omega)}^4 \, dt\right)^{\frac{1}{4}} \|\nabla \overline{\rho}_{\Delta t}\|_{L^2(Q)} \|\phi\|_{L^4(0,T;L^\infty(\Omega))} \\
\leqslant&c\epsilon \left(\|\overline{u}_{\Delta t}\|_{L^2(0,T;H^2(\Omega))}^{\frac{1}{2}} \|\overline{u}_{\Delta t}\|_{L^\infty(0,T;L^2(\Omega))}^{\frac{1}{2}} + \|\overline{u}_{\Delta t}\|_{L^4(0,T;L^2(\Omega))}\right) \|\nabla \overline{\rho}_{\Delta t}\|_{L^2(Q)} \|\phi\|_{L^4(0,T;L^\infty(\Omega))}.
\end{aligned} \tag{3.4.80}$$

Then we test the momentum equation (3.4.6) by an arbitrary function $\phi \in L^4(0,T;H^2_{0,\mathrm{div}}(\Omega))$. The inequality (3.4.80), in combination with the estimate (3.4.14) and the uniform bounds (3.4.31), (3.4.32), (3.4.33), (3.4.37) and (3.4.69), allows us to estimate

$$\begin{aligned}
&\left|\int_0^T \int_\Omega \partial_t \left(\rho_{\Delta t} u_{\Delta t}\right) \cdot \phi \, dxdt\right| \\
=&\left|\int_0^T \int_\Omega - \left(\overline{\rho}_{\Delta t} \overline{u}'_{\Delta t} \otimes \overline{u}_{\Delta t}\right) : \nabla \phi + 2\nu \mathbb{D}(\overline{u}_{\Delta t}) : \nabla \phi + \epsilon \left(\nabla \overline{u}_{\Delta t} \nabla \overline{\rho}_{\Delta t}\right) \cdot \phi + \epsilon \Delta \overline{u}_{\Delta t} \cdot \Delta \phi \right. \\
&\left. + m \overline{\rho}'_{\Delta t} \overline{\chi}_{\Delta t} \left(\overline{u}'_{\Delta t} - \overline{\Pi}'_{\Delta t}\right) \cdot \phi - \overline{\rho}'_{\Delta t} \overline{g}_{\Delta t} \cdot \phi - \frac{1}{\mu} \left(\mathrm{curl} \overline{B}'_{\Delta t} \times \overline{B}'_{\Delta t}\right) \cdot \phi \, dxdt\right| \\
\leqslant&\overline{\rho} \left\|\overline{u}'_{\Delta t}\right\|_{L^\infty(0,T;L^2(\Omega))} \|\overline{u}_{\Delta t}\|_{L^2(0,T;L^\infty(\Omega))} \|\nabla \phi\|_{L^2(Q)} + 2\nu \|\mathbb{D}\left(\overline{u}_{\Delta t}\right)\|_{L^2(Q)} \|\nabla \phi\|_{L^2(Q)} \\
&+ c\epsilon \left(\|\overline{u}_{\Delta t}\|_{L^2(0,T;H^2(\Omega))}^{\frac{1}{2}} \|\overline{u}_{\Delta t}\|_{L^\infty(0,T;L^2(\Omega))}^{\frac{1}{2}} + \|\overline{u}_{\Delta t}\|_{L^4(0,T;L^2(\Omega))}\right) \|\nabla \overline{\rho}_{\Delta t}\|_{L^2(Q)} \|\phi\|_{L^4(0,T;L^\infty(\Omega))} \\
&+ \epsilon \|\Delta \overline{u}_{\Delta t}\|_{L^2(Q)} \|\Delta \phi\|_{L^2(Q)} + \overline{\rho} m \left\|\overline{u}'_{\Delta t} - \overline{\Pi}'_{\Delta t}\right\|_{L^2(Q)} \|\phi\|_{L^2(Q)} + \overline{\rho} \|\overline{g}_{\Delta t}\|_{L^2(Q)} \|\phi\|_{L^2(Q)} \\
&+ \frac{1}{\mu} \left\|\mathrm{curl} \overline{B}'_{\Delta t}\right\|_{L^4(Q)} \left\|\overline{B}'_{\Delta t}\right\|_{L^4(Q)} \|\phi\|_{L^2(Q)} \\
\leqslant&c \|\phi\|_{L^4\left(0,T;H^2_{0,\mathrm{div}}\right)}
\end{aligned} \tag{3.4.81}$$

for a constant $c > 0$ independent of Δt. We point out that for this estimate the regularization term $\epsilon \Delta \overline{u}_{\Delta t}$ in the momentum equation, from which $L^2(0,T;H^2(\Omega))$-bound (3.4.33) of $\overline{u}_{\Delta t}$ resulted, is essential. We denote by P the orthogonal projection from $L^2(\Omega)$ onto the space $L^2_{\mathrm{div}}(\Omega)$ of weakly divergence-free $L^2(\Omega)$–functions and see, due to the identity $P(\phi) = \phi$ for all $\phi \in H^2_{0,\mathrm{div}}(\Omega)$, that

$$\left|\int_0^T \int_\Omega \partial_t \left(\rho_{\Delta t} u_{\Delta t}\right) \cdot \phi \, dxdt\right| = \left|\int_0^T \int_\Omega \partial_t P \left(\rho_{\Delta t} u_{\Delta t}\right) \cdot \phi \, dxdt\right| \qquad \forall \phi \in L^4(0,T;H^2_{0,\mathrm{div}}(\Omega)).$$

This identity together with the estimate (3.4.81) implies the dual estimate

$$\left\|\frac{P\left(\overline{\rho}_{\Delta t} \overline{u}_{\Delta t}\right)(\cdot) - P\left(\overline{\rho}_{\Delta t} \overline{u}_{\Delta t}\right)(\cdot - \Delta t)}{\Delta t}\right\|_{L^{\frac{4}{3}}\left(0,T;\left(H^2_{0,\mathrm{div}}(\Omega)\right)^*\right)} \leqslant c.$$

Due to the estimate

$$\|P\left(\overline{\rho}_{\Delta t} \overline{u}_{\Delta t}\right)\|_{L^2(Q)} \leqslant \|\overline{\rho}_{\Delta t} \overline{u}_{\Delta t}\|_{L^2(Q)} \leqslant c,$$

we may thus apply the discrete Aubin-Lions Lemma A.3.3 (i) in the appendix. This yields the strong convergence

$$P\left(\overline{\rho}_{\Delta t}\overline{u}_{\Delta t}\right) \to P(\rho u) \quad \text{in } L^2\left(0,T;\left(H^2_{0,\text{div}}(\Omega)\right)^*\right).$$

Since $\overline{u}_{\Delta t} \in L^2(0,T;H^2_{0,\text{div}}(\Omega))$ and hence $\overline{u}_{\Delta t} = P(\overline{u}_{\Delta t})$, it follows that

$$\begin{aligned}
&\left|\int_0^T\int_\Omega \overline{\rho}_{\Delta t}\left|\overline{u}_{\Delta t}\right|^2\, dxdt - \int_0^T\int_\Omega \rho|u|^2\, dxdt\right| \\
&= \left|\int_0^T\int_\Omega \left(\overline{\rho}_{\Delta t}\overline{u}_{\Delta t} - \rho u\right)\cdot \overline{u}_{\Delta t}\, dxdt + \int_0^T\int_\Omega \rho u\cdot\left(\overline{u}_{\Delta t} - u\right)\, dxdt\right| \\
&\leqslant \left\|P\left(\overline{\rho}_{\Delta t}\overline{u}_{\Delta t}\right) - P(\rho u)\right\|_{L^2(0,T;(H^2_{0,\text{div}}(\Omega))^*)}\left\|\overline{u}_{\Delta t}\right\|_{L^2(0,T;H^2_{0,\text{div}}(\Omega))} + \left|\int_0^T\int_\Omega \rho u\cdot\left(\overline{u}_{\Delta t} - u\right)\, dxdt\right| \to 0.
\end{aligned} \tag{3.4.82}$$

Further, the strong convergence (3.4.45) of the density together with its boundedness away from zero (cf. (3.2.2)) shows that

$$\sqrt{\overline{\rho}_{\Delta t}}\overline{u}_{\Delta t} \rightharpoonup \sqrt{\rho}u \quad \text{in } L^2\left(Q\right).$$

This, together with the convergence (3.4.82) of the $L^2(Q)$-norm of $\sqrt{\overline{\rho}_{\Delta t}}\overline{u}_{\Delta t}$ implies that

$$\sqrt{\overline{\rho}_{\Delta t}}\overline{u}_{\Delta t} \to \sqrt{\rho}u \quad \text{in } L^2\left(Q\right).$$

Consequently, the strong convergence (3.4.45) of the density and its bound away from zero in (3.2.2) as well as the $L^4(Q)$-bound (3.4.68) of $\overline{u}_{\Delta t}$ imply the desired strong convergence

$$\overline{u}_{\Delta t} \to u \quad \text{in } L^q(Q) \qquad \forall 1 \leqslant q < 4 \tag{3.4.83}$$

of the velocity field. In particular it follows that

$$\overline{\rho}_{\Delta t}\overline{u}'_{\Delta t}\otimes \overline{u}_{\Delta t} \rightharpoonup \rho u\otimes u \quad \text{in } L^2(Q). \tag{3.4.84}$$

We are now in the position to carry out the limit passage in the momentum equation (3.4.6). Indeed, using in particular the strong convergence (3.4.53) of the density gradient, the convergence (3.4.56), (3.4.65) of the penalization term, the convergence (3.4.72), (3.4.74), (3.4.78) of the Lorentz force and the converge (3.4.84) of the convective term, we obtain the relation

$$\begin{aligned}
&-\int_0^T\int_\Omega \rho u\cdot\partial_t\phi\, dxdt - \int_\Omega \rho_{0,m}u_{0,m}\cdot\phi(0,x)\, dx \\
&= \int_0^T\int_\Omega (\rho u\otimes u) : \nabla\phi - 2\nu\mathbb{D}(u) : \nabla\phi - m\rho\chi\left(u - \Pi_{[\chi,\rho,u]}\right)\cdot\phi \\
&\quad + \rho g\cdot\phi + \frac{1}{\mu}\left(\text{curl}B\times B\right)\cdot\phi - \epsilon\left(\nabla u\nabla\rho\right)\cdot\phi - \epsilon\Delta u\cdot\Delta\phi\, dxdt
\end{aligned}$$

for any $\phi \in \mathcal{D}([0,T)\times\Omega)$ with $\text{div}\phi = 0$.

3.4.6 Energy inequality

With the aim of deriving an energy inequality for the limit system we slightly modify the derivation of the energy estimate (3.4.29) on the Δt-level. We pick an arbitrary index $k \in \{1,...,\frac{T}{\Delta t}\}$ and some arbitrary value $s \in [0, \Delta t)$. Then we add to the first inequality in (3.4.26) a zero of the form

$$\begin{aligned}
0 = -s &\Bigg[\int_\Omega -m\rho_{\Delta t}^{k-1}\chi_{\Delta t}^k \left(u_{\Delta t}^{k-1} - \Pi_{\Delta t}^{k-1}\right)\cdot u_{\Delta t}^k + \rho_{\Delta t}^{k-1} g_{\Delta t}^k \cdot u_{\Delta t}^k + \frac{1}{\mu}\left(\mathrm{curl} B_{\Delta t}^{k-1} \times B_{\Delta t}^{k-1}\right)\cdot u_{\Delta t}^k \\
&+ \frac{1}{\mu}\left(u_{\Delta t}^k \times B_{\Delta t}^{k-1}\right)\cdot \mathrm{curl} B_{\Delta t}^k + \frac{1}{\sigma\mu} J_{\Delta t}^k \cdot \mathrm{curl} B_{\Delta t}^k \, dx\Bigg] \\
+s &\Bigg[\int_\Omega -m\rho_{\Delta t}^{k-1}\chi_{\Delta t}^k \left(u_{\Delta t}^{k-1} - \Pi_{\Delta t}^{k-1}\right)\cdot u_{\Delta t}^k + \rho_{\Delta t}^{k-1} g_{\Delta t}^k \cdot u_{\Delta t}^k + \frac{1}{\mu}\left(\mathrm{curl} B_{\Delta t}^{k-1} \times B_{\Delta t}^{k-1}\right)\cdot u_{\Delta t}^k \\
&+ \frac{1}{\mu}\left(u_{\Delta t}^k \times B_{\Delta t}^{k-1}\right)\cdot \mathrm{curl} B_{\Delta t}^k + \frac{1}{\sigma\mu} J_{\Delta t}^k \cdot \mathrm{curl} B_{\Delta t}^k \, dx\Bigg] \\
\leqslant -s &\Bigg[\int_\Omega -m\rho_{\Delta t}^{k-1}\chi_{\Delta t}^k \left(u_{\Delta t}^{k-1} - \Pi_{\Delta t}^{k-1}\right)\cdot u_{\Delta t}^k + \rho_{\Delta t}^{k-1} g_{\Delta t}^k \cdot u_{\Delta t}^k + \frac{1}{\mu}\left(\mathrm{curl} B_{\Delta t}^{k-1} \times B_{\Delta t}^{k-1}\right)\cdot u_{\Delta t}^k \\
&+ \frac{1}{\mu}\left(u_{\Delta t}^k \times B_{\Delta t}^{k-1}\right)\cdot \mathrm{curl} B_{\Delta t}^k + \frac{1}{\sigma\mu} J_{\Delta t}^k \cdot \mathrm{curl} B_{\Delta t}^k \, dx\Bigg] + c\left[\Delta t + (\Delta t)^{\frac{1}{2}}\right],
\end{aligned}$$

where the last estimate follows from the uniform bounds (3.4.14), (3.4.32), (3.4.37) and (3.4.69). As any $\tau \in (0, T]$ can be expressed as $\tau = k\Delta t - s$ for certain values $k \in \{1, ..., \frac{T}{\Delta t}\}$ and $s \in [0, \Delta t)$, this procedure results in the inequality

$$\begin{aligned}
&\int_\Omega \frac{1}{2}\overline{\rho}_{\Delta t}(\tau)\left|\overline{u}_{\Delta t}(\tau)\right|^2 + \frac{1}{2\mu}\left|\overline{B}_{\Delta t}(\tau)\right|^2 \, dx + \int_0^\tau \int_\Omega 2\nu\left|\nabla \overline{u}_{\Delta t}\right|^2 + \epsilon\left|\Delta \overline{u}_{\Delta t}\right|^2 \\
&+ \int_0^\tau \int_\Omega \frac{1}{\sigma\mu^2}\left|\mathrm{curl}\overline{B}_{\Delta t}\right|^2 + \frac{\epsilon}{\mu^3}\left|\mathrm{curl}\overline{B}_{\Delta t}\right|^4 + \frac{\epsilon}{\mu}\left|\nabla\,\mathrm{curl}\,\overline{B}_{\Delta t}\right|^2 \, dxdt \\
\leqslant &\int_\Omega \frac{1}{2}\rho_{0,m}|u_{0,m}|^2 + \frac{1}{2\mu}|B_{0,m}|^2 \, dx + \int_0^\tau \int_\Omega -m\overline{\rho}'_{\Delta t}\overline{\chi}_{\Delta t}\left(\overline{u}'_{\Delta t} - \overline{\Pi}'_{\Delta t}\right)\cdot \overline{u}_{\Delta t} + \overline{\rho}'_{\Delta t}\overline{g}_{\Delta t}\cdot \overline{u}_{\Delta t} \\
&+ \frac{1}{\mu}\left(\mathrm{curl}\overline{B}'_{\Delta t} \times \overline{B}'_{\Delta t}\right)\cdot \overline{u}_{\Delta t} + \frac{1}{\mu}\left(\overline{u}_{\Delta t} \times \overline{B}'_{\Delta t}\right)\cdot \mathrm{curl}\overline{B}_{\Delta t} + \frac{1}{\sigma\mu}\overline{J}_{\Delta t}\cdot \mathrm{curl}\overline{B}_{\Delta t} \, dxdt + c\left[\Delta t + (\Delta t)^{\frac{1}{2}}\right]
\end{aligned}$$

for all $\tau \in (0, T]$. On the right-hand side of this inequality we write

$$\frac{1}{\mu}\left(\overline{u}_{\Delta t} \times \overline{B}'_{\Delta t}\right)\cdot \mathrm{curl}\overline{B}_{\Delta t} = -\frac{1}{\mu}\left(\mathrm{curl}\overline{B}_{\Delta t} \times \overline{B}'_{\Delta t}\right)\cdot \overline{u}_{\Delta t}.$$

Then we pass to the limit under exploitation of in particular the convergence (3.4.56), (3.4.65) of the penalization term, the convergence (3.4.72), (3.4.73), (3.4.74), (3.4.78) of the Lorentz force and the strong convergence (3.4.83) of $\overline{u}_{\Delta t}$, as well as the weak lower semicontinuity of norms. This leads to the inequality

$$\begin{aligned}
&\int_\Omega \frac{1}{2}\rho(\tau)|u(\tau)|^2 + \frac{1}{2\mu}|B(\tau)|^2 \, dx + \int_0^\tau \int_\Omega 2\nu\left|\nabla u\right|^2 + \epsilon|\Delta u|^2 + \frac{1}{\sigma\mu^2}\left|\mathrm{curl} B\right|^2 + \frac{\epsilon}{\mu^3}\left|z\right|^{\frac{4}{3}} \\
&+ \frac{\epsilon}{\mu}\left|\nabla\,\mathrm{curl}\,B\right|^2 \, dxdt \\
\leqslant &\int_\Omega \frac{1}{2}\rho_{0,m}|u_{0,m}|^2 + \frac{1}{2\mu}|B_{0,m}|^2 \, dx + \int_0^\tau \int_\Omega -m\rho\chi\left(u - \Pi_{[\chi,\rho,u]}\right)\cdot u + \rho g\cdot u + \frac{1}{\sigma\mu} J\cdot \mathrm{curl} B \, dxdt
\end{aligned} \tag{3.4.85}$$

for almost all $\tau \in [0, T]$. Due to the orthogonality (3.1.7), we can further write

$$\int_0^\tau \int_\Omega m\rho\chi\left(u - \Pi_{[\chi,\rho,u]}\right)\cdot u \, dx = \int_0^\tau \int_{\mathbb{R}^3} m\rho\chi\left(u - \Pi_{[\chi,\rho,u]}\right)\cdot u \, dx = \int_0^\tau \int_{\mathbb{R}^3} m\rho\chi\left|u - \Pi_{[\chi,\rho,u]}\right|^2 \, dxdt.$$

Consequently, the inequality (3.4.85) finally turns into the desired energy inequality,

$$\begin{aligned}
&\int_\Omega \frac{1}{2}\rho(\tau)|u(\tau)|^2 + \frac{1}{2\mu}|B(\tau)|^2 \, dx + \int_0^\tau \int_\Omega 2\nu\left|\nabla u\right|^2 + \epsilon|\Delta u|^2 + \frac{1}{\sigma\mu^2}\left|\mathrm{curl} B\right|^2 + \frac{\epsilon}{\mu^3}\left|z\right|^{\frac{4}{3}} \\
&+ \frac{\epsilon}{\mu}\left|\nabla\,\mathrm{curl}\,B\right|^2 + m\rho\chi\left|u - \Pi_{[\chi,\rho,u]}\right|^2 \, dxdt \\
\leqslant &\int_\Omega \frac{1}{2}\rho_{0,m}|u_{0,m}|^2 + \frac{1}{2\mu}|B_{0,m}|^2 \, dx + \int_0^\tau \int_\Omega \rho g\cdot u + \frac{1}{\sigma\mu} J\cdot \mathrm{curl} B \, dxdt
\end{aligned}$$

for almost all $\tau \in [0, T]$. Altogether we have shown the following result.

Proposition 3.4.1. *Let all the assumptions of Theorem 3.1.1 be satisfied and let* $\epsilon, > 0$, $m \in \mathbb{N}$. *Assume in addition the regularized initial data* $\rho_{0,m}$, $u_{0,m}$, $B_{0,m}$ *to satisfy the conditions* (3.2.11). *Then, there exists an isometry* $\eta^{\Pi_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}}(s;t,\cdot) : \mathbb{R}^3 \to \mathbb{R}^3$, $s,t \in [0,T]$, *and*

$$\chi_\epsilon \in C\left([0,T]; L^p\left(\mathbb{R}^3\right)\right), \ 1 \leqslant p < \infty,$$

$$\rho_\epsilon \in \left\{\psi \in L^2\left(0,T;H^1(\Omega)\right) \bigcap L^\infty(Q) : \ \underline{\rho} \leqslant \psi \leqslant \overline{\rho} \ a.e. \ in \ Q\right\}, \tag{3.4.86}$$

$$u_\epsilon \in L^\infty\left(0,T;L^2(\Omega)\right) \bigcap L^2\left(0,T;H^2_{0,\mathrm{div}}(\Omega)\right) \tag{3.4.87}$$

$$B_\epsilon \in \Big\{b \in L^\infty\left(0,T;L^2(\Omega)\right) \bigcap L^2\left(0,T;H^1_{\mathrm{div}}(\Omega)\right) : \ \operatorname{curl} B \in L^2\left(0,T;H^1(\Omega)\right),$$
$$\operatorname{curl} b = 0 \ in \ Q^s(S_\epsilon) \bigcap Q, \ b \cdot \mathrm{n}\,|_{\partial\Omega} = 0\Big\}, \tag{3.4.88}$$

$$z_\epsilon \in L^{\frac{4}{3}}(Q)$$

for $S_\epsilon = S_\epsilon(\cdot) = \eta^{\Pi_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}}(0;\cdot,S_0)$, *such that*

$$\frac{d\eta^{\Pi_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}}(s;t,x)}{dt} = \Pi_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}\big(t,\eta^{\Pi_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}}(s;t,x)\big), \qquad \eta^{\Pi_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}}(s;s,x) = x, \tag{3.4.89}$$

$$-\int_0^T\int_{\mathbb{R}^3} \chi_\epsilon \partial_t \Theta \ dxdt - \int_{\mathbb{R}^3} \chi_0 \Theta(0,x) \ dx = \int_0^T\int_{\mathbb{R}^3} \left(\chi_\epsilon \Pi_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}\right) \cdot \nabla\Theta \ dxdt, \tag{3.4.90}$$

$$-\int_0^T\int_\Omega \rho_\epsilon \partial_t \psi \ dxdt - \int_\Omega \rho_{0,m}\psi(0,x) \ dx = \int_0^T\int_\Omega (\rho_\epsilon u_\epsilon) \cdot \nabla\psi + \epsilon\rho_\epsilon \Delta\psi \ dxdt, \tag{3.4.91}$$

$$\begin{aligned}
-\int_0^T\int_\Omega \rho_\epsilon u_\epsilon \cdot \partial_t \phi \ dxdt - \int_\Omega \rho_{0,m} u_{0,m} \cdot \phi(0,x) \ dx = \int_0^T\int_\Omega & (\rho_\epsilon u_\epsilon \otimes u_\epsilon) : \nabla\phi - 2\nu\mathbb{D}(u_\epsilon) : \nabla\phi \\
& - m\rho_\epsilon\chi_\epsilon\left(u_\epsilon - \Pi_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}\right) \cdot \phi + \rho_\epsilon g \cdot \phi \\
& + \frac{1}{\mu}\left(\operatorname{curl} B_\epsilon \times B_\epsilon\right) \cdot \phi - \epsilon\left(\nabla u_\epsilon \nabla\rho_\epsilon\right) \cdot \phi \\
& - \epsilon\Delta u_\epsilon \cdot \Delta\phi \ dxdt,
\end{aligned} \tag{3.4.92}$$

$$\begin{aligned}
-\int_0^T\int_\Omega B_\epsilon \cdot \partial_t b \ dxdt - \int_\Omega B_{0,m} \cdot b(0,x) \ dx = \int_0^T\int_\Omega & \Big[-\frac{1}{\sigma\mu}\operatorname{curl} B_\epsilon + u_\epsilon \times B_\epsilon + \frac{1}{\sigma} J \\
& -\frac{\epsilon}{\mu^2} z_\epsilon\Big] \cdot \operatorname{curl} b - \epsilon\left(\nabla \operatorname{curl} B_\epsilon\right) : \left(\nabla \operatorname{curl} b\right) \ dxdt
\end{aligned} \tag{3.4.93}$$

for all $\Theta \in \mathcal{D}([0,T) \times \mathbb{R}^3)$, $\psi, \phi \in \mathcal{D}([0,T) \times \Omega)$ *with* $\operatorname{div}\phi = 0$ *and all* $b \in Y(S_\epsilon)$. *Moreover, these functions satisfy the energy inequality*

$$\begin{aligned}
& \int_\Omega \frac{1}{2}\rho_\epsilon(\tau)|u_\epsilon(\tau)|^2 + \frac{1}{2\mu}|B_\epsilon(\tau)|^2 \ dx + \int_0^\tau\int_\Omega 2\nu\left|\nabla u_\epsilon\right|^2 + \epsilon|\Delta u_\epsilon|^2 + \frac{1}{\sigma\mu^2}\left|\operatorname{curl} B_\epsilon\right|^2 + \frac{\epsilon}{\mu^3}\left|z_\epsilon\right|^{\frac{4}{3}} \\
& + \frac{\epsilon}{\mu}\left|\nabla \operatorname{curl} B_\epsilon\right|^2 + m\rho_\epsilon\chi_\epsilon\left|u_\epsilon - \Pi_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}\right|^2 \ dxdt \\
\leqslant & \int_\Omega \frac{1}{2}\rho_{0,m}|u_{0,m}|^2 + \frac{1}{2}|B_{0,m}|^2 \ dx + \int_0^\tau\int_\Omega \rho_\epsilon g \cdot u_\epsilon + \frac{1}{\sigma\mu} J \cdot \operatorname{curl} B_\epsilon \ dxdt
\end{aligned} \tag{3.4.94}$$

for almost all $\tau \in [0,T]$ *and the characteristic function* χ_ϵ *is connected to the isometry* $\eta^{\Pi_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}}$ *by the formula*

$$\chi(t,x) = \chi_0\left(\eta^{\Pi_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}}(t;0,x)\right).$$

3.5 Limit passage with respect to $\epsilon \to 0$

Applying the Gronwall inequality to the energy inequality (3.4.94) we find a constant $c > 0$, uniform with respect to ϵ, such that

$$\|u_\epsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|u_\epsilon\|_{L^2(0,T;H^1(\Omega))} + \|B_\epsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|B_\epsilon\|_{L^2(0,T;H^1(\Omega))} \leqslant c, \tag{3.5.1}$$

$$\epsilon^{\frac{1}{2}} \|\Delta u_\epsilon\|_{L^2(Q)} + \epsilon^{\frac{1}{2}} \|\nabla \operatorname{curl} B_\epsilon\|_{L^2(Q)} + \epsilon^{\frac{3}{4}} \|z_\epsilon\|_{L^{\frac{4}{3}}(Q)} \leqslant c. \tag{3.5.2}$$

Further, recalling the identity (3.4.49), which was obtained by using the density as a test function in the continuity equation on the ϵ-level, we infer that

$$\epsilon^{\frac{1}{2}} \|\rho_\epsilon\|_{L^2(0,T;H^1(\Omega))} \leqslant c. \tag{3.5.3}$$

The uniform bounds (3.5.1)–(3.5.3) together with the upper and lower bounds (3.4.86) for the density allow us to find functions

$$\rho \in \left\{\psi \in L^\infty(Q) : \ \underline{\rho} \leqslant \psi \leqslant \overline{\rho} \quad \text{a.e. in } Q\right\},$$

$$u \in L^\infty\left(0,T;L^2(\Omega)\right) \bigcap L^2\left(0,T;H^1_{0,\mathrm{div}}(\Omega)\right), \tag{3.5.4}$$

$$B \in \left\{b \in L^\infty\left(0,T;L^2(\Omega)\right) \bigcap L^2\left(0,T;H^1_{\mathrm{div}}(\Omega)\right) : \ b\cdot \mathrm{n}|_{\partial\Omega} = 0\right\} \tag{3.5.5}$$

such that, possibly after the extraction of a subsequence,

$$\rho_\epsilon \overset{*}{\rightharpoonup} \rho \quad \text{in } L^\infty(Q), \tag{3.5.6}$$

$$u_\epsilon \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty\left(0,T;L^2(\Omega)\right), \qquad u_\epsilon \rightharpoonup u \quad \text{in } L^2\left(0,T;H^1(\Omega)\right), \tag{3.5.7}$$

$$B_\epsilon \overset{*}{\rightharpoonup} B \quad \text{in } L^\infty\left(0,T;L^2(\Omega)\right), \qquad B_\epsilon \rightharpoonup B \quad \text{in } L^2\left(0,T;H^1(\Omega)\right), \tag{3.5.8}$$

$$\epsilon\nabla\rho_\epsilon,\ \epsilon\Delta u_\epsilon,\ \epsilon\nabla \operatorname{curl} B_\epsilon \to 0 \quad \text{in } L^2(Q), \qquad \epsilon z_\epsilon \to 0 \quad \text{in } L^{\frac{4}{3}}(Q).$$

The boundary conditions of u and B in (3.5.4) and (3.5.5) follow directly from the corresponding boundary conditions of the velocity field and the magnetic induction on the ϵ-level, cf. the inclusions (3.4.87) and (3.4.88).

3.5.1 Continuity equation

We consider arbitrary functions $\psi \in \mathcal{D}(0,T)$, $\Phi \in \mathcal{D}(\Omega)$ and test the continuity equation (3.4.91) on the ϵ-level by $\psi\Phi$. Under exploitation of the upper bound (3.4.86) of ρ_ϵ and the $L^\infty(0,T;L^2(\Omega))$-bound (3.5.1) of u_ϵ this leads to the dual estimate

$$\begin{aligned}\left\|\partial_t \int_\Omega \rho_\epsilon \Phi dx\right\|_{L^2(0,T)} &= \left\|\int_\Omega (\rho_\epsilon u_\epsilon)\cdot\nabla\Phi + \epsilon\rho_\epsilon\Delta\Phi dx\right\|_{L^2(0,T)} \\ &\leqslant \overline{\rho}\,\|u_\epsilon\|_{L^2(Q)}\|\nabla\Phi\|_{L^2(\Omega)} + \epsilon\|\rho_\epsilon\|_{L^2(Q)}\|\Delta\Phi\|_{L^2(\Omega)} \leqslant c\end{aligned}$$

with a constant $c > 0$ dependent on Φ but not on ϵ. Since $L^2(\Omega)$ is embedded compactly into $(H^1(\Omega))^*$ we may thus apply Lemma A.4.1 from the Appendix and infer that

$$\rho_\epsilon \to \rho \quad \text{in } C_{\mathrm{weak}}\left([0,T];L^2(\Omega)\right) \text{ and hence in } L^p\left(0,T;\left(H^1(\Omega)\right)^*\right) \qquad \forall 1 \leqslant p < \infty.$$

In particular, the weak convergence (3.5.7) of u_ϵ and the weak-* convergence (3.5.6) of ρ_ϵ now imply that

$$\rho_\epsilon u_\epsilon \rightharpoonup \rho u \quad \text{in } L^2(Q).$$

This allows us to pass to the limit in the continuity equation (3.4.91) and obtain the identity

$$-\int_0^T\int_\Omega \rho\partial_t\psi dxdt - \int_\Omega \rho_{0,m}\psi(0,x)\ dx = \int_0^T\int_\Omega (\rho u)\cdot\nabla\psi\ dxdt \qquad \forall\psi \in \mathcal{D}([0,T)\times\Omega). \tag{3.5.9}$$

Next, we exploit this identity in order to derive strong convergence of the density for the limit passage in the momentum equation in Section 3.5.4 below. More precisely, as $\rho \in L^2(Q)$, the identity (3.5.9) shows that ρ solves the renormalized continuity equation (3.1.22) for all $\beta \in C^1(\mathbb{R})$ of the form (3.1.23) according to the transport theory by DiPerna and Lions [35]. Since ρ is bounded from below by $\underline{\rho}$, we do not need to care about the behavior of β close to zero and hence, without loss of generality, we may choose $\beta(z) = z\ln(z)$. This yields the identity

$$\int_\Omega \rho(\tau)\ln\left(\rho(\tau)\right)\ dx = \int_\Omega \rho_{0,m}\ln\left(\rho_{0,m}\right)\ dx \qquad \text{for a.a. } \tau \in 0, m0, T]. \tag{3.5.10}$$

Using the same choice $\beta(z) = z\ln(z)$ in the renormalized continuity equation (3.4.55) on the ϵ-level we further obtain the inequality

$$\int_\Omega \rho_\epsilon(\tau)\ln\left(\rho_\epsilon(\tau)\right)\ dx \leqslant \int_\Omega \rho_{0,m}\ln\left(\rho_{0,m}\right)\ dx \qquad \text{for a.a. } \tau \in [0,T].$$

We subtract the equation (3.5.10) and infer that

$$\lim_{\epsilon\to 0}\int_\Omega \rho_\epsilon(\tau)\ln(\rho_\epsilon(\tau))\ dx \leqslant \int_\Omega \rho(\tau)\ln(\rho(\tau))\ dx \qquad \text{for a.a. } \tau \in [0,T]. \tag{3.5.11}$$

Moreover, due to the strict convexity of the mapping $z \mapsto z\ln(z)$, we may use the well known relations between weak convergence and convex functions (cf. [45, Theorem 10.20]) to conclude that

$$\rho\ln(\rho) \leqslant \underline{\rho\ln(\rho)} \qquad \text{a.e. in } Q,$$

where $\underline{\rho\ln(\rho)}$ denotes a weak $L^1(Q)$-limit of $\rho_\epsilon\ln(\rho_\epsilon)$. In combination with the inequality (3.5.11) it follows that

$$\rho\ln(\rho) = \underline{\rho\ln(\rho)} \qquad \text{a.e. in } Q.$$

Exploiting once more the relations between weak convergence and strictly convex functions given by [45, Theorem 10.20], we infer from this identity that

$$\rho_\epsilon \to \rho \quad \text{a.e. in } Q. \tag{3.5.12}$$

In particular - for ρ extended by $\underline{\rho}$ outside of Ω - it follows that

$$\rho_\epsilon \to \rho \quad \text{in } L^p\left((0,T)\times\mathbb{R}^3\right) \quad \forall 1 \leqslant p < \infty, \qquad \underline{\rho} \leqslant \rho \leqslant \overline{\rho} \qquad \text{a.e. in } [0,T]\times\mathbb{R}^3. \tag{3.5.13}$$

3.5.2 Transport equation

From the lower bound (3.4.60) for the total mass of the solid we deduce, similarly to the corresponding bounds (3.4.13) on the Δt-level, the estimates

$$|v_\epsilon(t)|\,,\ |w_\epsilon(t)| \leqslant c\|u_\epsilon(t)\|_{L^2(\Omega)} \qquad \text{for a.a. } t \in [0,T] \tag{3.5.14}$$

with $c > 0$ independent of t and ϵ, where

$$v_\epsilon := (u_G)_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]} - \omega_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]} \times a_{[\chi_\epsilon,\rho_\epsilon]}, \qquad w_\epsilon := \omega_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}$$

and therefore

$$\|\Pi_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}(t)\|_{L^\infty(\Omega)} = \|v_\epsilon(t) + w_\epsilon(t)\times(\cdot)\,\|_{L^\infty(\Omega)} \leqslant c\|u_\epsilon(t)\|_{L^2(\Omega)} \qquad \text{for a.a. } t \in [0,T]. \tag{3.5.15}$$

The bounds (3.5.14), together with the $L^\infty(0,T;L^2(\Omega))$-bound (3.5.1) of u_ϵ, the transport equation (3.4.90) and the equation (3.4.89) for the associated characteristics guarantee us the conditions of Lemma A.4.2 in the appendix. From this and Remark A.4.2 we infer that

$$\Pi_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]} \overset{*}{\rightharpoonup} \Pi_{[\chi,\rho,u]} \quad \text{in } L^\infty\left(0,T;W^{1,\infty}_{\text{loc}}(\mathbb{R}^3)\right), \tag{3.5.16}$$

$$\eta^{\Pi_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}} \to \eta^{\Pi_{[\chi,\rho,u]}} \quad \text{in } C\left([0,T]\times[0,T];C_{\text{loc}}\left(\mathbb{R}^3\right)\right), \tag{3.5.17}$$

$$\chi_\epsilon \to \chi \quad \text{in } C\left([0,T];L^p\left(\mathbb{R}^3\right)\right) \ \forall 1 \leqslant p < \infty, \quad \chi(t,x) = \chi_0\left(\eta^{\Pi_{[\chi,\rho,u]}}(t;0,x)\right), \tag{3.5.18}$$

where $\eta^{\Pi_{[\chi,\rho,u]}}$ and χ denote the unique solutions to the initial value problem

$$\frac{d\eta^{\Pi_{[\chi,\rho,u]}}(s;t,x)}{dt} = \Pi_{[\chi,\rho,u]}\left(t, \eta^{\Pi_{[\chi,\rho,u]}}(s;t,x)\right), \qquad \eta^{\Pi_{[\chi,\rho,u]}}(s;s,x) = x$$

for all $x \in \mathbb{R}^3$, $s,t \in [0,T]$ and the transport equation

$$-\int_0^T \int_{\mathbb{R}^3} \chi \partial_t \Theta dx dt - \int_{\mathbb{R}^3} \chi_0 \Theta(0,x)\, dx = \int_0^T \int_{\mathbb{R}^3} \left(\chi \Pi_{[\chi,\rho,u]}\right) \cdot \nabla\Theta \, dx dt$$

for all $\Theta \in \mathcal{D}([0,T) \times \mathbb{R}^3)$ respectively. The identification of the limit function in the convergence (3.5.16) as the rigid velocity field $\Pi_{[\chi,\rho,u]}$ can be seen, exactly as the corresponding identification (3.4.65) on the ϵ-level, from the strong convergence (3.5.13) of the density and the strong convergence (3.5.18) of the characteristic function. For the limit passage in the induction equation in Section 3.5.3 below we keep record of the fact that the uniform convergence (3.5.17) allows us to find, for any $\kappa > 0$, some value $\delta(\kappa) > 0$ with

$$(S(t))_\kappa \subset S_\epsilon(t) \subset (S(t))^{\frac{\kappa}{2}} \subset (S(t))^\kappa \qquad \forall t \in [0,T],\ \epsilon < \delta(\kappa), \tag{3.5.19}$$

where

$$S(t) := \eta^{\Pi_{[\chi,\rho,u]}}(0;t,S_0) = \left\{x \in \mathbb{R}^3 :\ \chi(t,x) = 1\right\}.$$

3.5.3 Induction equation

The first inclusion in (3.5.19) shows that, exactly as the corresponding relation (3.4.67) in the limit passage with respect to $\Delta t \to 0$, the magnetic induction B in the limit is curl-free in the solid region,

$$\text{curl} B = 0 \qquad \text{a.e. in } Q^s(S) \bigcap Q. \tag{3.5.20}$$

Next, we improve the bounds for u_ϵ and B_ϵ via an interpolation of their bounds (3.5.1) in $L^\infty(0,T;L^2(\Omega))$ and $L^2(0,T;H^1(\Omega))$. This shows that

$$\begin{aligned} \|u_\epsilon\|_{L^3(Q)} &\leqslant \left[\int_0^T \|u_\epsilon(t)\|_{L^6(\Omega)}^{\frac{3}{2}} \|u_\epsilon(t)\|_{L^2(\Omega)}^{\frac{3}{2}} dt\right]^{\frac{1}{3}} \\ &\leqslant \|u_\epsilon\|^{\frac{1}{2}}_{L^{\frac{3}{2}}(0,T;L^6(\Omega))} \|u_\epsilon\|^{\frac{1}{2}}_{L^\infty(0,T;L^2(\Omega))} \leqslant c\, \|u_\epsilon\|^{\frac{1}{2}}_{L^2(0,T;H^1(\Omega))} \|u_\epsilon\|^{\frac{1}{2}}_{L^\infty(0,T;L^2(\Omega))} \leqslant c \end{aligned} \tag{3.5.21}$$

and in the same way

$$\|B_\epsilon\|_{L^3(Q)} \leqslant c \tag{3.5.22}$$

for a constant $c > 0$ independent of ϵ. This, in combination with the Hölder inequality, implies the existence of functions $z_4, z_5 \in L^{\frac{6}{5}}(Q)$ such that, possibly after the extraction of a subsequence,

$$u_\epsilon \times B_\epsilon \rightharpoonup z_5 \quad \text{in } L^{\frac{6}{5}}(Q), \qquad \text{curl} B_\epsilon \times B_\epsilon \rightharpoonup z_6 \quad \text{in } L^{\frac{6}{5}}(Q). \tag{3.5.23}$$

For the identification of the limit functions z_4 and z_5 we again study the solid and the fluid domain separately. In the solid domain it is sufficient to remark that, due to the identity $\text{curl}\, B_\epsilon = 0$ in $Q^s(S_\epsilon) \bigcap Q$ and the first inclusion in (3.5.19),

$$z_5 \cdot \text{curl} b = 0 = (u \times B) \cdot \text{curl} b, \quad z_6 = 0 = \text{curl} B \times B \qquad \text{a.e. in } Q^s(S) \bigcap Q \tag{3.5.24}$$

for any $b \in Y(S)$. For the corresponding identification in the fluid domain we need to show strong convergence of B_ϵ. To this end we first note that for any $\kappa > 0$ and any function $b \in Y(S)$ being curl-free in a κ-neighborhood of $\overline{Q^s(S)}$ it also holds that

$$\text{curl}\, b = 0 \quad \text{in a } \frac{\kappa}{2}\text{-neighborhood of } \overline{Q^s(S_\epsilon)} \quad \text{and thus} \quad b \in Y(S_\epsilon) \quad \forall \epsilon < \delta(\kappa), \tag{3.5.25}$$

where $\delta(\kappa)$ denotes the parameter specified in the inclusions (3.5.19). We fix an arbitrary interval $I \subset (0,T)$ and an arbitrary ball $U \subset \Omega$ with the property $\overline{I \times U} \subset Q^f(S)$. Since $\operatorname{dist}(\overline{I \times U}, Q^s(S)) > \kappa$ for some real number $\kappa > 0$, (3.5.25) implies that

$$\psi b \in Y(S_\epsilon) \qquad \forall \psi \in \mathcal{D}(I),\ b \in \mathcal{D}(U),\ \epsilon < \delta(\kappa), \tag{3.5.26}$$

where ψ and b are extended by 0 outside of I and U respectively. Therefore, for any $\epsilon < \delta(\kappa)$, any function ψb of the form (3.5.26) is an admissible test function in the induction equation (3.4.93) on the ϵ-level. Using it as such we infer the dual estimate

$$\begin{aligned}
&\left\| \partial_t \int_U B_\epsilon \cdot b \, dx \right\|_{L^{\frac{6}{5}}(I)} \\
&= \left\| \int_U \left[-\frac{1}{\sigma\mu} \operatorname{curl} B_\epsilon + u_\epsilon \times B_\epsilon + \frac{1}{\sigma} J - \frac{\epsilon}{\mu^2} z_\epsilon \right] \cdot \operatorname{curl} b - \epsilon \left(\nabla \operatorname{curl} B_\epsilon \right) \cdot \left(\nabla \operatorname{curl} b \right) \, dx \right\|_{L^{\frac{6}{5}}(I)} \\
&\leqslant \frac{1}{\sigma\mu} \|\operatorname{curl} B_\epsilon\|_{L^{\frac{6}{5}}(Q)} \|\operatorname{curl} b\|_{L^6(U)} + \|u_\epsilon\|_{L^2(Q)} \|B_\epsilon\|_{L^3(Q)} \|\operatorname{curl} b\|_{L^6(U)} \\
&\quad + \frac{1}{\sigma} \|J\|_{L^{\frac{6}{5}}(Q)} \|\operatorname{curl} b\|_{L^6(U)} + \frac{\epsilon}{\mu^2} \|z_\epsilon\|_{L^{\frac{6}{5}}(Q)} \|\operatorname{curl} b\|_{L^6(U)} + \epsilon \|\nabla \operatorname{curl} B_\epsilon\|_{L^{\frac{6}{5}}(Q)} \|\nabla \operatorname{curl} b\|_{L^6(U)} \leqslant c.
\end{aligned}$$

with a constant $c > 0$ depending on b but independent of ϵ due to the uniform bounds (3.5.1), (3.5.2) and (3.5.22). Since $L^2(U)$ is embedded compactly into $H^{-1}(U)$ we thus infer from Lemma A.4.1 in the appendix that

$$B_\epsilon \to B \quad \text{in } C_{\text{weak}}\left(\overline{I}; L^2(U)\right) \text{ and thus in } L^p\left(I; H^{-1}(U)\right) \qquad \forall 1 \leqslant p < \infty. \tag{3.5.27}$$

For the identification of z_5, we now choose an arbitrary test function $b \in \mathcal{D}(I \times U)$. Then the strong convergence (3.5.27) and the weak $L^2(0,T; H^1(\Omega))$-convergence (3.5.8) of B_ϵ show that

$$\begin{aligned}
\int_I \int_U (\operatorname{curl} B_\epsilon \times B_\epsilon) \cdot b \, dxdt &= \int_I \int_U -\left(B_\epsilon \otimes B_\epsilon\right) : \nabla b + \frac{1}{2} |B_\epsilon|^2 \operatorname{div} b \, dxdt \\
&\to \int_I \int_U -\left(B \otimes B\right) : \nabla b + \frac{1}{2} |B|^2 \operatorname{div} b \, dxdt = \int_I \int_U (\operatorname{curl} B \times B) \cdot b \, dxdt.
\end{aligned}$$

This, together with the strong convergence (3.5.27) of B_ϵ and the weak $L^2(0,T; H^1(\Omega))$-convergence (3.5.7) of u_ϵ for the identification of z_4, allows us to identify, as desired,

$$z_5 = u \times B \quad \text{a.e. in } Q^f(S), \qquad z_6 = \operatorname{curl} B \times B \quad \text{a.e. in } Q^f(S). \tag{3.5.28}$$

We are now in the position to carry out the limit passage in the induction equation. To this end we consider an arbitrary test function $b \in Y(S)$. From the inclusion (3.5.25) we immediately see that b is also an admissible test function in the induction equation (3.4.93) on the ϵ-level for all sufficiently small $\epsilon > 0$. Letting ϵ tend to zero and making use of the convergences (3.5.23) as well as of the identities (3.5.24) and (3.5.28) we obtain the relation

$$- \int_0^T \int_\Omega B \cdot \partial_t b \, dxdt - \int_\Omega B_{0,m} \cdot b(0) \, dx = \int_0^T \int_\Omega \left[-\frac{1}{\sigma\mu} \operatorname{curl} B + u \times B + \frac{1}{\sigma} J \right] \cdot \operatorname{curl} b \, dxdt$$

for any $b \in Y(S)$.

3.5.4 Momentum equation

For the limit passage in the momentum equation it remains to prove strong convergence of the velocity field. To this end we first choose an arbitrary function $\Phi \in \mathcal{D}(\Omega)$. We estimate, under exploitation of the Gagliardo-Nirenberg interpolation inequality as in (3.4.79),

$$\begin{aligned}
&\left\| \int_\Omega \epsilon \left(\nabla u_\epsilon \nabla \rho_\epsilon\right) \cdot \Phi \; dx \right\|_{L^{\frac{4}{3}}(0,T)} \\
\leqslant & \epsilon^{\frac{1}{2}} \left\|\nabla u_\epsilon\right\|_{L^4(0,T;L^2(\Omega))} \epsilon^{\frac{1}{2}} \left\|\nabla \rho_\epsilon\right\|_{L^2(Q)} \|\Phi\|_{L^\infty(\Omega)} \\
\leqslant & c\epsilon^{\frac{1}{2}} \left(\int_0^T \left\|\nabla^2 u_\epsilon(t)\right\|^2_{L^2(\Omega)} \|u_\epsilon(t)\|^2_{L^2(\Omega)} + \|u_\epsilon(t)\|^4_{L^2(\Omega)} \; dt \right)^{\frac{1}{4}} \epsilon^{\frac{1}{2}} \left\|\nabla \rho_\epsilon\right\|_{L^2(Q)} \|\Phi\|_{L^\infty(\Omega)} \\
\leqslant & c\epsilon^{\frac{1}{2}} \left(\|u_\epsilon\|^{\frac{1}{2}}_{L^2(0,T;H^2(\Omega))} \|u_\epsilon\|^{\frac{1}{2}}_{L^\infty(0,T;L^2(\Omega))} + \|u_\epsilon\|_{L^4(0,T;L^2(\Omega))} \right) \epsilon^{\frac{1}{2}} \left\|\nabla \rho_\epsilon\right\|_{L^2(Q)} \|\Phi\|_{L^\infty(\Omega)} \\
\leqslant & c\epsilon^{\frac{1}{2}} \|\Delta u_\epsilon\|^{\frac{1}{2}}_{L^2(Q)} \|u_\epsilon\|^{\frac{1}{2}}_{L^\infty(0,T;L^2(\Omega))} \epsilon^{\frac{1}{2}} \left\|\nabla \rho_\epsilon\right\|_{L^2(Q)} \|\Phi\|_{L^\infty(\Omega)} \\
& + c\epsilon^{\frac{1}{2}} \|u_\epsilon\|_{L^\infty(0,T;L^2(\Omega))} \epsilon^{\frac{1}{2}} \left\|\nabla \rho_\epsilon\right\|_{L^2(Q)} \|\Phi\|_{L^\infty(\Omega)} . \qquad (3.5.29)
\end{aligned}$$

Here, the last inequality is due to the estimate for the $H^2(\Omega)$-norm given by Lemma A.2.1 in the appendix. Next, we test the momentum equation (3.4.92) on the ϵ-level by $\psi\Phi$ with $\psi \in \mathcal{D}(0,T)$ and $\Phi \in \mathcal{D}(\Omega)$ such that $\operatorname{div}\Phi = 0$ in Ω. The inequality (3.5.29) together with the identity $P(\Phi) = \Phi$ for the orthogonal projection P from $L^2(\Omega)$ onto $L^2_{\text{div}}(\Omega)$ allows us to estimate

$$\begin{aligned}
&\left\| \partial_t \int_\Omega P(\rho_\epsilon u_\epsilon) \cdot \Phi dx \right\|_{L^{\frac{4}{3}}(0,T)} \\
=&\Bigg\| \int_\Omega \left(\rho_\epsilon u_\epsilon \otimes u_\epsilon\right) : \nabla\Phi - 2\nu \mathbb{D}\left(u_\epsilon\right) : \nabla \Phi - m\rho_\epsilon\chi_\epsilon \left(u_\epsilon - \Pi_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}\right) \cdot \Phi + \rho_\epsilon g \cdot \Phi + \frac{1}{\mu}\left(\operatorname{curl} B_\epsilon \times B_\epsilon\right) \cdot \Phi \\
& - \epsilon \left(\nabla u_\epsilon \nabla \rho_\epsilon\right) \cdot \Phi - \epsilon \Delta u_\epsilon \cdot \Delta \Phi \; dxdt \Bigg\|_{L^{\frac{4}{3}}(0,T)} \\
\leqslant & \overline{\rho} \|u_\epsilon\|^2_{L^{\frac{8}{3}}(0,T;L^2(\Omega))} \|\Phi\|_{L^\infty(\Omega)} + 2\nu \left\|\mathbb{D}\left(u_\epsilon\right)\right\|_{L^{\frac{4}{3}}(Q)} \|\nabla\Phi\|_{L^4(\Omega)} + \overline{\rho} m \left\|u_\epsilon - \Pi_{[\chi_\epsilon,\rho_\epsilon,u_\epsilon]}\right\|_{L^{\frac{4}{3}}(Q)} \|\Phi\|_{L^4(\Omega)} \\
& + \overline{\rho} \|g\|_{L^{\frac{4}{3}}(Q)} \|\Phi\|_{L^4(\Omega)} + \frac{1}{\mu} \|B_\epsilon\|_{L^\infty(0,T;L^2(\Omega))} \left\|\operatorname{curl} B_\epsilon\right\|_{L^{\frac{4}{3}}(0,T;L^2(\Omega))} \|\Phi\|_{L^\infty(\Omega)} \\
& + c\epsilon^{\frac{1}{2}} \|\Delta u_\epsilon\|^{\frac{1}{2}}_{L^2(Q)} \|u_\epsilon\|^{\frac{1}{2}}_{L^\infty(0,T;L^2(\Omega))} \epsilon^{\frac{1}{2}} \left\|\nabla \rho_\epsilon\right\|_{L^2(Q)} \|\Phi\|_{L^\infty(\Omega)} \\
& + c\epsilon^{\frac{1}{2}} \|u_\epsilon\|_{L^\infty(0,T;L^2(\Omega))} \epsilon^{\frac{1}{2}} \left\|\nabla \rho_\epsilon\right\|_{L^2(Q)} \|\Phi\|_{L^\infty(\Omega)} + \epsilon \|\Delta u_\epsilon\|_{L^{\frac{4}{3}}(Q)} \|\Delta\Phi\|_{L^4(\Omega)} \\
\leqslant & c
\end{aligned}$$

for a constant $c > 0$ depending on Φ but not on ϵ according to the uniform bounds (3.5.1)–(3.5.3) and the estimate (3.5.15). Because of the compactness of the embedding of $L^2_{\text{div}}(\Omega)$ into $(H^1_{0,\text{div}}(\Omega))^*$ we thus conclude from Lemma A.4.1 in the Appendix that

$$P\left(\rho_\epsilon u_\epsilon\right) \to P(\rho u) \quad \text{in } C_{\text{weak}}\left([0,T]; L^2_{\text{div}}(\Omega)\right) \text{ and hence in } L^2\left(0,T; \left(H^1_{0,\text{div}}(\Omega)\right)^*\right).$$

This convergence, exactly as in the deduction of the strong convergence (3.4.83) of the velocity field in the limit passage with respect to $\Delta t \to 0$, first yields

$$\int_0^T \int_\Omega \rho_\epsilon |u_\epsilon|^2 \; dxdt \to \int_0^T \int_\Omega \rho |u|^2 \; dxdt$$

and subsequently, together with the strong convergence (3.5.13) of the density and the $L^3(Q)$-bound (3.5.21) of u_ϵ, the desired strong convergence

$$u_\epsilon \to u \quad \text{in } L^q\left(Q\right) \qquad \forall 1 \leqslant q < 3. \qquad (3.5.30)$$

In particular we conclude that

$$\rho_\epsilon u_\epsilon \otimes u_\epsilon \rightharpoonup \rho u \otimes u \qquad \text{in } L^{\frac{3}{2}}\left(Q\right).$$

Combining this with the convergence (3.5.23), (3.5.24), (3.5.28) of the Lorentz force, we are now in the position to pass to the limit in the momentum equation (3.4.92). Altogether we infer the relation

$$
\begin{aligned}
&-\int_0^T\int_\Omega \rho u\cdot\partial_t\phi\;dxdt-\int_\Omega \rho_{0,m}u_{0,m}\cdot\phi(0,x)\;dx\\
&=\int_0^T\int_\Omega (\rho u\otimes u):\nabla\phi-2\nu\mathbb{D}(u):\nabla\phi-m\rho\chi\left(u-\Pi_{[\chi,\rho,u]}\right)\cdot\phi+\rho g\cdot\phi+\frac{1}{\mu}\left(\mathrm{curl}B\times B\right)\cdot\phi\;dxdt
\end{aligned}
$$

for any $\phi\in\mathcal{D}([0,T)\times\Omega)$ with $\mathrm{div}\phi=0$.

3.5.5 Energy inequality

We drop the (nonnegative) regularization terms from the left-hand side of the energy inequality (3.4.94). Using the weak lower semicontinuity of norms, we then let ϵ tend to 0 and obtain

$$
\begin{aligned}
&\int_\Omega \frac{1}{2}\rho(\tau)|u(\tau)|^2+\frac{1}{2}|B(\tau)|^2\;dx+\int_0^\tau\int_\Omega 2\nu\left|\nabla u\right|^2+\frac{1}{\sigma\mu^2}\left|\mathrm{curl}B\right|^2+m\rho\chi\left|\left(u-\Pi_{[\chi,\rho,u]}\right)\right|^2\;dxdt\\
&\leqslant\int_\Omega \frac{1}{2}\rho_{0,m}|u_{0,m}|^2+\frac{1}{2}|B_{0,m}|^2\;dx+\int_0^\tau\int_\Omega \rho g\cdot u+\frac{1}{\sigma\mu}J\cdot\mathrm{curl}B\;dxdt
\end{aligned}
$$

for almost all $\tau\in[0,T]$. Altogether we have shown

Proposition 3.5.1. *Let all the assumptions of Theorem 3.1.1 be satisfied and let $m\in\mathbb{N}$. Assume in addition the regularized initial data $\rho_{0,m}$, $u_{0,m}$, $B_{0,m}$ to satisfy the conditions* (3.2.11). *Then, there exists an isometry $\eta^{\Pi_{[\chi_m,\rho_m,u_m]}}(s;t,\cdot):\mathbb{R}^3\to\mathbb{R}^3$, $s,t\in[0,T]$, and*

$$
\begin{aligned}
&\chi_m\in C\left([0,T];L^p\left(\mathbb{R}^3\right)\right),\;1\leqslant p<\infty,\\
&\rho_m\in\left\{\psi\in L^\infty(Q):\;\underline{\rho}\leqslant\psi\leqslant\overline{\rho}\;a.e.\;in\;Q\right\}, && (3.5.31)\\
&u_m\in L^\infty\left(0,T;L^2(\Omega)\right)\bigcap L^2\left(0,T;H^1_{0,\mathrm{div}}(\Omega)\right) && (3.5.32)\\
&B_m\in\left\{b\in L^\infty\left(0,T;L^2(\Omega)\right)\bigcap L^2\left(0,T;H^1_{\mathrm{div}}(\Omega)\right):\;\mathrm{curl}\,b=0\;in\;Q^s(S_m)\bigcap Q,\;b\cdot\mathrm{n}\,|_{\partial\Omega}=0\right\}, && (3.5.33)
\end{aligned}
$$

for $S_m=S_m(\cdot)=\eta^{\Pi_{[\chi_m,\rho_m,u_m]}}(0;\cdot,S_0)$, such that

$$
\frac{d\eta^{\Pi_{[\chi_m,\rho_m,u_m]}}(s;t,x)}{dt}=\Pi_{[\chi_m,\rho_m,u_m]}\left(t,\eta^{\Pi_{[\chi_m,\rho_m,u_m]}}(s;t,x)\right),\qquad \eta^{\Pi_{[\chi_m,\rho_m,u_m]}}(s;s,x)=x \tag{3.5.34}
$$

$$
-\int_0^T\int_{\mathbb{R}^3}\chi_m\partial_t\Theta dxdt-\int_{\mathbb{R}^3}\chi_0\Theta(0,x)\;dx=\int_0^T\int_{\mathbb{R}^3}\left(\chi_m\Pi_{[\chi_m,\rho_m,u_m]}\right)\cdot\nabla\Theta\;dxdt, \tag{3.5.35}
$$

$$
-\int_0^T\int_\Omega \rho_m\partial_t\psi dxdt-\int_\Omega \rho_{0,m}\psi(0,x)\;dx=\int_0^T\int_\Omega (\rho_m u_m)\cdot\nabla\psi\;dxdt, \tag{3.5.36}
$$

$$
\begin{aligned}
-\int_0^T\int_\Omega \rho_m u_m\cdot\partial_t\phi\;dxdt-\int_\Omega \rho_{0,m}u_{0,m}\cdot\phi(0,x)\;dx=&\int_0^T\int_\Omega (\rho_m u_m\otimes u_m):\nabla\phi-2\nu\mathbb{D}(u_m):\nabla\phi\\
&-m\rho_m\chi_m\left(u_m-\Pi_{[\chi_m,\rho_m,u_m]}\right)\cdot\phi+\rho_m g\cdot\phi\\
&+\frac{1}{\mu}\left(\mathrm{curl}\,B_m\times B_m\right)\cdot\phi\;dxdt,
\end{aligned} \tag{3.5.37}
$$

$$
\begin{aligned}
-\int_0^T\int_\Omega B_m\cdot\partial_t b\;dxdt-\int_\Omega B_{0,m}\cdot b(0,x)\;dx=\int_0^T\int_\Omega\Bigg[&-\frac{1}{\sigma\mu}\mathrm{curl}\,B_m+u_m\times B_m\\
&+\frac{1}{\sigma}J\Bigg]\cdot\mathrm{curl}\,b\;dxdt
\end{aligned} \tag{3.5.38}
$$

for all $\Theta \in \mathcal{D}([0,T) \times \mathbb{R}^3)$, $\psi, \phi \in \mathcal{D}([0,T) \times \Omega)$ *with* $\operatorname{div} \phi = 0$ *and all* $b \in Y(S_m)$. *Moreover, these functions satisfy the energy inequality*

$$\begin{aligned}
&\int_\Omega \frac{1}{2}\rho_m(\tau)|u_m(\tau)|^2 + \frac{1}{2\mu}|B_m(\tau)|^2 \, dx + \int_0^\tau \int_\Omega 2\nu \, |\nabla u_m|^2 + \frac{1}{\sigma\mu^2} \, |\operatorname{curl} B_m|^2 \\
&+ m\rho_m\chi_m \left|u_m - \Pi_{[\chi_m,\rho_m,u_m]}\right|^2 \, dxdt \\
\leqslant &\int_\Omega \frac{1}{2}\rho_{0,m}|u_{0,m}|^2 + \frac{1}{2}|B_{0,m}|^2 \, dx + \int_0^\tau \int_\Omega \rho_m g \cdot u_m + \frac{1}{\sigma\mu} J \cdot \operatorname{curl} B_m \, dxdt
\end{aligned} \tag{3.5.39}$$

for almost all $\tau \in [0,T]$ *and the characteristic function* χ_m *is connected to the isometry* $\eta^{\Pi_{[\chi_m,\rho_m,u_m]}}$ *by the formula*

$$\chi(t,x) = \chi_0 \left(\eta^{\Pi_{[\chi_m,\rho_m,u_m]}}(t;0,x)\right).$$

3.6 Limit passage with respect to $m \to \infty$

In order to prove the main result Theorem 3.1.1 of this chapter, we now assume the regularized initial data on the m-level to satisfy

$$\rho_{0,m} \to \rho_0 \quad \text{in } L^2(\Omega), \qquad u_{0,m} \to u_0 \quad \text{in } L^2(\Omega), \qquad B_{0,m} \to B_0 \quad \text{in } L^2(\Omega), \tag{3.6.1}$$

where ρ_0, u_0, B_0 denote the initial data in Theorem 3.1.1. The energy inequality (3.5.39) implies the existence of a constant $c > 0$, independent of m, such that

$$\|u_m\|_{L^\infty(0,T;L^2(\Omega))} + \|u_m\|_{L^2(0,T;H^1(\Omega))} + \|B_m\|_{L^\infty(0,T;L^2(\Omega))} + \|B_m\|_{L^2(0,T;H^1(\Omega))} \leqslant c, \tag{3.6.2}$$

$$m^{\frac{1}{2}} \left\|\chi_m \left(u_m - \Pi_{[\chi_m,\rho_m,u_m]}\right)\right\|_{L^2(Q)} \leqslant c. \tag{3.6.3}$$

The above bounds, together with the uniform bounds for the density in (3.5.31), allow us to find functions

$$\begin{aligned}
&\rho \in \left\{\psi \in L^\infty(Q) : \ \underline{\rho} \leqslant \psi \leqslant \overline{\rho} \quad \text{a.e. in } Q\right\}, \\
&u \in L^\infty\left(0,T;L^2(\Omega)\right) \bigcap L^2\left(0,T;H^1_{0,\text{div}}(\Omega)\right),
\end{aligned} \tag{3.6.4}$$

$$B \in \left\{ b \in L^\infty\left(0,T;L^2(\Omega)\right) \bigcap L^2\left(0,T;H^1_{\text{div}}(\Omega)\right) : \ b \cdot \mathrm{n}|_{\partial\Omega} = 0 \right\} \tag{3.6.5}$$

such that for extracted subsequences

$$\begin{aligned}
&\rho_m \overset{*}{\rightharpoonup} \rho \quad \text{in } L^\infty(0,T;L^\infty(\Omega)), \\
&u_m \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty\left(0,T;L^2(\Omega)\right), \qquad u_m \rightharpoonup u \quad \text{in } L^2\left(0,T;H^1(\Omega)\right), \\
&B_m \overset{*}{\rightharpoonup} B \quad \text{in } L^\infty\left(0,T;L^2(\Omega)\right), \qquad B_m \rightharpoonup B \quad \text{in } L^2\left(0,T;H^1(\Omega)\right).
\end{aligned} \tag{3.6.6}$$

The boundary conditions of u and B in (3.6.4) and (3.6.5) follow directly from the corresponding boundary conditions of the velocity field and the magnetic induction on the m-level given by the inclusions (3.5.32) and (3.5.33).

3.6.1 Continuity equation

With the regularization term in the continuity equation gone, the proof of the strong convergence of the density is greatly simplified in comparison to the limit passage with respect to $\epsilon \to 0$. More precisely, we can directly apply the classical compactness results for the incompressible Navier-Stokes equations, cf. [86, Theorem 2.4, Remark 2.4 3)], and infer that

$$\rho_m \to \rho \quad \text{in } C\left([0,T]; L^p\left(\mathbb{R}^3\right)\right) \quad \forall 1 \leqslant p < \infty, \tag{3.6.7}$$

with ρ once again extended by $\underline{\rho}$ outside of Ω. This, together with the weak convergence (3.6.6) of u_m and the convergence (3.6.1) of the initial data, suffices to pass to the limit in the continuity equation (3.5.36) and obtain the identity

$$-\int_0^T \int_\Omega \rho \partial_t \psi dxdt - \int_\Omega \rho_0 \psi(0,x) \, dx = \int_0^T \int_\Omega (\rho u) \cdot \nabla\psi \, dxdt \quad \forall \psi \in \mathcal{D}([0,T) \times \Omega). \tag{3.6.8}$$

3.6.2 Transport equation

Similarly to the corresponding estimates (3.4.13) on the Δt-level we deduce that

$$|v_m(t)|\,,\ |w_m(t)| \leqslant c\|u_m(t)\|_{L^2(\Omega)} \qquad \text{for a.a. } t \in [0,T]$$

with $c > 0$ independent of t and m, where

$$v_m := (u_G)_{[\chi_m,\rho_m,u_m]} - \omega_{[\chi_m,\rho_m,u_m]} \times a_{[\chi_m,\rho_m]}, \qquad w_m := \omega_{[\chi_m,\rho_m,u_m]}$$

and thus

$$\left\|\Pi_{[\chi_m,\rho_m,u_m]}(t)\right\|_{L^\infty(\Omega)} = \|v_m(t) + w_m(t) \times (\cdot)\,\|_{L^\infty(\Omega)} \leqslant c\|u_m(t)\|_{L^2(\Omega)} \qquad \text{for a.a. } t \in [0,T], \quad (3.6.9)$$

$$\left\|\nabla\Pi_{[\chi_m,\rho_m,u_m]}(t)\right\|_{L^\infty(\Omega)} \leqslant c\|u_m(t)\|_{L^2(\Omega)} \qquad \text{for a.a. } t \in [0,T]. \quad (3.6.10)$$

In combination with the transport equation (3.5.35) and the equation (3.5.34) these bounds allow us once more to apply Lemma A.4.2 and Remark A.4.2, which yield

$$\Pi_{[\chi_m,\rho_m,u_m]} \overset{*}{\rightharpoonup} \Pi_{[\chi,\rho,u]} \quad \text{in } L^\infty\left(0,T;W^{1,\infty}_{\text{loc}}(\mathbb{R}^3)\right), \quad (3.6.11)$$

$$\eta_m^{\Pi_{[\chi_m,\rho_m,u_m]}} \to \eta^{\Pi_{[\chi,\rho,u]}} \quad \text{in } C([0,T] \times [0,T]; C_{\text{loc}}\left(\mathbb{R}^3\right)), \quad (3.6.12)$$

$$\chi_m \to \chi \quad \text{in } C\left([0,T]; L^p\left(\mathbb{R}^3\right)\right) \ \ \forall 1 \leqslant p < \infty, \quad \chi(t,x) = \chi_0\left(\eta^{\Pi_{[\chi,\rho,u]}}(t;0,x)\right), \quad (3.6.13)$$

where $\eta^{\Pi_{[\chi,\rho,u]}}$ and χ denote the unique solutions to

$$\frac{d\eta^{\Pi_{[\chi,\rho,u]}}(s;t,x)}{dt} = \Pi_{[\chi,\rho,u]}\left(t, \eta^{\Pi_{[\chi,\rho,u]}}(s;t,x)\right), \quad \eta^{\Pi_{[\chi,\rho,u]}}(s;s,x) = x, \quad (3.6.14)$$

$$-\int_0^T \int_{\mathbb{R}^3} \chi\partial_t\Theta dxdt - \int_{\mathbb{R}^3} \chi_0\Theta(0,x)\ dx = \int_0^T \int_{\mathbb{R}^3} \left(\chi\Pi_{[\chi,\rho,u]}\right) \cdot \nabla\Theta\ dxdt \quad (3.6.15)$$

for all $x \in \mathbb{R}^3$, all $s,t \in [0,T]$ and all $\Theta \in \mathcal{D}\left([0,T) \times \mathbb{R}^3\right)$. The identification of the limit function in (3.6.11) as $\Pi_{[\chi,\rho,u]}$ is, just like the corresponding identity (3.4.65) on the Δt-level, a consequence of the strong convergence (3.6.7) of the density and the strong convergence (3.6.13) of the characteristic function. As a consequence of the uniform convergence (3.6.12) we find, for any $\kappa > 0$, some number $M(\kappa) > 0$ such that

$$(S(t))_\kappa \subset S_m(t) \subset (S(t))^{\frac{\kappa}{2}} \subset (S(t))^\kappa \qquad \forall t \in [0,T],\ m \geqslant M(\kappa), \quad (3.6.16)$$

where

$$S(t) := \eta^{\Pi_{[\chi,\rho,u]}}(0;t,S_0) = \left\{x \in \mathbb{R}^3 :\ \chi(t,x) = 1\right\}.$$

3.6.3 Induction equation

In the induction equation, all the approximation terms already vanished during the previous limit passage. Thus the limit passage with respect to $m \to \infty$ works by the same arguments as before. Indeed, with the inclusions (3.6.16) at hand we can argue as in the derivation of the corresponding relation (3.4.67) on the Δt-level to conclude that

$$\text{curl}B = 0 \quad \text{a.e. in } Q^s(S) \bigcap Q \quad (3.6.17)$$

and, as in Section 3.5.3 on the ϵ-level,

$$(u_m \times B_m) \cdot \text{curl}\, b \rightharpoonup (u \times B) \cdot \text{curl}\, b \quad \text{in } L^{\frac{6}{5}}(Q)\,,$$

$$\text{curl}\, B_m \times B_m \rightharpoonup \text{curl}\, B \times B \quad \text{in } L^{\frac{6}{5}}(Q) \quad (3.6.18)$$

for all $b \in Y(S)$. Exploiting further the convergence (3.6.1) of the initial data, we can pass to the limit in the induction equation (3.5.38) and obtain

$$-\int_0^T \int_\Omega B \cdot \partial_t b\ dxdt - \int_\Omega B_0 \cdot b(0,x)\ dx = \int_0^T \int_\Omega \left[-\frac{1}{\sigma\mu}\text{curl}B + u \times B + \frac{1}{\sigma}J\right] \cdot \text{curl}b\ dxdt \quad (3.6.19)$$

for all $b \in Y(S)$.

3.6.4 Momentum equation

Let now T' be given by (3.1.21), i.e. T' denotes the first time at which the rigid body $S(\chi(\cdot))$ collides with $\partial\Omega$ or, if this never happens in $[0, T]$, then $T' = T$. Since the initial distance between the body and $\partial\Omega$ is positive by (3.1.8), the uniform convergence (3.6.12) implies $T' > 0$ and, for any $T_0 < T'$, there exists some $\kappa > 0$ such that

$$\text{dist}\Big(\partial\Omega,\ S(t)\Big) > \kappa \quad \forall t \in [0, T_0]. \tag{3.6.20}$$

Our first goal in this section is to show that the limit velocity u indeed coincides with its own projection $\Pi_{[\chi,\rho,u]}$ onto a rigid velocity field in the solid region. To this end we consider an arbitrary compact set $\overline{I \times U} \subset Q^s(S, T')$ with an interval $I \subset (0, T')$ and some ball $U \subset \Omega$. From the first inclusion in (3.6.16) we see that for sufficiently large m it holds that

$$\overline{I \times U} \subset Q^s(S_m, T') \bigcap Q \quad \Leftrightarrow \quad \chi_m = 1 \quad \text{in } \overline{I \times U}.$$

Consequently the estimate (3.6.3) implies that

$$u_m - \Pi_{[\chi_m,\rho_m,u_m]} \to 0 \quad \text{in } L^2(I \times U),$$

and as $\overline{I \times U}$ was chosen arbitrarily we get, as desired,

$$u = \Pi_{[\chi,\rho,u]} \quad \text{a.e. in } Q^s(S, T'). \tag{3.6.21}$$

Next, we show that the penalization term vanishes in the limit of the momentum equation (3.5.37). We fix some arbitrary function ϕ from the test function space $Z(S, T')$, defined in (3.1.1). In particular, $\phi \in \mathcal{D}([0, T') \times \Omega)$ and we can choose $T_0 < T'$ such that

$$\text{supp}\phi \subset [0, T_0] \times \Omega \tag{3.6.22}$$

and a corresponding $\kappa > 0$ according to (3.6.20). The inclusion $\phi \in Z(S, T')$ further implies the existence of $0 < \sigma < \kappa$ such that

$$\mathbb{D}(\phi) = 0 \quad \text{in } \Big\{(t, x) \in Q\left(T'\right) :\ \text{dist}\left((t, x), Q^s(S, T')\right) < \sigma\Big\},$$

cf. (3.1.2). Consequently, for all $t \in [0, T_0]$, the function $\phi(t, \cdot)$ coincides with a rigid velocity field $\phi^s(t, \cdot)$ on $(S(t))^\sigma \subset \Omega$ and, by the inclusion (3.6.16), also on $S_m(t)$ for all $m \geqslant M(\sigma)$. As

$$\chi_m(t, x) = 0 \quad \text{for } x \in \Omega \backslash S_m\left(t\right),$$

we infer that for such m it holds that

$$\int_0^{T'} \int_\Omega -m\rho_m\chi_m\left(u_m - \Pi_{[\chi_m,\rho_m,u_m]}\right) \cdot \phi\ dxdt = \int_0^{T'} \int_\Omega -m\rho_m\chi_m\left(u_m - \Pi_{[\chi_m,\rho_m,u_m]}\right) \cdot \phi^s\ dxdt = 0, \tag{3.6.23}$$

where the second equality is a consequence of the fact that $\Pi_{[\chi_m,\rho_m,u_m]}(t, \cdot)$ is the orthogonal projection of $u_m(t, \cdot)$ onto rigid velocity fields on $S_m(t)$, cf. (3.1.7). The final ingredient we require for the limit passage in the momentum equation is strong convergence of the velocity field,

$$u_m \to u \quad \text{in } L^q\left(Q\left(T_0\right)\right) \qquad \forall 1 \leqslant q < 3,\ 0 < T_0 < T', \tag{3.6.24}$$

which in consequence implies that

$$\rho_m u_m \otimes u_m \rightharpoonup \rho u \otimes u \qquad \text{in } L^{\frac{3}{2}}\left(Q\left(T_0\right)\right) \qquad \forall 0 < T_0 < T'. \tag{3.6.25}$$

Exactly as in the deduction of the corresponding strong convergence (3.5.30) in the ϵ-limit, the convergence (3.6.24) will follow provided we can show that

$$\int_0^{T_0}\int_\Omega \rho_m|u_m|^2\,dxdt \to \int_0^{T_0}\int_\Omega \rho|u|^2\,dxdt \tag{3.6.26}$$

for arbitrary $0 < T_0 < T'$. The proof of (3.6.26) is achieved by following mostly [15] and using further arguments from [44]. More precisely, for fixed $0 < T_0 < T'$, we choose $\kappa_{\sup} = \kappa_{\sup}(T_0) > 0$ as the supremum over all κ which satisfy the estimate (3.6.20). By the second inclusion in (3.6.16) we find some $m_{\min} = m_{\min}(\kappa_{\sup}) > 0$ such that

$$\operatorname{dist}\Big(\partial\Omega,\ S_m(t)\Big) \geqslant \frac{\kappa_{\sup}}{2} \qquad \forall t \in [0,T_0],\ m \geqslant m_{\min}. \tag{3.6.27}$$

Then for any $0 \leqslant \kappa \leqslant \frac{\kappa_{\sup}}{4}$, $t \in [0,T_0]$ and $r \in [0,1]$ we define

$$\begin{aligned} K^r_{t,\kappa}(\Omega) &:= \left\{v(t) \in H^r_{0,\mathrm{div}}(\Omega):\ \mathbb{D}(v(t)) = 0 \text{ in } \mathcal{D}'\left((S(t))^\kappa\right)\right\},\\ K^r_{t,\kappa,m}(\Omega) &:= \left\{v(t) \in H^r_{0,\mathrm{div}}(\Omega):\ \mathbb{D}(v(t)) = 0 \text{ in } \mathcal{D}'\left((S_m(t))^\kappa\right)\right\} \end{aligned} \tag{3.6.28}$$

together with the associated orthogonal projections

$$P^r_\kappa(t):\ H^r(\Omega) \to K^r_{t,\kappa}(\Omega), \qquad P^r_{\kappa,m}(t):\ H^r(\Omega) \to K^r_{t,\kappa,m}(\Omega).$$

By the triangle inequality we estimate, for arbitrary $\psi \in \mathcal{D}(0,T_0)$, $r \in (0,1)$ and $\kappa \in (0, \frac{\kappa_{\sup}}{4}]$,

$$\begin{aligned} &\left|\int_0^{T_0}\int_\Omega \psi\rho_m|u_m|^2\,dxdt - \int_0^{T_0}\int_\Omega \psi\rho|u|^2\,dxdt\right|\\ \leqslant&\left|\int_0^{T_0}\int_\Omega \psi\rho_m u_m\left(u_m - P^r_\kappa u_m\right)dxdt\right| + \left|\int_0^{T_0}\int_\Omega \psi\left(\rho_m u_m\cdot P^r_\kappa u_m - \rho u\cdot P^r_\kappa u\right)dxdt\right|\\ &+\left|\int_0^{T_0}\int_\Omega \psi\rho u\cdot\left(P^r_\kappa u - u\right)dxdt\right|\\ \leqslant&\overline{\rho}\,\|\psi\|_{L^\infty(0,T_0)}\,\|u_m\|_{L^2(Q(T_0))}\,\|P^r_\kappa u_m - u_m\|_{L^2(Q(T_0))} + \left|\int_0^{T_0}\int_\Omega \psi\left(\rho_m u_m\cdot P^r_\kappa u_m - \rho u\cdot P^r_\kappa u\right)\,dxdt\right|\\ &+\overline{\rho}\,\|\psi\|_{L^\infty(0,T_0)}\,\|u\|_{L^2(Q(T_0))}\,\|P^r_\kappa u - u\|_{L^2(Q(T_0))}\,. \end{aligned} \tag{3.6.29}$$

Our goal is to show that the right-hand side of this estimate vanishes. We first focus on the second term on the right-hand side. The vanishing of this term is shown by the following version of [44, Lemma 3.4] (cf. [15, Lemma 3.8] for a related result), the proof of which we outline for the convenience of the reader.

Lemma 3.6.1. *For any $\kappa \in (0, \frac{\kappa_{sup}}{4}]$ and any $0 < r < 1$, it holds that*

$$\left|\int_0^{T_0}\int_\Omega \psi\left(\rho_m u_m\cdot P^r_\kappa u_m - \rho u\cdot P^r_\kappa u\right)\,dxdt\right| \to 0 \qquad \text{for } m \to \infty.$$

Proof

The argument of the proof is the same as in [44, Lemma 3.4]. We consider some arbitrary $\tau \in [0,T_0]$ and a neighborhood $I(\tau)$ of τ, which by the inclusion (3.6.16) and the fact that $\eta^{\Pi_{[\chi,\rho,u]}} \in C([0,T]\times[0,T];C_{\mathrm{loc}}(\mathbb{R}^3))$ (cf. (3.6.12)), can be chosen sufficiently small such that

$$\left(S_m\left(t\right)\right)^{\frac{\kappa}{8}} \subset \left(S(\tau)\right)^{\frac{\kappa}{4}} \qquad \text{and} \qquad P^r_\kappa u_m \in L^2\left(I(\tau); K^r_{\tau,\frac{\kappa}{2}}(\Omega)\right) \tag{3.6.30}$$

for all $t \in I(\tau)$ and all sufficiently large $m \in \mathbb{N}$. Then we test the momentum equation (3.5.37) on the m-level by test functions of the form $\tilde{\psi}\Phi$, where $\Phi \in K^0_{\tau,\frac{\kappa}{4}}(\Omega)\bigcap\mathcal{D}(\Omega)$ and $\tilde{\psi} \in \mathcal{D}(I(\tau))$. In particular, by the first inclusion in (3.6.30), these test functions satisfy $\mathbb{D}(\tilde{\psi}\Phi) = 0$ in $(S_m(t))^{\frac{\kappa}{8}}$ for all $t \in I(\tau)$

and all sufficiently large $m \in \mathbb{N}$. By the same arguments as in the derivation of the identity (3.6.23) it then follows that, for all such t and m,

$$\int_\Omega -m\rho_m(t)\chi_m(t)\left(u_m(t) - \Pi_{[\chi_m,\rho_m,u_m]}(t)\right)\cdot\left(\tilde{\psi}(t)\Phi\right)\, dx = 0.$$

This allows us to deduce the dual estimate

$$\begin{aligned}
&\left\|\partial_t \int_\Omega \rho_m u_m \cdot \Phi\, dx\right\|_{L^{\frac{4}{3}}(I(\tau))}\\
=&\left\|\int_\Omega (\rho_m u_m \otimes u_m) : \nabla\Phi - 2\nu\mathbb{D}(u_m) : \nabla\Phi + \rho_m g\cdot\Phi + \frac{1}{\mu}(\mathrm{curl}B_m \times B_m)\cdot\Phi\, dxdt\right\|_{L^{\frac{4}{3}}(I(\tau))}\\
\leqslant&\overline{\rho}\left\|u_m\right\|^2_{L^{\frac{8}{3}}(0,T;L^2(\Omega))}\|\Phi\|_{L^\infty(\Omega)} + 2\nu\|\mathbb{D}(u_m)\|_{L^{\frac{4}{3}}(Q)}\|\nabla\Phi\|_{L^4(\Omega)}\\
&+\overline{\rho}\|g\|_{L^{\frac{4}{3}}(Q)}\|\Phi\|_{L^4(\Omega)} + \frac{1}{\mu}\|B_m\|_{L^\infty(0,T;L^2(\Omega))}\|\mathrm{curl}\, B_m\|_{L^{\frac{4}{3}}(0,T;L^2(\Omega))}\|\Phi\|_{L^\infty(\Omega)} \leqslant c
\end{aligned}$$

for a constant $c > 0$ depending on Φ but not on m. Due to the density of $K^0_{\tau,\frac{\kappa}{4}}(\Omega)\bigcap\mathcal{D}(\Omega)$ in $K^0_{\tau,\frac{\kappa}{2}}(\Omega)$ and the compactness of $K^r_{\tau,\frac{\kappa}{2}}(\Omega)$ in $K^0_{\tau,\frac{\kappa}{2}}(\Omega)$, we thus infer from Lemma A.4.1 in the appendix that

$$\rho_m u_m \to \rho u \quad \text{in } C_{\text{weak}}\left(I(\tau);\left(K^0_{\tau,\frac{\kappa}{2}}(\Omega)\right)^*\right) \quad \text{and thus in} \quad L^2\left(I(\tau);\left(K^r_{\tau,\frac{\kappa}{2}}(\Omega)\right)^*\right). \tag{3.6.31}$$

Moreover, due to the second inclusion in (3.6.30), we have

$$\|P^r_\kappa u_m\|_{L^2(I(\tau);K^r_{\tau,\frac{\kappa}{2}}(\Omega))} = \|P^r_\kappa u_m\|_{L^2(I(\tau);H^r(\Omega))} \leqslant \|u_m\|_{L^2(I(\tau);H^r(\Omega))} \leqslant c,$$

and in particular

$$P^r_\kappa u_m \rightharpoonup P^r_\kappa u \quad \text{in } L^2(I(\tau)\times\Omega).$$

The previous two relations, together with the strong convergence (3.6.31), imply that

$$\begin{aligned}
&\left|\int_{I(\tau)}\int_\Omega \psi\left(\rho_m u_m \cdot P^r_\kappa u_m - \rho u\cdot P^r_\kappa u\right) dxdt\right|\\
\leqslant&\|\psi P^r_\kappa u_m\|_{L^2(I(\tau);K^r_{\tau,\frac{\kappa}{2}}(\Omega))}\|\rho_m u_m - \rho u\|_{L^2(I(\tau);(K^r_{\tau,\frac{\kappa}{2}}(\Omega))^*)} + \left|\int_{I(\tau)}\int_\Omega \psi\rho u\left(P^r_\kappa u_m - P^r_\kappa u\right) dxdt\right| \to 0.
\end{aligned} \tag{3.6.32}$$

Finally, the compact interval $[0, T_0]$ can be covered by finitely many intervals $I(\tau)$ on which (3.6.32) holds true, which concludes the proof.

□

In order to show the vanishing of the first and the third term on the right-hand side of the inequality (3.6.29) we use the following version of [15, Lemma 3.6, Lemma 3.7]. Again we outline the proof for the convenience of the reader.

Lemma 3.6.2. *For any fixed $r \in (0,1)$ it holds that*

$$\begin{aligned}
(i)\quad &\lim_{\kappa\to 0}\lim_{m\to\infty}\|P^r_\kappa u_m - u_m\|_{L^2(Q(T_0))} = 0,\\
(ii)\quad &\lim_{\kappa\to 0}\|P^r_\kappa u - u\|_{L^2(Q(T_0))} = 0.
\end{aligned}$$

Proof We give the proof of (i), which follows [15, Lemma 3.7]. For almost all $t \in [0, T_0]$ we define $v_{\kappa m}(t,\cdot) \in H^1(\Omega\backslash(S_m(t))^\kappa)$, $p_{\kappa m}(t,\cdot) \in L^2(\Omega\backslash(S_m(t))^\kappa)$ as the solution to the Stokes problem

$$\begin{aligned}
-\Delta v_{\kappa m}(t,\cdot) + \nabla p_{\kappa m}(t,\cdot) &= -\Delta u_m(t,\cdot) \quad \text{in } \Omega\backslash(S_m(t))^\kappa\\
\mathrm{div} v_{\kappa m}(t,\cdot) &= 0 \quad \text{in } \Omega\backslash(S_m(t))^\kappa\\
v_{\kappa m}(t,\cdot) &= \begin{cases}\Pi_{[\chi_m,\rho_m,u_m]}(t,\cdot) & \text{on } \partial(S_m(t))^\kappa,\\ 0 & \text{on } \partial\Omega.\end{cases}
\end{aligned}$$

The existence of this solution is guaranteed by [112, Proposition 2.3]. We extend the function $v_{\kappa m}(t,\cdot)$ by $\Pi_{[\chi_m,\rho_m,u_m]}(t,\cdot)$ in $(S_m(t))^\kappa$ and note that $e_{\kappa m}(t,\cdot) := v_{\kappa m}(t,\cdot) - u_m(t,\cdot)$ solves a corresponding Stokes problem in $\Omega\backslash(S_m(t))^\kappa$ with 0-right-hand side. For any sufficiently small $\kappa < \frac{\kappa_{\sup}}{4}$ we estimate

$$\begin{aligned}
&\int_0^{T_0} \|e_{\kappa m}(t,\cdot)\|^2_{L^2((S_m(t))^\kappa \backslash S_m(t))}\ dt\\
\leqslant & c\int_0^{T_0} \kappa \|e_{\kappa m}(t,\cdot)\|^{\frac{1}{2}}_{L^2(S_m(t))} \|e_{\kappa m}(t,\cdot)\|^{\frac{3}{2}}_{H^1(S_m(t))} + \kappa^2 \|\nabla e_{\kappa m}(t,\cdot)\|^2_{L^2((S_m(t))^\kappa)}\ dt\\
\leqslant & c\kappa \left(\int_0^{T_0} \|e_{\kappa m}(t,\cdot)\|^2_{L^2(S_m(t))}\, dt\right)^{\frac{1}{4}} \left(\int_0^{T_0} \|e_{\kappa m}(t,\cdot)\|^2_{H^1(S_m(t))}\ dt\right)^{\frac{3}{4}} + c\kappa^2 \leqslant c\left(\kappa + \kappa^2\right), \qquad (3.6.33)
\end{aligned}$$

using the trace inequality (A.5.5) and the Poincaré-type estimate (A.5.6) in Lemma A.5.2 in the appendix for the first inequality as well as the Hölder inequality and the uniform estimates (3.6.2), (3.6.9) and (3.6.10) for the second and the third inequality. Next, by applying the estimate (A.5.9) in Lemma A.5.3 in the appendix to the solution $e_{\kappa m}$ to the Stokes problem in $\Omega\backslash(S_m(t))^\kappa$ with 0-right-hand side, we estimate

$$\begin{aligned}
&\int_0^{T_0} \|e_{\kappa m}(t,\cdot)\|^2_{L^2(\Omega\backslash(S_m(t))^\kappa)}\ dt\\
\leqslant & \int_0^{T_0} \|e_{\kappa m}(t,\cdot)\|^{\frac{1}{2}}_{L^2((S_m(t))^\kappa)} \|e_{\kappa m}(t,\cdot)\|^{\frac{3}{2}}_{H^1((S_m(t))^\kappa)}\ dt\\
\leqslant & c\left(\int_0^{T_0} \|e_{\kappa m}(t,\cdot)\|^2_{L^2((S_m(t))^\kappa)}\, dt\right)^{\frac{1}{4}} \left(\int_0^{T_0} \|e_{\kappa m}(t,\cdot)\|^2_{H^1((S_m(t))^\kappa)}\, dt\right)^{\frac{3}{4}}\\
\leqslant & c\left(\int_0^{T_0} \|e_{\kappa m}(t,\cdot)\|^2_{L^2(S_m(t))}\, dt + c\left(\kappa+\kappa^2\right)\right)^{\frac{1}{4}} \leqslant c\left(\frac{1}{m} + \kappa + \kappa^2\right)^{\frac{1}{4}}, \qquad (3.6.34)
\end{aligned}$$

where we further used the estimates (3.6.2), (3.6.9), (3.6.10) and (3.6.33) for the third inequality and the uniform bound (3.6.3) for the fourth one. Now a combination of the uniform bound (3.6.3) and the estimates (3.6.33) and (3.6.34) gives us

$$\begin{aligned}
&\int_0^{T_0} \|e_{\kappa m}(t,\cdot)\|^2_{L^2(\Omega)}\, dt\\
= & \int_0^{T_0} \|e_{\kappa m}(t,\cdot)\|^2_{L^2(S_m(t))} + \|e_{\kappa m}(t,\cdot)\|^2_{L^2((S_m(t))^\kappa\backslash S_m(t))} + \|e_{\kappa m}(t,\cdot)\|^2_{L^2(\Omega\backslash(S_m(t))^\kappa)}\, dt\\
\leqslant & c\left(\frac{1}{m} + \kappa + \kappa^2 + \left(\frac{1}{m} + \kappa + \kappa^2\right)^{\frac{1}{4}}\right). \qquad (3.6.35)
\end{aligned}$$

Next, we estimate, under exploitation of the classical estimates for the Stokes problem (cf. [112, Proposition 2.2, Proposition 2.3]) and the trace inequality,

$$\int_0^{T_0} \|e_{\kappa m}(t,\cdot)\|^2_{H^1(\Omega\backslash(S_m(t))^\kappa)}\ dt \leqslant c\int_0^{T_0} \|e_{\kappa m}(t,\cdot)\|^2_{H^{\frac{1}{2}}(\partial(S_m(t))^\kappa)}\ dt \leqslant c\int_0^{T_0} \|e_{\kappa m}(t,\cdot)\|^2_{H^1((S_m(t))^\kappa)}\ dt \leqslant c,$$

which yields that

$$\int_0^{T_0} \|e_{\kappa m}(t,\cdot)\|^2_{H^1(\Omega)}\ dt = \int_0^{T_0} \|e_{\kappa m}(t,\cdot)\|^2_{H^1(\Omega\backslash(S_m(t))^\kappa)} + \|e_{\kappa m}(t,\cdot)\|^2_{H^1((S_m(t))^\kappa)}\ dt \leqslant c. \qquad (3.6.36)$$

The estimates (3.6.35) and (3.6.36), together with an interpolation between $L^2(\Omega)$ and $H^1(\Omega)$, yield

$$\lim_{\kappa\to 0}\lim_{m\to\infty} \|e_{\kappa m}\|_{L^2(0,T_0;H^r(\Omega))} \leqslant \lim_{\kappa\to 0}\lim_{m\to\infty} \|e_{\kappa m}\|^{1-r}_{L^2(Q(T_0))} \|e_{\kappa m}\|^r_{L^2(0,T_0;H^1(\Omega))} = 0. \qquad (3.6.37)$$

Since $v_{\kappa m}(t,\cdot)$ coincides with a rigid velocity field in $(S_m(t))^\kappa$ for almost all $t \in [0,T_0]$, i.e. $v_{\kappa m}(t,\cdot)$ is contained in the space $K^r_{t,\kappa,m}(\Omega)$ defined in (3.6.28), the equation (3.6.37) implies that

$$\lim_{\kappa\to 0}\lim_{m\to\infty}\left\|P^r_{\kappa,m}u_m - u_m\right\|_{L^2(0,T_0;H^r(\Omega))} \leqslant \lim_{\kappa\to 0}\lim_{m\to\infty}\|v_{\kappa m} - u_m\|_{L^2(0,T_0;H^r(\Omega))} = 0.$$

Moreover, by the first inclusion in (3.6.16) we have $(S(t))^\kappa \subset (S_m(t))^{2\kappa}$ for all sufficiently large $m \in \mathbb{N}$. Hence, for such m, it holds that $P^r_{2\kappa,m}(t)u_m(t,\cdot) \in K^r_{t,\kappa}(\Omega)$, which then yields

$$\begin{aligned}&\lim_{\kappa\to 0}\lim_{m\to\infty}\|P^r_\kappa u_m - u_m\|_{L^2(Q(T_0))}\\ \leqslant &\lim_{\kappa\to 0}\lim_{m\to\infty}\|P^r_\kappa u_m - u_m\|_{L^2(0,T_0;H^r(\Omega))} \leqslant \lim_{\kappa\to 0}\lim_{m\to\infty}\left\|P^r_{2\kappa,m}u_m - u_m\right\|_{L^2(0,T_0;H^r(\Omega))} = 0,\end{aligned}$$

i.e. (i). The assertion (ii) follows by similar arguments, cf. also [15, Lemma 3.6].

□

With Lemma 3.6.1 and Lemma 3.6.2 at hand we can finally return to the estimate (3.6.29). Keeping $r \in (0,1)$ and $\kappa \in (0, \frac{\kappa_{\sup}}{4}]$ fixed, we let first m tend to ∞. During this procedure, the second term on the right-hand side of (3.6.29) vanishes due to Lemma 3.6.1. Subsequently, by letting κ tend to 0, also the first and the last term on the right-hand side of (3.6.29) vanish according to Lemma 3.6.2. Finally, replacing ψ by a suitable sequence of cut-off functions on $[0,T_0]$, we infer the convergence (3.6.26) and hence the desired convergences (3.6.24) and (3.6.25). Since for any arbitrary but fixed test function $\phi \in Z(S,T')$ we can find $T_0 < T'$ such that the inclusion (3.6.22) holds true, the convergence (3.6.25) suffices to pass to the limit in the $\rho_m u_m \otimes u_m$-term in the momentum equation. Combining this with the convergence (3.6.1) of the initial data, the convergence (3.6.18) of the Lorentz force and the identity (3.6.23), we can pass to the limit in the momentum equation (3.5.37) and obtain

$$\begin{aligned}-\int_0^{T'}\int_\Omega \rho u\cdot\partial_t\phi\,dxdt - \int_\Omega \rho_0 u_0\cdot\phi(0,x)\,dx = \int_0^{T'}\int_\Omega &(\rho u\otimes u) : \nabla\phi - 2\nu\mathbb{D}(u) : \nabla\phi\\ &+ \rho g\cdot\phi + \frac{1}{\mu}\left(\operatorname{curl}B\times B\right)\cdot\phi\,dxdt\end{aligned} \tag{3.6.38}$$

for any $\phi \in Z(S,T')$.

3.6.5 Proof of the main result

Summarizing the results from Section 3.6 we can now finish the proof of Theorem 3.1.1. The isometry $\eta(s;t,\cdot)$ in (3.1.10) is given by $\eta = \eta^{\Pi_{[\chi,\rho,u]}}$. Indeed, by (3.6.12), $\eta^{\Pi_{[\chi,\rho,u]}}$ is the (pointwise) limit of a sequence of isometries and hence an isometry itself. Moreover, from the continuity of $\eta^{\Pi_{[\chi,\rho,u]}}$ and the fact that $\eta^{\Pi_{[\chi,\rho,u]}}(s;s,\cdot) = \mathrm{id}$ it follows that $\eta^{\Pi_{[\chi,\rho,u]}}$ is orientation preserving. The regularity of χ and ρ in (3.1.11) and (3.1.12) follows from the choice of the spaces in (3.6.13) and (3.6.7). As

$$\mathbb{D}\left(\Pi_{[\chi,\rho,u]}\right) = 0,$$

the properties of u in (3.1.13) follow from (3.6.4) and the relation (3.6.21) between u and $\Pi_{[\chi,\rho,u]}$, while the properties of B in (3.1.14) are given by (3.6.5) and (3.6.17). The transport equations (3.1.15) and (3.1.16) were shown in (3.6.15) and (3.6.8), where in (3.6.15) the function $\Pi_{[\chi,\rho,u]}$ can indeed be replaced by u due to the relation (3.6.21) between these two functions and the fact that $\chi = 0$ outside of $Q^s(S,T')$. The momentum equation (3.1.17) is satisfied according to (3.6.38). The induction equation (3.1.18) was shown to hold true in (3.6.19). By the group property [35, (76)], which is satisfied by $\eta^{\Pi_{[\chi,\rho,u]}}$ as a solution to the initial value problem (3.6.14), it holds that

$$\begin{aligned}S(t) &= \left\{x \in \mathbb{R}^3 :\ \chi(t,x) = 1\right\} = \left\{\eta^{\Pi_{[\chi,\rho,u]}}(0;t,x) :\ x \in S_0\right\} = \eta^{\Pi_{[\chi,\rho,u]}}(0;t,S_0)\\ &= \eta^{\Pi_{[\chi,\rho,u]}}\left(s;t,\left\{\eta^{\Pi_{[\chi,\rho,u]}}(0;s,S_0)\right\}\right) = \eta^{\Pi_{[\chi,\rho,u]}}(s;t,S(s))\end{aligned}$$

for all $s,t \in [0,T']$. This yields the identity (3.1.19). Finally, the energy inequality (3.1.20) follows by dropping the nonnegative penalization term in the energy inequality (3.5.39) on the m-level and exploiting the weak lower semicontinuity of norms. This concludes the proof of Theorem 3.1.1.

Chapter 4

Fluid-rigid body interaction in a compressible electrically conducting fluid

In this chapter we extend the local-in-time existence result proved in Chapter 3 for weak solutions to a fluid-rigid body interaction problem with an electrically conducting incompressible fluid to the compressible case. In addition we further generalize the result to the setting of multiple rigid bodies and we prove the existence independently of any potential collisions between the bodies or a body and the domain boundary within the studied time interval. More precisely we prove the global-in-time existence of weak solutions to the system (1.3.15)–(1.3.28) of partial differential equations presented in Section 1.3.2, which models the interaction between a viscous non-homogeneous compressible and electrically conducting fluid, one or more insulating rigid bodies moving through the fluid and the electromagnetic fields living in both of these materials. This result - including essentially the same proof which we give in the present chapter - has been published by the author of this thesis in the article [91].

The main difficulty in the proof lies in a suitable choice of the approximate system: As in the incompressible setting, the test functions in the weak formulation of the induction equation depend on the moving solid domain and hence on the solution to the system itself. Thus the problem of the high coupling in the system carries over to the compressible setting, forcing us once more to discretize the system with respect to the time variable in order to be able to solve the induction equation with given test functions after first determining the position of the solid bodies at each fixed discrete time.

In the compressible case, however, the discretization of the mechanical part of the system gives rise to a new problem: It appears not to be possible to discretize the Navier-Stokes equations in such a way that non-negativity of the density can be guaranteed. As the latter property is essential for obtaining uniform bounds from the energy inequality we thus pick up the idea of a hybrid discrete-continuous in time approximation scheme from Section 3.2 again. This time, however, we only discretize the induction equation while we treat the whole mechanical subsystem as a continuous problem on the small intervals between the discrete time points. This allows us to construct a non-negative density by the classical techniques for the compressible Navier-Stokes system. A proper choice of the coupling terms on the approximate level then assures us the possibility to combine the discrete and the continuous part of the system into a suitable energy inequality. Furthermore, in order to deal with the solution-dependent test functions in the momentum equation, we resort to a well-investigated penalization method similar to the Brinkman penalization used in Section 3.2. However, as the Brinkman penalization is designed specifically for the case of incompressible fluids, we switch to the penalization method from [43, 103] in the present setting.

Possible (biomechanical) applications of our results include capsule endoscopy ([59]) or remote drug delivery ([58, Section 4.4]), for more details cf. Section 1.1. Despite the fact that blood is generally considered as incompressible, red blood cells are in fact slightly compressible, cf. [102]. For this reason, blood can also be modeled as a compressible fluid, which makes the results in the present chapter interesting for those applications.

4.1 Weak solutions and main result

4.1.1 Notation

The system we consider in the present chapter describes the interaction between a viscous non-homogeneous compressible and electrically conducting fluid and multiple rigid bodies, both confined to some bounded domain $\Omega \subset \mathbb{R}^3$, over a time interval $[0,T]$, $T > 0$. The notation used to characterize the motion of the bodies in this setting closely resembles the corresponding notation we used for the case of one rigid body inside of an incompressible fluid, cf. Section 3.1.1. However, due to some small but important differences, we consider it more convenient for the reader to reintroduce the whole notation in the following: By some bounded domains $\varnothing \neq S_0^i \subset \Omega$, $i = 1, ..., N$, $N \in \mathbb{N}$ we denote the initial positions of the bodies. Since the motion of the bodies is rigid, we can associate to each body an orientation preserving isometry $\eta^i(t,\cdot) : \mathbb{R}^3 \to \mathbb{R}^3$, $t \in [0,T]$, such that its position $S^i(t)$ at an arbitrary time $t \in [0,T]$ is given via the set-valued function

$$S^i : \ [0,T] \to 2^{\mathbb{R}^3}, \quad S^i(t) := \eta^i\left(t, S_0^i\right).$$

In particular, with the notation

$$\eta(t,\cdot) : \ S_0 := \bigcup_{i=1}^{N} S_0^i \to \mathbb{R}^3, \quad \eta(t,\cdot)|_{S_0^i} := \eta^i(t,\cdot) \qquad \forall i = 1, ..., N, \ t \in [0,T],$$

the solid region at the time t is expressed via the set-valued function

$$S : \ [0,T] \to 2^{\mathbb{R}^3}, \qquad S(t) := \eta\left(t, S_0\right).$$

In the compressible setting we further make use of the concept of Carathéodory solutions (cf. Section A.1 in the appendix), in order to connect the motion of the rigid bodies to the velocity field u of our system. More precisely, we require u to be compatible with the system $\{S_0^i, \eta^i\}_{i=1}^N$, i.e. we require the existence of rigid velocity fields $u^{s^i}(t,\cdot)$, $i = 1, ..., N$, such that

$$u(t,x) = u^{s^i}(t,x) \quad \text{for a.a. } t \in [0,T] \text{ and a.a. } x \in S^i(t) \tag{4.1.1}$$

and $\eta^i(\cdot,x)$ is the unique Carathéodory solution (cf. Theorem A.1.1 in the appendix) to the initial value problem

$$\frac{d\eta^i(t,x)}{dt} = u^{s^i}\left(t, \eta^i(t,x)\right), \quad \eta^i(0,x) = x \tag{4.1.2}$$

for all $x \in \mathbb{R}^3$. Moreover, we define the time-space domain $Q := (0,T) \times \Omega$, which we again divide into a solid part and a fluid part,

$$Q^s(S) := \left\{(t,x) \in (0,T) \times \mathbb{R}^3 : \ x \in S(t)\right\}, \qquad Q^f(S) := Q \backslash \overline{Q^s(S)}.$$

Furthermore, we denote by $\mathcal{Z}(S)$ and $\mathcal{Y}(S)$ the test function spaces

$$\mathcal{Z}(S) := \left\{\phi \in \mathcal{D}\left([0,T) \times \Omega\right) : \ \mathbb{D}(\phi) = 0 \quad \text{in a neighborhood of } \overline{Q^s(S)}\right\}, \tag{4.1.3}$$

$$\mathcal{Y}(S) := \left\{b \in \mathcal{D}\left([0,T) \times \Omega\right) : \ \operatorname{curl} b = 0 \quad \text{in a neighborhood of } \overline{Q^s(S)}\right\} \tag{4.1.4}$$

for our variational formulations of the momentum equation and the induction equation, respectively, in Definition 4.1.1 below. Furthermore, for arbitrary sets $S \subset \mathbb{R}^3$ and arbitrary values $\kappa > 0$ we once more denote by S^κ the κ-neighborhood of S and by S_κ the κ-kernel of S, i.e.

$$S^\kappa := \left\{x \in \mathbb{R}^3 : \ \operatorname{dist}(x,S) < \kappa\right\}, \qquad S_\kappa := \left\{x \in S : \ \operatorname{dist}(x, \partial S) > \kappa\right\}.$$

For the later use we remark that if $S \subset \mathbb{R}^3$ is of class C^2, it is possible to choose $\kappa > 0$ sufficiently small, such that the κ-neighborhood of the κ-kernel of S coincides with S itself,

$$(S_\kappa)^\kappa = S, \tag{4.1.5}$$

cf. [103, Proposition 2.1]. Finally, we again denote by

$$H^r_{\text{div}}(\Omega) := \{v \in H^r(\Omega) : \ \operatorname{div} v = 0 \text{ in } \mathcal{D}'(\Omega)\} \qquad \text{for } r \geqslant 0$$

the Sobolev spaces of functions in $H^r(\Omega)$ which are in addition divergence-free.

Remark 4.1.1. *In the compressible case, as opposed to in the incompressible one, we forego characterizing the position of the solid bodies via their characteristic functions, which has the following reason: If the velocity field u is divergence-free, such characteristic functions satisfy a transport equation associated to u. This guarantees the correct relation between the velocity u and the motion of the rigid bodies, which is described by the corresponding characteristic curves. In the setting of a compressible fluid, however, this is not the case. Instead we recover the desired relation between u and the motion of the rigid bodies directly via the characteristics of u in the solid domain, which we achieve by imposing the compatibility condition* (4.1.1), (4.1.2).

4.1.2 Weak solutions

We are now in the position to present our variational formulation of the system (1.3.15)–(1.3.28), which describes the interplay between a compressible electrically conducting fluid and multiple insulating rigid bodies contained in the fluid. With a slight abuse of notation we will write here and in the following sections $\sigma = \sigma^f > 0$, since the quantities containing $\sigma^s = 0$ are not visible in this weak formulation due to the non-conductivity of the solids.

Definition 4.1.1. *Let $T > 0$, let $\Omega \subset \mathbb{R}^3$ be a bounded domain and let $S_0 = \bigcup_{i=1}^N S_0^i$, where $S_0^i \subset \Omega$ for $i = 1, ..., N \in \mathbb{N}$ are bounded domains such that*

$$\varnothing \neq S_0^i \text{ is open and connected, } |\partial S_0^i| = 0 \text{ and } \overline{S_0^i} \bigcap \overline{S_0^j} = \varnothing \qquad \forall i, j = 1, ..., N, \ i \neq j. \tag{4.1.6}$$

Assume $\nu, \lambda, a, \gamma, \sigma, \mu \in \mathbb{R}$ to satisfy

$$\nu, a, \sigma, \mu > 0, \quad \nu + \lambda \geqslant 0, \quad \gamma > \frac{3}{2}, \tag{4.1.7}$$

consider some external data $g, J \in L^\infty(Q)$ and consider some initial data $0 \leqslant \rho_0 \in L^\gamma(\Omega)$, $(\rho u)_0 \in L^1(\Omega)$ and $B_0 \in L^2_{\text{div}}(\Omega)$ satisfying

$$\frac{|(\rho u)_0|^2}{\rho_0} \in L^1(\Omega), \quad (\rho u)_0 = 0 \ \text{ a.e. in } \{x \in \Omega : \ \rho_0(x) = 0\}, \quad B_0 \cdot \mathrm{n} = 0 \text{ on } \partial\Omega. \tag{4.1.8}$$

Then the system (1.3.15)–(1.3.28) *is said to admit a weak solution on $[0, T]$ if there exists a function*

$$\eta : \ [0, T] \times S_0 \to \mathbb{R}^3, \quad \eta(t, \cdot)|_{S_0^i} = \eta^i(t, \cdot) \qquad \forall i = 1, ..., N, \ t \in [0, T], \tag{4.1.9}$$

where each $\eta^i(t, \cdot) : \mathbb{R}^3 \to \mathbb{R}^3$ denotes an orientation preserving isometry, and if there exist functions

$$0 \leqslant \rho \in L^\infty(0, T; L^\gamma(\Omega; \mathbb{R})) \bigcap C\left([0, T]; L^1(\Omega; \mathbb{R})\right), \tag{4.1.10}$$

$$u \in \left\{\phi \in L^2\left(0, T; H^1_0\left(\Omega; \mathbb{R}^3\right)\right) : \ \mathbb{D}(\phi) = 0 \text{ in } Q^s(S)\right\}, \tag{4.1.11}$$

$$B \in \left\{b \in L^\infty\left(0, T; L^2\left(\Omega; \mathbb{R}^3\right)\right) \bigcap L^2\left(0, T; H^1_{\text{div}}\left(\Omega; \mathbb{R}^3\right)\right) : \ \operatorname{curl} b = 0 \text{ in } Q^s(S), \ b \cdot \mathrm{n} = 0 \text{ on } \partial\Omega\right\}, \tag{4.1.12}$$

where $S = S(\cdot) = \eta(\cdot, S_0)$, such that ρ and u, extended by 0 in $\mathbb{R}^3 \backslash \Omega$, satisfy the continuity equation

$$-\int_0^T \int_\Omega \rho \partial_t \psi dx dt - \int_\Omega \rho_0 \psi(0, x) \ dx = \int_0^T \int_\Omega (\rho u) \cdot \nabla \psi \ dx dt \tag{4.1.13}$$

for all $\psi \in \mathcal{D}([0, T) \times \Omega)$ as well as the renormalized continuity equation

$$\partial_t \zeta(\rho) + \operatorname{div}\left(\zeta(\rho) u\right) + \left[\zeta'(\rho) \rho - \zeta(\rho)\right] \operatorname{div} u = 0 \quad \text{in } \mathcal{D}'\left((0, T) \times \mathbb{R}^3\right), \tag{4.1.14}$$

for any

$$\zeta \in C^1([0,\infty)) : \quad |\zeta'(r)| \leqslant c r^{\lambda_1} \quad \forall r \geqslant 1, \qquad \text{where } c > 0,\ \lambda_1 > -1, \tag{4.1.15}$$

such that the momentum equation and the induction equation,

$$\begin{aligned} -\int_0^T \int_\Omega \rho u \cdot \partial_t \phi \; dxdt - \int_\Omega (\rho u)_0 \cdot \phi(0,x) \; dx = \int_0^T \int_\Omega & (\rho u \otimes u) : \mathbb{D}(\phi) + a\rho^\gamma \operatorname{div} \phi - 2\nu \mathbb{D}(u) : \mathbb{D}(\phi) \\ & - \lambda \operatorname{div} u \operatorname{div} \phi + \rho g \cdot \phi + \frac{1}{\mu} (\operatorname{curl} B \times B) \cdot \phi \; dxdt, \end{aligned} \tag{4.1.16}$$

$$-\int_0^T \int_\Omega B \cdot \partial_t b \; dxdt - \int_\Omega B_0 \cdot b(0,x) \; dx = \int_0^T \int_\Omega \left[-\frac{1}{\sigma\mu} \operatorname{curl} B + u \times B + \frac{1}{\sigma} J \right] \cdot \operatorname{curl} b \; dxdt, \tag{4.1.17}$$

are satisfied for any $\phi \in \mathcal{Z}(S)$ and any $b \in \mathcal{Y}(S)$ and, finally, such that the system $\{S_0^i, \eta^i\}_{i=1}^N$ is compatible with the velocity field u.

In this definition, the compatibility of the velocity field u and the system of rigid bodies leads to some vivid consequences for the solids. First of all, while the bodies are able to touch each other or the domain boundary, the possibility of interpenetrations is ruled out, cf. [43, Lemma 3.1, Corollary 3.1]. Moreover, even though the density does not satisfy a transport equation in the case of a compressible fluid, it still travels along the characteristics of u in the solid part of the domain, cf. [43, Lemma 3.2]. For definiteness we present these results in the following lemma.

Lemma 4.1.1 ([43]). *Let $\Omega \subset \mathbb{R}^3$ and $S_0^i \subset \mathbb{R}^3$, $i = 1, ..., N \in \mathbb{N}$, be bounded domains and let further $u \in L^2(0,T; H_0^{1,2}(\Omega))$ be extended by 0 outside of Ω. Moreover, assume u to be compatible with the system $\{S_0^i, \eta^i\}_{i=1}^N$, where each $\eta^i(t) : \mathbb{R}^3 \to \mathbb{R}^3$, $t \in [0,T]$, $i = 1, ..., N$, denotes an isometry. Then it holds:*

(i) If, for $i \neq j \in \{1, ..., N\}$, there exists $\tau \in [0,T]$ such that $S^i(\tau) \bigcap S^j(\tau) \neq \varnothing$, then $\eta^i(t) = \eta^j(t)$ for all $t \in [0,T]$. Further, if there exists $\tau \in [0,T]$ such that $S^i(\tau) \not\subset \Omega$, then $\eta^i(t) = \mathrm{id}$ for all $t \in [0,T]$.

(ii) If $\rho \in L^\infty(0,T; L^\gamma(\Omega))$, $\gamma > 1$, extended by 0 outside of Ω, satisfies

$$\partial_t \rho + \operatorname{div}(\rho u) = 0 \qquad \text{in } \mathcal{D}'\left((0,T) \times \mathbb{R}^3\right)$$

then

$$\rho\left(t, \eta^i(t,x)\right) = \rho(0,x) \qquad \text{for all } t \in [0,T], \quad i = 1, ..., N \quad \text{and a.a. } x \in S_0^i. \tag{4.1.18}$$

Proof

A detailed proof of the assertions (i) and (ii) is given in [43, Lemma 3.1, Corollary 3.1] and [43, Lemma 3.2], respectively. The first part of assertion (i) can be shown directly from the fact that $\{S_0^i, \eta^i\}$ and $\{S_0^j, \eta^j\}$ are compatible with the same velocity field u, the second part then follows by regarding $\mathbb{R}^3 \backslash \Omega$ as a rigid body with the associated rigid velocity field 0. The proof of the assertion (ii) is achieved via a regularization of ρ with respect to the spatial variable and a subsequent application of the regularization method by DiPerna and Lions, cf. [35], to the continuity equation (4.1.18) on compact subsets of the solid time-space domain.

□

4.1.3 Main result

Our main result in this chapter yields the existence of weak solutions as introduced in Definition 4.1.1.

Theorem 4.1.1. *[105, Theorem 2.3] Let $T > 0$, assume $\Omega \subset \mathbb{R}^3$ to be a simply connected domain of class $C^{2,\xi}$, $\xi \in (0,1)$, and assume $S_0^i \subset \Omega$, $i = 1,...,N \in \mathbb{N}$ to be bounded domains of class C^2 which satisfy the conditions* (4.1.6)*. Assume moreover the coefficients $\sigma, \mu, \nu, \lambda, a, \gamma \in \mathbb{R}$ to satisfy the conditions* (4.1.7) *and the data $g, J \in L^\infty(Q)$, $\rho_0 \in L^\gamma(\Omega)$, $(\rho u)_0 \in L^1(\Omega)$ and $B_0 \in L^2_{\text{div}}(\Omega)$ to satisfy the conditions* (4.1.8)*. Then the system* (1.3.15)*–*(1.3.28) *admits a weak solution (η, ρ, u, B) on $[0,T]$ in the sense of Definition 4.1.1 which in addition satisfies the energy inequality*

$$\int_\Omega \frac{1}{2}\rho(\tau)\,|u(\tau)|^2 + \frac{a}{\gamma-1}\rho^\gamma(t) + \frac{1}{2\mu}\,|B(\tau)|^2 \;dx + \int_0^\tau \int_\Omega 2\nu|\mathbb{D}\,(u)\,|^2 + \lambda\,|\operatorname{div} u|^2 + \frac{1}{\sigma\mu^2}\,|\operatorname{curl} B|^2 \;dxdt$$

$$\leqslant \int_\Omega \frac{1}{2}\frac{|(\rho u)_0|^2}{\rho_0} + \frac{a}{\gamma-1}\rho_0^\gamma + \frac{1}{2\mu}\,|B_0|^2 \;dx + \int_0^\tau \int_\Omega \rho g\cdot u + \frac{1}{\sigma\mu} J\cdot \operatorname{curl} B \;dxdt \tag{4.1.19}$$

for almost all $\tau \in [0,T]$.

Remark 4.1.2. *We point out that, for a slight improvement of the above result, it might be desirable to consider test functions with non-compact support in the spatial domain in the induction equation* (4.1.17)*. This is possible exactly as explained in Remark 3.1.1 in the incompressible case.*

Remark 4.1.3. *The reason why we require Ω to be of class $C^{2,\xi}$ instead of only C^2 as in the incompressible setting lies in the fact that, in order to solve the continuity equation, we now need to resort to the theory for the parabolic Neumann problem. This theory requires $C^{2,\xi}$-regularity of the domain, cf. Lemma A.6.1 in the appendix.*

The remainder of this chapter is devoted to the proof of Theorem 4.1.1. Large proportions of this proof are adopted from the article [105] by the author of this thesis, in which Theorem 4.1.1 was published. However, from time to time we explicate the proof in greater detail in order to make it easier to understand. Furthermore, we shorten a few steps which are identical with the corresponding steps in the proof given in Chapter 3 for the incompressible setting. In the following section we begin with an outline of the proof, which is subsequently carried out in the Sections 4.3–4.8.

4.2 Approximate system

The biggest challenge in the extension of the proof in the incompressible case in Chapter 3 to the compressible case lies in an appropriate construction of the approximate problem. We fix five parameters $\Delta t > 0$, $n, m \in \mathbb{N}$ and $\epsilon, \alpha > 0$ and introduce an approximation which consists of five different approximation levels, each of which corresponds to one of the parameters. Again, a solution to the original problem will be obtained by first solving the approximate system and passing to the limit in all approximation levels afterwards.

- On the Δt-level, the induction equation is discretized with respect to the time variable via the Rothe method, cf. [99, Section 8.2], while the mechanical part of the problem is split up into a series of time-dependent problems on the small intervals between the discrete times. To this end we fix $\Delta t > 0$ with $\frac{T}{\Delta t} \in \mathbb{N}$ and split up the interval $[0,T]$ into the discrete times $k\Delta t$, $k = 1, ..., \frac{T}{\Delta t}$.

- On the n-level, a Galerkin method is carried out in order to solve the approximate momentum equation.

- The approximation levels associated to m, ϵ and α correspond to the approximation used in [43] for the purely mechanical problem: The m-level describes a penalization method which allows us to pass from a fluid-only system to a system containing both a fluid and rigid bodies. On the ϵ- and α-levels, the system is regularized through the addition of multiple regularization terms as well as an artificial pressure.

In the following we present the complete approximate system, containing all five approximation levels, and subsequently give a more explicit description of each included level and its purpose: Let $\beta > 0$ be sufficiently large such that it satisfies in particular $\beta > \max\{4, \gamma\}$ as in [43, Section 6]. Let

$$V_n := \operatorname{span}\{\phi_1, ..., \phi_n\}, \quad n \in \mathbb{N}, \tag{4.2.1}$$

denote the n-dimensional Galerkin space spanned by the first n eigenfunctions of the Lamé equation in Ω, which constitute an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $H_0^1(\Omega)$, cf. [94, Lemma 4.33]. Then, provided that the approximate system has already been solved up to the (discrete) time $(k-1)\Delta t$ for some $k =\in \{1, ..., \frac{T}{\Delta t}\}$, we seek a solution

$$\rho_{\Delta t,k} \in \left\{\psi \in C\left([(k-1)\Delta t, k\Delta t]; C^{2,\xi}\left(\overline{\Omega}\right)\right) \bigcap C^1\left([(k-1)\Delta t, k\Delta t]; C^{0,\xi}\left(\overline{\Omega}\right)\right) : \ \nabla\psi \cdot \mathrm{n}|_{\partial\Omega} = 0\right\}, \tag{4.2.2}$$

$$u_{\Delta t,k} \in C\left([(k-1)\Delta t, k\Delta t]; V_n\right), \tag{4.2.3}$$

$$B_{\Delta t}^k \in Y^k\left(S_{\Delta t,k}\right) := \left\{b \in H^1_{\mathrm{div}}(\Omega) : \operatorname{curl} b \in H^1(\Omega),\ \operatorname{curl} b = 0 \text{ in } S_{\Delta t,k}(k\Delta t) \bigcap \Omega,\ b \cdot \mathrm{n}|_{\partial\Omega} = 0\right\} \tag{4.2.4}$$

to the system

$$\partial_t \rho_{\Delta t,k} + \operatorname{div}\left(\rho_{\Delta t,k} u_{\Delta t,k}\right) = \epsilon \Delta \rho_{\Delta t,k} \quad \text{in } [(k-1)\Delta t, k\Delta t] \times \Omega, \tag{4.2.5}$$

$$\begin{aligned}
\int_{(k-1)\Delta t}^{k\Delta t}\int_\Omega \partial_t\left(\rho_{\Delta t,k} u_{\Delta t,k}\right) \cdot \phi \, dxdt = \int_{(k-1)\Delta t}^{k\Delta t}\int_\Omega & \left(\rho_{\Delta t,k} u_{\Delta t,k} \otimes u_{\Delta t,k}\right) : \mathbb{D}(\phi) + \left(a\rho_{\Delta t,k}^\gamma + \alpha\rho_{\Delta t,k}^\beta\right) \operatorname{div}\phi \\
& - 2\nu\left(\chi_{\Delta t}^{k-1}\right) \mathbb{D}\left(u_{\Delta t,k}\right) : \mathbb{D}(\phi) - \lambda\left(\chi_{\Delta t}^{k-1}\right) \operatorname{div}\left(u_{\Delta t,k}\right) \operatorname{div}\phi \\
& + \rho_{\Delta t,k} g \cdot \phi + \frac{1}{\mu}\left(\operatorname{curl} B_{\Delta t}^{k-1} \times B_{\Delta t}^{k-1}\right) \cdot \phi - \epsilon \left|u_{\Delta t,k}\right|^2 u_{\Delta t,k} \cdot \phi \\
& - \epsilon\left(\nabla u_{\Delta t,k} \nabla \rho_{\Delta t,k}\right) \cdot \phi \, dxdt,
\end{aligned} \tag{4.2.6}$$

$$\begin{aligned}
-\int_\Omega \frac{B_{\Delta t}^k - B_{\Delta t}^{k-1}}{\Delta t} \cdot b \, dx = \int_\Omega & \left[\frac{1}{\sigma\mu} \operatorname{curl} B_{\Delta t}^k - \tilde{u}_{\Delta t}^{k-1} \times B_{\Delta t}^{k-1} + \frac{\epsilon}{\mu^2}\left|\operatorname{curl} B_{\Delta t}^k\right|^2 \operatorname{curl} B_{\Delta t}^k \right. \\
& \left. - \frac{1}{\sigma} J_{\Delta t}^k\right] \cdot \operatorname{curl} b + \epsilon\left(\nabla \operatorname{curl} B_{\Delta t}^k\right) \cdot \left(\nabla \operatorname{curl} b\right) \, dx
\end{aligned} \tag{4.2.7}$$

for all $\phi \in C([(k-1)\Delta t, k\Delta t]; V_n)$ and

$$b \in W^k\left(S_{\Delta t,k}\right) := \left\{b \in H^1(\Omega) : \operatorname{curl} b \in H^1(\Omega),\ \operatorname{curl} b = 0 \text{ in } S_{\Delta t,k}(k\Delta t) \bigcap \Omega,\ b \cdot \mathrm{n}|_{\partial\Omega} = 0\right\}, \tag{4.2.8}$$

which in addition satisfies the initial conditions

$$\rho_{\Delta t,k}((k-1)\Delta t, x) = \rho_{\Delta t,k-1}((k-1)\Delta t, x), \quad \rho_{\Delta t,1}(0,x) = \rho_0(x), \qquad \forall x \in \Omega, \tag{4.2.9}$$

$$u_{\Delta t,k}((k-1)\Delta t, x) = u_{\Delta t,k-1}((k-1)\Delta t, x), \quad u_{\Delta t,1}(0,x) = u_0(x), \qquad \forall x \in \Omega, \tag{4.2.10}$$

$$B_{\Delta t}^0(x) = B_0(x), \qquad \forall x \in \Omega. \tag{4.2.11}$$

Before we provide an explanation of the different approximation levels in (4.2.2)–(4.2.11), we shed light on the notation used in these equations: The spaces $Y^k(S_{\Delta t,k})$ and $W^k(S_{\Delta t,k})$ in (4.2.4) and (4.2.8) are equipped with the norm

$$\|\cdot\|_{Y^k(S_{\Delta t,k})} := \|\cdot\|_{W^k(S_{\Delta t,k})} := \|\cdot\|_{H^1(\Omega)} + \|\operatorname{curl}(\cdot)\|_{H^1(\Omega)}.$$

For the definition of the set $S_{\Delta t}(k\Delta t)$ in (4.2.4) and (4.2.8), we first denote by

$$O^i := \left(S_0^i\right)_\delta$$

the δ-kernel of the initial domain S_0^i of the i-th body, where $\delta > 0$ is chosen sufficiently small such that for all $i = 1, ..., N$ the δ-neighborhood $(O^i)^\delta$ of O^i coincides with S_0^i,

$$\left(O^i\right)^\delta = \left(\left(S_0^i\right)_\delta\right)^\delta = S_0^i \qquad \forall i = 1, ..., N. \tag{4.2.12}$$

Such $\delta > 0$ exists due to the C^2-regularity of S_0^i, cf. (4.1.5). Then we set

$$O := \bigcup_{i=1}^{N} O^i.$$

Moreover, we denote by $\eta_{\Delta t,k}$ the unique Carathéodory solution (cf. Theorem A.1.1 in the appendix) to the initial value problem

$$\frac{d}{dt}\eta_{\Delta t,k}(t,x) = R_\delta\left[u_{\Delta t,k}\right]\left(t,\eta_{\Delta t,k}(t,x)\right), \quad t \in [(k-1)\Delta t, k\Delta t], \tag{4.2.13}$$

$$\eta_{\Delta t,k}((k-1)\Delta t, x) = \eta_{\Delta t,k-1}((k-1)\Delta t, x), \quad \eta_{\Delta t,1}(0,x) = x, \quad x \in \mathbb{R}^3, \tag{4.2.14}$$

where $R_\delta[u_{\Delta t,k}](t,\cdot) := u_{\Delta t,k}(t,\cdot) * \Theta_\delta(\cdot)$ and Θ_δ denotes a radially symmetric and non-increasing mollifier with respect to the spatial variable. With this notation at hand we define the domain $S^i_{\Delta t,k}(t)$ of the i-th approximate solid at an arbitrary time $t \in [(k-1)\Delta t, k\Delta t] \subset [0,T]$ by

$$S^i_{\Delta t,k}(t) := \left(O^i_{\Delta t,k}(t)\right)^\delta = \left\{x \in \mathbb{R}^3 : \ \operatorname{dist}\left(x, O^i_{\Delta t,k}(t)\right) < \delta\right\}, \quad O^i_{\Delta t,k}(t) := \eta_{\Delta t,k}\left(t, O^i\right). \tag{4.2.15}$$

Consequently, the approximate solid region at the time t is given by

$$S_{\Delta t,k}(t) := \bigcup_{i=1}^{N} S^i_{\Delta t,k}(t),$$

which in particular defines the set $S_{\Delta t,k}(k\Delta t)$ in (4.2.4) and (4.2.8). For the later use we remark that $S^i_{\Delta t,k}(t)$, as the δ-neighborhood of a bounded set, satisfies the cone condition and thus has the property

$$\left|\partial S^i_{\Delta t,k}(t)\right| = 0 \quad \text{for all } t \in [0,T], \ i = 1, ..., N. \tag{4.2.16}$$

Next, for the definition of the variable viscosity coefficients $\nu(\chi^{k-1}_{\Delta t})$ and $\lambda(\chi^{k-1}_{\Delta t})$ in the momentum equation (4.2.6) we denote the signed distance function of arbitrary sets $U \subset \mathbb{R}^3$ by

$$\mathbf{db}_U(x) := \operatorname{dist}\left(x, \overline{\mathbb{R}^3 \setminus U}\right) - \operatorname{dist}\left(x, \overline{U}\right). \tag{4.2.17}$$

Further we introduce the signed distance function of the approximate solid area,

$$\chi_{\Delta t,k}(t,x) := \mathbf{db}_{S_{\Delta t,k}(t)}(x), \qquad \chi^{k-1}_{\Delta t}(t) := \chi_{\Delta t,k}((k-1)\Delta t, \cdot) \quad \text{for } t \in [(k-1)\Delta t, k\Delta t].$$

Choosing a convex function $H \in C^\infty(\mathbb{R})$ such that

$$H(z) = 0 \quad \text{for } z \in (-\infty, 0], \qquad H(z) > 0 \quad \text{for } z \in (0, +\infty), \tag{4.2.18}$$

we then define the variable viscosity coefficients by

$$\nu\left(\chi^{k-1}_{\Delta t}\right) := \nu + mH\left(\chi^{k-1}_{\Delta t}\right), \qquad \lambda\left(\chi^{k-1}_{\Delta t}\right) := \lambda + mH\left(\chi^{k-1}_{\Delta t}\right). \tag{4.2.19}$$

Moreover, in the induction equation (4.2.7) the function $\tilde{u}^{k-1}_{\Delta t}$ is defined by

$$\tilde{u}^{k-1}_{\Delta t}(x) := \begin{cases} \frac{1}{\Delta t}\displaystyle\int_{(k-2)\Delta t}^{(k-1)\Delta t} u_{\Delta t,k-1}(t,x)dt & \text{if } k \geqslant 2, \\ u_{0,\alpha}(x) & \text{if } k = 1, \end{cases} \tag{4.2.20}$$

while the discretized external force $J^k_{\Delta t}$ is defined by

$$J^k_{\Delta t} := J_\omega(k\Delta t), \quad J_\omega(t) := \int_0^T \theta_\omega\left(t + \omega\frac{T-2t}{T} - s\right) J(s)\, ds, \tag{4.2.21}$$

for another mollifier $\theta_\omega : \mathbb{R} \to \mathbb{R}$ and a suitable choice of $\omega = \omega(\Delta t)$, $\omega(\Delta t) \to 0$ for $\Delta t \to 0$. Finally, the functions $\rho_{0,\alpha}$, $u_{0,\alpha}$, $B_{0,\alpha}$ in the initial conditions (4.2.9)–(4.2.11) denote regularizations of the initial data in Theorem 4.1.1 satisfying

$$\rho_{0,\alpha} \in C^{2,\xi}\left(\overline{\Omega}\right), \qquad (\rho u)_{0,\alpha} \in C^2\left(\overline{\Omega}\right), \qquad u_{0,\alpha} := P_n\left(\frac{(\rho u)_0}{\rho_0}\right) \in V_n, \qquad B_{0,\alpha} \in H^2_{\mathrm{div}}(\Omega), \tag{4.2.22}$$

$$0 < \alpha \leqslant \rho_{0,\alpha} \leqslant \alpha^{-\frac{1}{2\beta}}, \qquad \nabla \rho_{0,\alpha} \cdot \mathrm{n}|_{\partial\Omega} = 0, \qquad B_{0,\alpha} \cdot \mathrm{n}|_{\partial\Omega} = 0, \tag{4.2.23}$$

where P_n denotes the orthogonal projection of $L^2(\Omega)$ onto V_n. We are now in the position to discuss the several approximation levels and the reasons why they are required. We start from the highest level.

The Δt-level constitutes the level which contains most of the difficulties. It is here where the main novelties of our proof enter, compared to the incompressible setting in Chapter 3. The precarious situation presents itself in the following way: On the one hand, we face the same problem as in the incompressible case, namely, the dependence of the test functions (4.1.3), (4.1.4) for the induction equation and the momentum equation on the solution of the system. This hinders the effort to solve all of the equations in the system simultaneously. Again we can deal with the test functions in the momentum equation by means of a penalization method (cf. the m-level below), whereas the unavailability of such a method for the setting in the induction equation suggests to decouple the system by the use of a classical time discretization. The latter procedure enables us, at each fixed discrete time, to first determine a velocity field and, from this, the position of the approximate solid. This in turn determines the test functions (4.2.8) and solving the discretized induction equation (4.2.7) becomes a routine matter. On the other hand, however, the various functions evaluated at different discrete times in a fully discretized system complicate the derivation of a meaningful energy inequality. The author could not find a way to transfer several of the techniques known for the continuous compressible Navier-Stokes system (cf. [94, Sections 7.6.5, 7.6.6, 7.7.4.2]) - in particular, the proof of the non-negativity of the density - to the discrete case and it did not seem to be possible to derive the uniform bounds required for the limit passage with respect to $\Delta t \to 0$.

Our solution to this dilemma consists of consolidating the approach of a hybrid approximate system used in the incompressible setting (cf. Section 3.2), consisting of both discrete equations and equations which are continuous on small time intervals. More precisely, instead of a strictly discretized system, we consider a hybrid system in which the induction equation (4.2.7) is indeed discretized by the Rothe method, while the continuity equation and the momentum equation are solved as continuous equations on the small intervals between each pair of consecutive discrete times, c.f (4.2.5) and (4.2.6). Through this, the solution dependence of the test functions in the induction equation can be handled as in the fully discrete system, while the mechanical part of the energy inequality - with the density bounded away from zero - can be derived as in the strictly continuous case. The difference between the hybrid approximation here and the one used in the incompressible case lies in the weighting of the discrete and the continuous part of the system. While in the incompressible case the whole system except for the transport equation for the characteristic function of the solid region was discretized, we consider a continuous approximation of the whole mechanical part of the system in the present setting. This leads to an additional difficulty in the derivation of the energy inequality, for which a combination of the discrete and the time-dependent part of the system is necessary. The latter is achieved under the consideration of piecewise linear interpolants of the discrete functions, which allows us to deduce a continuous energy estimate from the discrete induction equation (4.2.7). This can subsequently be added together with the corresponding mechanical estimate to obtain the full energy inequality, cf. Section 4.4.1.

The Galerkin method carried out on the n-level is used to solve the continuous momentum equation (4.2.5) on the small time intervals from the Δt-level by a standard procedure. The Galerkin-regularity of the velocity field furthermore helps us during the limit passage with respect to $\Delta t \to 0$, cf. (4.4.29) below.

After letting n tend to ∞ we find ourselves in the same situation as in the approximation of the exclusively mechanical system in [43, Section 6]. Indeed, the remaining three approximation levels correspond directly to the three level approximation scheme used in that article. Hence, for the

mechanical part of our problem we can follow exactly the strategy used therein. Moreover, the limit passages in the induction equation from here on do not contain any new difficulties anymore. Consequently, after the limit passage in n the rest of the proof will become a routine matter.
The penalization method on the m-level differs from the Brinkman penalization used in the incompressible setting, cf. Section 3.2. This is because the Brinkman penalization is designed specifically for the case of incompressible fluids and presupposes a certain bound of the density away from zero, which is not guaranteed anymore once we leave the Galerkin level in the compressible case. The penalization method we use instead is the same as the one used for the fluid-rigid bodies system in [43] and was, before that, also used for example for a corresponding two dimensional problem in [103]. The idea behind it is to approximate the entirety of the fluid and the rigid bodies by a fluid in the whole domain with viscosity tending to infinity in the later solid regions. Mathematically this is implemented through the variable viscosity coefficients (4.2.19). Due to the choice of the function H in (4.2.18) these coefficients blow up in the approximate solid region once we let m tend to ∞ and, thanks to the energy inequality, this will cause the limit velocity field u to coincide with a rigid velocity field in each body. Moreover, the positions $S^i(t)$ of the bodies in the m-limit are determined through the flow curves of $R_\delta[u]$, cf. (4.2.13) and (4.2.15). This regularized velocity field has the useful property that, for any domain $U \subset \mathbb{R}^3$, it holds

$$\mathbb{D}(u(t,\cdot)) = 0 \quad \text{in } U \qquad \Rightarrow \qquad R_\delta[u](t,\cdot) = u(t,\cdot) \quad \text{in } U_\delta = \{x \in U : \operatorname{dist}(x, \partial U) > \delta\}, \tag{4.2.24}$$

cf. [43, Remark 6.1]. Hence $R_\delta[u]$ coincides with u itself in the sets $O^i(t) \subset S^i(t)$, in which $\mathbb{D}(u) = 0$. Consequently, the rigid velocity fields coinciding with u in $S^i(t)$ also coincide, in $O^i(t)$, with the velocity field $R_\delta[u]$ which determines the motion of the bodies. In particular, this shows that the bodies $S^i(t)$ are indeed rigid.
On the ϵ-level, the continuity equation (4.2.5) is regularized through the additional Laplacian $\epsilon\Delta\rho_{\Delta t,k}$. The additional quantity $\epsilon(\nabla u_{\Delta t,k}\nabla\rho_{\Delta t,k})$ in the momentum equation (4.2.6) ensures that the energy inequality is preserved under this regularization. This procedure, which is classical in the theory of the compressible Navier-Stokes equations (cf. [94, Section 7.3.8]) and which we already transferred to the discretized system in the incompressible setting (cf. Section 3.2), is what guarantees us the non-negativity of the density. The other regularization term $\epsilon|u_{\Delta t,k}|^2 u_{\Delta t,k}$ in (4.2.6) is needed on the time discrete level where, as opposed to the continuous case, the mixed terms from the momentum equation and the induction equation do not annihilate each other in the energy inequality, which prevents a direct application of the Gronwall lemma. The quantity $\epsilon|u_{\Delta t,k}|^2 u_{\Delta t,k}$ can be used to control the velocity part of these mixed terms. The 4-double-curl $\epsilon \operatorname{curl}(\left|\operatorname{curl} B^k_{\Delta t}\right|^2 \operatorname{curl} B^k_{\Delta t})$ in the induction equation (4.2.7) fulfills, as in the incompressible setting in Section 3.2, the same purpose for the magnetic part of the mixed terms so that we are able to derive uniform bounds from the energy inequality nevertheless. The reason why, as opposed to in the incompressible case, the term $\epsilon|u_{\Delta t,k}|^2 u_{\Delta t,k}$ is required in this procedure stems from the bound of the density away from zero. In the incompressible case, such a bound can be shown before the derivation of the energy inequality, which allows us to control the velocity field on the right-hand side of the energy inequality via the (discrete) Gronwall lemma. In the present setting, however, a (uniform) energy estimate needs to be derived first, forcing us to absorb the velocity field into the left-hand side of the energy inequality. We remark that the control of the mixed terms is also the motivation for the definition of the velocity field (4.2.20) in the discrete induction equation: Indeed, defining this quantity as a mean value of the velocity field obtained from the momentum equation on the intervals $[(k-2)\Delta t, (k-1)\Delta t]$, we can absorb it into the left-hand side of the energy inequality thanks to the above-mentioned regularization terms. If instead the term was defined, more intuitively, as a pointwise evaluation of $u_{\Delta t,k}$, we would not be able to handle it. The last regularization term in (4.2.7), the quantity $\operatorname{curl}(\Delta(\operatorname{curl} B^k_{\Delta t}))$, enables us to construct $B^k_{\Delta t}$ via a weakly continuous coercive operator as in the incompressible case, cf. again Section 3.2.
Finally, on the α-level, the artificial pressure term $\alpha\rho^\beta_{\Delta t,k}$ is added to the momentum equation (4.2.6). Again this method is already well-known from the general existence theory for the compressible Navier-Stokes system, cf. [94, Section 7.3.8]. The artificial pressure gives us an additional amount of integrability of the density and its gradient, required to pass to the limit in the term $\epsilon(\nabla u_{\Delta t,k}\nabla\rho_{\Delta t,k})$ from

the ϵ-level, cf. [94, Section 7.8.2]. It furthermore simplifies the limit passage with respect to $\epsilon \to 0$, since the additional integrability allows for the use of the regularization technique by DiPerna and Lions, cf. [94, Lemma 6.8, Lemma 6.9].

4.3 Existence of the approximate solution

We begin the proof of Theorem 4.1.1 by showing the existence of a solution to the approximate problem (4.2.2)–(4.2.11) on the highest approximation level.

4.3.1 Existence of the density and the velocity field

The existence of the density and the velocity field on the Galerkin level can be shown by classical methods, cf. for example [94, Section 7.7]. More precisely, the continuity equation (4.2.5) and the momentum equation (4.2.6) can be solved simultaneously by means of a fixed point argument: For fixed $w \in C([(k-1)\Delta t, k\Delta t]; V_n)$ we consider the Neumann problem

$$\partial_t \rho + \operatorname{div}(\rho w) = \epsilon \Delta \rho \qquad \text{in } [(k-1)\Delta t, k\Delta t] \times \Omega, \tag{4.3.1}$$

$$\nabla \rho \cdot \mathrm{n}|_{\partial\Omega} = 0, \quad \rho((k-1)\Delta t, \cdot) = \rho_{\Delta t,k-1}((k-1)\Delta t, \cdot) \qquad \text{in } \Omega, \tag{4.3.2}$$

$$0 < \underline{\rho} \leqslant \rho_{\Delta t,k-1}((k-1)\Delta t, \cdot) \leqslant \overline{\rho} < \infty, \qquad \text{in } \Omega \tag{4.3.3}$$

for some constants $0 < \underline{\rho} \leqslant \overline{\rho} < \infty$. It is well known (cf. Lemma A.6.1 in the appendix) that (4.3.1)–(4.3.3) admits a unique solution

$$\rho = \rho(w) \in C\left([(k-1)\Delta t, k\Delta t]; C^{2,\xi}\left(\overline{\Omega}\right)\right) \bigcap C^1\left([(k-1)\Delta t, k\Delta t]; C^{0,\xi}\left(\overline{\Omega}\right)\right),$$

which satisfies the estimate

$$0 < \underline{\rho} \exp\left(-\int_{(k-1)\Delta t}^{t} \|w(\tau)\|_{V_n} \, d\tau\right) \leqslant \rho(w)(t,x) \leqslant \overline{\rho} \exp\left(\int_{(k-1)\Delta t}^{t} \|w(\tau)\|_{V_n} \, d\tau\right) < \infty \tag{4.3.4}$$

for all $(t,x) \in [(k-1)\Delta t, k\Delta t] \times \overline{\Omega}$. Further, we consider a linearized version of the momentum equation (4.2.6): Given $w \in C([(k-1)\Delta t, k\Delta t]; V_n)$ and the associated solution $\rho(w)$ to the Neumann problem (4.3.1)–(4.3.3), we seek $u \in C([(k-1)\Delta t, k\Delta t]; V_n)$ such that

$$\begin{aligned}
\int_\Omega \partial_t (\rho(w) u) \cdot \phi \, dx = \int_\Omega & (\rho(w) w \otimes u) : \mathbb{D}(\phi) + \left(a\rho^\gamma(w) + \alpha\rho^\beta(w)\right) \operatorname{div} \phi \\
& - 2\nu \left(\chi_{\Delta t}^{k-1}\right) \mathbb{D}(u) : \mathbb{D}(\phi) - \lambda \left(\chi_{\Delta t}^{k-1}\right) \operatorname{div}(u) \operatorname{div} \phi \\
& + \rho(w) g \cdot \phi + \frac{1}{\mu} \left(\operatorname{curl} B_{\Delta t}^{k-1} \times B_{\Delta t}^{k-1}\right) \cdot \phi \\
& - \epsilon \left(\nabla u \nabla \rho(w)\right) \cdot \phi - \epsilon |w|^2 u \cdot \phi \, dx \qquad \text{in } [(k-1)\Delta t, k\Delta t],
\end{aligned} \tag{4.3.5}$$

$$u((k-1)\Delta t, \cdot) = u_{\Delta t,k-1}((k-1)\Delta t, \cdot) \qquad \text{in } \Omega, \tag{4.3.6}$$

for all $\phi \in C([(k-1)\Delta t, k\Delta t]; V_n)$. Under exploitation of the fact that, by (4.3.4), $\rho(w)$ is bounded away from 0 and the linearity of the problem, it follows from classical results on ordinary differential equations that this problem admits a unique solution $u = u(w) \in C([(k-1)\Delta t, k\Delta t]; V_n)$. We can thus define an operator

$$\mathbb{T} : C\left([(k-1)\Delta t, k\Delta t]; V_n\right) \to C\left([(k-1)\Delta t, k\Delta t]; V_n\right), \quad \mathbb{T}(w) := u.$$

The desired solution $u_{\Delta t,k}$ to the momentum equation (4.2.6) can now be understood as a fixed point of $\mathbb{T}$. We use the variant [40, Section 9.2.2, Theorem 4] of the Schauder fixed point theorem to show that $\mathbb{T}$ possesses such a fixed point. For this we first require continuity and compactness of $\mathbb{T}$. We define another operator

$$\mathcal{M}_{\rho(w)(t)} : V_n \to (V_n)^*, \qquad \left\langle \mathcal{M}_{\rho(w)(t)} v, \phi \right\rangle_{(V_n)^* \times V_n} := \int_\Omega \rho(w)(t) v \cdot \phi \, dx$$

for all $t \in [(k-1)\Delta t, k\Delta t]$ and all $v, \phi \in V_n$. As, by the estimate (4.3.4), $\rho(w)$ is bounded away from zero, the operator $\mathcal{M}_{\rho(w)(t)}$ is invertible with an inverse $\mathcal{M}^{-1}_{\rho(w)(t)} : (V_n)^* \to V_n$, which enjoys the properties

$$\begin{aligned} &\partial_t \left\langle \mathcal{M}^{-1}_{\rho(w)(t)} v(t), \phi \right\rangle_{(V_n)^* \times V_n} \\ =& \left\langle \mathcal{M}^{-1}_{\rho(w)(t)} \mathcal{M}_{\partial_t \rho(w)(t)} \mathcal{M}^{-1}_{\rho(w)(t)} v(t) + \mathcal{M}^{-1}_{\rho(w)(t)} \partial_t v(t), \phi \right\rangle_{(V_n)^* \times V_n} \quad \text{in } \mathcal{D}'((k-1)\Delta t, k\Delta t), \end{aligned} \tag{4.3.7}$$

and

$$\left\| \mathcal{M}^{-1}_{\rho(w)(t)} \right\|_{\mathcal{L}((V_n)^*, V_n)} \leqslant \frac{1}{a_n(w)}, \tag{4.3.8}$$

$$\left\| \mathcal{M}^{-1}_{\rho(w^1)(t)} - \mathcal{M}^{-1}_{\rho(w^2)(t)} \right\|_{\mathcal{L}((V_n)^*, V_n)} \leqslant \frac{c(n)}{\min\left\{a_n\left(w^1\right), a_n\left(w^2\right)\right\}^2} \left\| \rho\left(w^1\right)(t) - \rho\left(w^2\right)(t) \right\|_{L^1(\Omega)}, \tag{4.3.9}$$

$$\left\| \mathcal{M}^{-1}_{\rho(w)(t)} \mathcal{M}_{\partial_t \rho(w)(t)} \mathcal{M}^{-1}_{\rho(w)(t)} \right\|_{\mathcal{L}((V_n)^*, V_n)} \leqslant \frac{c(n)}{a_n^2(w)} \left\| \partial_t \rho(w)(t) \right\|_{L^1(\Omega)} \tag{4.3.10}$$

for a constant $c(n) > 0$, for all $t \in [(k-1)\Delta t, k\Delta t]$, $w, w^1, w^2, v \in C([(k-1)\Delta t, k\Delta t]; V_n)$, $\phi \in V_n$ and

$$a_n(w) := \underline{\rho} \exp\left(- \int_{(k-1)\Delta t}^{k\Delta t} \|w(\tau)\|_{V_n} \, d\tau \right),$$

cf. [94, Section 7.7.1]. We may write the solution $u = \mathbb{T}(w)$ to the linearized problem (4.3.5), (4.3.6) in the form

$$u(t) = \mathcal{M}^{-1}_{\rho(w)(t)} \left[\left(\rho_{\Delta t, k-1}((k-1)\Delta t) u_{\Delta t, k-1}((k-1)\Delta t)\right)^* + \int_0^t \mathcal{N}(w, \rho(w), u) \, d\tau \right], \tag{4.3.11}$$

where

$$\begin{aligned} \left\langle \mathcal{N}(w, \rho, u), \phi \right\rangle_{(V_n)^* \times V_n} := \int_\Omega & (\rho(w) w \otimes u) : \mathbb{D}(\phi) + \left(a\rho^\gamma(w) + \alpha \rho^\beta(w) \right) \operatorname{div} \phi \\ & - 2\nu \left(\chi^{k-1}_{\Delta t} \right) \mathbb{D}(u) : \mathbb{D}(\phi) - \lambda \left(\chi^{k-1}_{\Delta t} \right) \operatorname{div} u \operatorname{div} \phi + \rho(w) g \cdot \phi \\ & + \frac{1}{\mu} \left(\operatorname{curl} B^{k-1}_{\Delta t} \times B^{k-1}_{\Delta t} \right) \cdot \phi - \epsilon \left(\nabla u \nabla \rho(w) \right) \cdot \phi - \epsilon |w|^2 u \cdot \phi \, dx \end{aligned}$$

and

$$\begin{aligned} &\left\langle \left(\rho_{\Delta t, k-1}((k-1)\Delta t) u_{\Delta t, k-1}((k-1)\Delta t)\right)^*, \phi \right\rangle_{(V_n)^* \times V_n} \\ :=& \int_\Omega \rho_{\Delta t, k-1}((k-1)\Delta t) u_{\Delta t, k-1}((k-1)\Delta t) \cdot \phi \, dx \end{aligned}$$

for all $\phi \in V_n$. On the one hand, a combination of the identity (4.3.11), the estimates (4.3.4), (4.3.8), (4.3.9) and the estimates (A.6.4) and (A.6.5) given by Lemma A.6.1 in the appendix for the solution to the Neumann problem (4.3.1)–(4.3.3) yields the desired continuity of the operator $\mathbb{T}$. On the other hand, a combination of the identities (4.3.7) and (4.3.11) yields the representation

$$\begin{aligned} &\partial_t u(t) \\ =& \mathcal{M}^{-1}_{\rho(w)(t)} \mathcal{M}_{\partial_t \rho(w)(t)} \mathcal{M}^{-1}_{\rho(w)(t)} \left[\left(\rho_{\Delta t, k-1}((k-1)\Delta t) u_{\Delta t, k-1}((k-1)\Delta t)\right)^* + \int_0^t \mathcal{N}(w, \rho(w)), u) \, d\tau \right] \\ &+ \mathcal{M}^{-1}_{\rho(w)(t)} \left[\mathcal{N}(w, \rho(w)), u)(t) \right] \end{aligned}$$

of $\partial_t u$. This, together with the estimates (4.3.4), (4.3.8), (4.3.10) and the uniform bound (A.6.4) for the solution to the Neumann problem (4.3.1)–(4.3.3) yields the bound

$$\|\partial_t u\|_{L^2((k-1)\Delta t, k\Delta t); V_n)} \leqslant c(n, w), \tag{4.3.12}$$

where $c(n, w) > 0$ denotes a constant which remains bounded for bounded values of w in $C([(k-1)\Delta t, k\Delta t]; V_n)$. The latter bound then suffices to infer also the desired compactness of the operator $\mathbb{T}$. As a final condition for the application of the fixed point theorem [40, Section 9.2.2, Theorem 4] we need to show that fixed points $u \in C([(k-1)\Delta t, k\Delta t]; V_n)$ of the operator $s\mathbb{T}$, with $s \in [0, 1]$ are bounded in $C([(k-1)\Delta t, k\Delta t]; V_n)$, uniformly with respect to s. For the choice $w = u$ we test the regularized continuity equation (4.3.1) by $\frac{1}{2}|u|^2$ and the linearized momentum equation (4.3.5) by u. As a difference between the resulting identities we obtain the energy equation

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{1}{2}\rho(u)(t)|u(t)|^2 + s\frac{a}{\gamma-1}\rho^\gamma(u)(t) + s\frac{\alpha}{\beta-1}\rho^\beta(u)(t)\; dx + \int_\Omega 2\nu\left(\chi_{\Delta t}^{k-1}\right)|\mathbb{D}(u(t))|^2 \\
&+ \lambda\left(\chi_{\Delta t}^{k-1}\right)|\operatorname{div} u(t)|^2 + sa\epsilon\gamma\rho^{\gamma-2}(u)(t)\,|\nabla\rho(u)(t)|^2 + s\alpha\epsilon\beta\rho^{\beta-2}(u)(t)\,|\nabla\rho(u)(t)|^2 \; + \epsilon\,|u(t)|^4\, dx \\
=&s\int_\Omega \rho(u)(t)g(t)\cdot u(t) + \frac{1}{\mu}\left(\operatorname{curl} B_{\Delta t}^{k-1} \times B_{\Delta t}^{k-1}\right)\cdot u(t)\; dx
\end{aligned} \tag{4.3.13}$$

for all $t \in [(k-1)\Delta t, k\Delta t]$. Applying the Gronwall Lemma to this equation we conclude the desired bound of all fixed points u of $s\mathbb{T}$ in $C([(k-1)\Delta t, k\Delta t]; V_n)$, uniform in $s \in [0, 1]$. Since we have already shown continuity and compactness of $\mathbb{T}$ we may thus apply the fixed point theorem [40, Section 9.2.2, Theorem 4] to infer the existence of a fixed point $u_{\Delta t,k} \in C([(k-1)\Delta t, k\Delta t]; V_n)$ of $\mathbb{T}$, which constitutes the desired solution to the initial value problem (4.2.6), (4.2.10). Furthermore, by construction, the associated density $\rho_{\Delta t,k} := \rho(u_{\Delta t,k})$ is the desired solution to the corresponding initial value problem (4.2.5), (4.2.9) for the density.

4.3.2 Existence of the magnetic induction

The existence of the magnetic induction is obtained exactly as in the incompressible case in Section 3.3.4. Indeed, we first consider the problem

$$\begin{aligned}
&\left\langle A\left(B_{\Delta t}^k\right), b\right\rangle_{(Y^k(S_{\Delta t,k}))^* \times Y^k(S_{\Delta t,k})} \\
&= \int_\Omega \frac{B_{\Delta t}^{k-1}}{\Delta t}\cdot b + \left[\tilde{u}_{\Delta t}^{k-1} \times B_{\Delta t}^{k-1} + \frac{1}{\sigma}J_{\Delta t}^k\right]\cdot \operatorname{curl} b\; dx \qquad \forall b \in Y^k\left(S_{\Delta t,k}\right),
\end{aligned} \tag{4.3.14}$$

where $A : Y^k(S_{\Delta t,k}) \to (Y^k(S_{\Delta t,k}))^*$ denotes the operator defined via the formula (3.3.7). As shown in Section 3.3.4, the operator A is both coercive and weakly continuous and therefore surjective on $Y^k(S_{\Delta t,k})$, cf. [49, Theorem 1.2]. Consequently there exists a function $B_{\Delta t}^k \in Y^k(S_{\Delta t,k})$ which satisfies the identity (4.3.14) for all $b \in Y^k(S_{\Delta t,k})$ and, as seen in Section 3.3.4, due to the Helmholtz decomposition (cf. Lemma A.2.2 in the appendix), also for all non-solenoidal test functions $b \in W^k(S_{\Delta t,k}) \supset Y^k(S_{\Delta t,k})$. Therefore $B_{\Delta t}^k$ constitutes the desired solution to the discrete induction equation (4.2.7). Altogether, we have shown the following result.

Proposition 4.3.1. *Let all the assumptions of Theorem 4.1.1 be satisfied, let $n, m \in \mathbb{N}$, let $\Delta t, \epsilon, \alpha > 0$, let $\beta > \max\{4, \gamma\}$ be sufficiently large and let $\delta > 0$ be as in* (4.2.12). *Let further $J_{\Delta t}^k$ be given by* (4.2.21) *for any $k = 0, ..., \frac{T}{\Delta t}$ and assume the regularized initial data $\rho_{0,\alpha}$, $u_{0,\alpha}$, $B_{0,\alpha}$ to satisfy the conditions* (4.2.22), (4.2.23). *Then, for all $k = 1, ..., \frac{T}{\Delta t}$, there exist functions*

$$0 \leqslant \rho_{\Delta t,k} \in \left\{\psi \in C\left([(k-1)\Delta t, k\Delta t]; C^{2,\xi}\left(\overline{\Omega}\right)\right)\bigcap C^1\left([(k-1)\Delta t, k\Delta t]; C^{0,\xi}\left(\overline{\Omega}\right)\right) : \; \nabla\psi\cdot \mathrm{n}|_{\partial\Omega} = 0\right\},$$

$$u_{\Delta t,k} \in C\left([(k-1)\Delta t, k\Delta t]; V_n\right), \qquad B_{\Delta t}^k \in Y^k\left(S_{\Delta t,k}\right),$$

which satisfy the continuity equation (4.2.5), *the momentum equation* (4.2.6) *for all test functions $\phi \in C\left([(k-1)\Delta t, k\Delta t]; V_n\right)$ and the induction equation* (4.2.7) *for all test functions $b \in W^k(S_{\Delta t,k})$ as well as the initial conditions* (4.2.9)–(4.2.11).

4.4 Limit passage with respect to $\Delta t \to 0$

We continue by passing to the limit with respect to $\Delta t \to 0$, i.e. we pass to the limit in the time discretization in order to return to the realm of (fully) continuous equations. As in the incompressible case in Section 3.4, we first assemble the functions constructed in Section 4.3, defined only on small time intervals or in discrete time points, to functions defined on the whole time interval $[0,T]$. More precisely, for functions $f_{\Delta t,k}$, defined on $[(k-1)\Delta t, k\Delta t] \times \Omega$ for $k = 1, ..., \frac{T}{\Delta t}$, we define the assembled functions

$$f_{\Delta t}(t) := f_{\Delta t,k}(t) \qquad \forall t \in ((k-1)\Delta t, k\Delta t], \quad k = 1, ..., \frac{T}{\Delta t} \tag{4.4.1}$$

while for discrete functions $h^k_{\Delta t}$, defined on Ω for $k = 0, ..., \frac{T}{\Delta t}$, we define the piecewise affine and piecewise constant interpolants

$$h_{\Delta t}(t) := \left(\frac{t}{\Delta t} - (k-1)\right) h^k_{\Delta t} + \left(k - \frac{t}{\Delta t}\right) h^{k-1}_{\Delta t} \qquad \forall t \in ((k-1)\Delta t, k\Delta t], \quad k = 1, ..., \frac{T}{\Delta t}, \tag{4.4.2}$$

$$\overline{h}_{\Delta t}(t) := h^k_{\Delta t} \qquad \forall t \in ((k-1)\Delta t, k\Delta t], \quad k = 0, ..., \frac{T}{\Delta t}, \tag{4.4.3}$$

$$\overline{h}'_{\Delta t}(t) := h^{k-1}_{\Delta t} \qquad \forall t \in ((k-1)\Delta t, k\Delta t], \quad k = 1, ..., \frac{T}{\Delta t}. \tag{4.4.4}$$

Moreover, in order to derive a suitable energy inequality in Section 4.4.1 below, we also introduce a piecewise affine interpolation of the square of the $L^2(\Omega)$-norm,

$$h_{\Delta t,\|\cdot\|}(t) := \left(\frac{t}{\Delta t} - (k-1)\right) \left\|h^k_{\Delta t}\right\|^2_{L^2(\Omega)} + \left(k - \frac{t}{\Delta t}\right) \left\|h^{k-1}_{\Delta t}\right\|^2_{L^2(\Omega)}, \qquad \forall t \in ((k-1)\Delta t, k\Delta t]$$

for any $k = 1, ..., \frac{T}{\Delta t}$. Similarly we assemble the sets describing the approximate solid region,

$$\begin{aligned}
S_{\Delta t}(t) &:= S_{\Delta t,k}(t) = (\eta_{\Delta t}(t, O))^\delta & \forall t \in ((k-1)\Delta t, k\Delta t], \quad k = 1, ..., \frac{T}{\Delta t},\\
\overline{\overline{S}}_{\Delta t}(t) &:= S_{\Delta t,k}(k\Delta t) = (\eta_{\Delta t}(k\Delta t, O))^\delta & \forall t \in ((k-1)\Delta t, k\Delta t], \quad k = 0, ..., \frac{T}{\Delta t},\\
\overline{\overline{S}}'_{\Delta t}(t) &:= S_{\Delta t,k}((k-1)\Delta t) = (\eta_{\Delta t}((k-1)\Delta t, O))^\delta & \forall t \in ((k-1)\Delta t, k\Delta t], \quad k = 1, ..., \frac{T}{\Delta t}.
\end{aligned}$$

As in the incompressible case we use the notation $\overline{\overline{S}}_{\Delta t}(t)$ instead of $\overline{S}_{\Delta t}(t)$ for the piecewise constant interpolants in order to avoid confusion with the notation for the closure of sets. With the above notation we are able to express the hybrid system on the Δt-level as a fully continuous system on the time interval $[0,T]$: Since, by Proposition 4.3.1, the functions $\rho_{\Delta t,k}$ and $u_{\Delta t,k}$ satisfy the continuity equation (4.2.5), the momentum equation (4.2.6), and the initial conditions (4.2.9), (4.2.10), it follows from the definition of $\rho_{\Delta t}$ and $u_{\Delta t}$ in (4.4.1) as well as of $\overline{B}_{\Delta t}$, $\overline{B}'_{\Delta t}$ and $\overline{\chi}'_{\Delta t}$ in and (4.4.3) and (4.4.4) that these functions satisfy the continuity equation

$$\partial_t \rho_{\Delta t} + \operatorname{div}(\rho_{\Delta t} u_{\Delta t}) = \epsilon \rho_{\Delta t} \quad \text{a.e. in } Q, \tag{4.4.5}$$

the momentum equation

$$\begin{aligned}
\int_0^T \int_\Omega \partial_t (\rho_{\Delta t} u_{\Delta t}) \cdot \phi \, dxdt = \int_0^T \int_\Omega & (\rho_{\Delta t} u_{\Delta t} \otimes u_{\Delta t}) : \mathbb{D}(\phi) + \left(a\rho^\gamma_{\Delta t} + \alpha \rho^\beta_{\Delta t}\right) \operatorname{div} \phi\\
&- 2\nu \left(\overline{\chi}'_{\Delta t}\right) \mathbb{D}(u_{\Delta t}) : \mathbb{D}(\phi) - \lambda \left(\overline{\chi}'_{\Delta t}\right) \operatorname{div} u_{\Delta t} \operatorname{div} \phi + \rho_{\Delta t} g \cdot \phi\\
&+ \frac{1}{\mu} \left(\operatorname{curl} \overline{B}'_{\Delta t} \times \overline{B}'_{\Delta t}\right) \cdot \phi - \epsilon \left|u_{\Delta t}\right|^2 u_{\Delta t} \cdot \phi - \epsilon \left(\nabla u_{\Delta t} \nabla \rho_{\Delta t}\right) \cdot \phi \, dxdt
\end{aligned} \tag{4.4.6}$$

for any $\phi \in C([0,T]; V_n)$ and the initial conditions

$$\rho_{\Delta t}(0) = \rho_{0,\alpha}, \quad u_{\Delta t}(0) = u_{0,\alpha}.$$

Furthermore, from $\eta_{\Delta t,k}$ being the unique Carathéodory solution to the initial value problem (4.2.13), (4.2.14), it follows that $\eta_{\Delta t}$ is the unique Carathéodory solution to

$$\frac{d\eta_{\Delta t}(t,x)}{dt} = R_\delta \left[u_{\Delta t}\right](t, \eta_{\Delta t}(t,x)), \quad \text{for } t \in [0,T], \qquad \eta_{\Delta t}(0,x) = x, \quad \text{for } x \in \mathbb{R}^3. \tag{4.4.7}$$

Finally, as in the incompressible case (cf. the identity (3.4.8)), we infer from the discrete induction equation (4.2.7) that the interpolations (4.4.2)–(4.4.4) of the magnetic induction $B^k_{\Delta t}$ satisfy the induction equation

$$\begin{aligned} -\int_0^T \int_\Omega \partial_t B_{\Delta t} \cdot b \, dxdt = \int_0^T \int_\Omega & \left[\frac{1}{\sigma\mu} \operatorname{curl} \overline{B}_{\Delta t} - \overline{\tilde{u}}'_{\Delta t} \times \overline{B}'_{\Delta t} + \frac{\epsilon}{\mu^2} \left|\operatorname{curl} \overline{B}_{\Delta t}\right|^2 \operatorname{curl} \overline{B}_{\Delta t} - \frac{1}{\sigma} \overline{J}_{\Delta t} \right] \cdot \operatorname{curl} b \\ & + \epsilon \left(\nabla \operatorname{curl} \overline{B}_{\Delta t}\right) \cdot (\nabla \operatorname{curl} b) \, dxdt \end{aligned} \tag{4.4.8}$$

for all test functions

$$b \in L^4\left(0,T; H_0^{2,2}(\Omega)\right) \quad \text{such that} \quad b(t) \in W^k\left(S_{\Delta t,k}\right) \quad \text{for a.a. } t \in [(k-1)\Delta t, k\Delta t], \; k = 1, ..., \frac{T}{\Delta t}.$$

4.4.1 Energy inequality on the Δt-level

We derive an energy inequality in order to obtain bounds, uniform in Δt, for the solution to the hybrid system on the Δt-level. As opposed to in the incompressible setting in Section 3.4.1 we have to combine the discrete induction equation (4.2.7) with the continuous Navier-Stokes equations (4.2.5), (4.2.6) in a suitable way in order to achieve this goal. We pick an arbitrary time $\tau \in (0,T]$ and choose $k \in \left\{1, ..., \frac{T}{\Delta t}\right\}$, $s \in [0, \Delta t)$ such that $\tau = k\Delta t - s$. For the magnetic part of the energy inequality we test the discrete induction equation (4.2.7) at any time $l\Delta t$, $l = 1, ..., k$, by $\frac{1}{\mu} B^l_{\Delta t}$. This leads to the estimate

$$\begin{aligned} & \frac{1}{2\mu} \partial_t B_{\Delta t, \|\cdot\|}(t) + \int_\Omega \frac{1}{\sigma\mu^2} \left|\operatorname{curl} B^l_{\Delta t}\right|^2 + \frac{\epsilon}{\mu^3} \left|\operatorname{curl} B^l_{\Delta t}\right|^4 + \frac{\epsilon}{\mu} \left|\nabla \operatorname{curl} B^l_{\Delta t}\right|^2 \, dx \\ = & \int_\Omega \frac{1}{2\mu\Delta t} \left|B^l_{\Delta t}\right|^2 - \frac{1}{2\mu\Delta t} \left|B^{l-1}_{\Delta t}\right|^2 + \frac{1}{\sigma\mu^2} \left|\operatorname{curl} B^l_{\Delta t}\right|^2 + \frac{\epsilon}{\mu^3} \left|\operatorname{curl} B^l_{\Delta t}\right|^4 + \frac{\epsilon}{\mu} \left|\nabla \operatorname{curl} B^l_{\Delta t}\right|^2 \, dx \\ \leqslant & \int_\Omega \frac{1}{\mu} \left(\tilde{u}^{l-1}_{\Delta t} \times B^{l-1}_{\Delta t}\right) \cdot \operatorname{curl} B^l_{\Delta t} + \frac{1}{\sigma\mu} J^l_{\Delta t} \cdot \operatorname{curl} B^l_{\Delta t} \, dx \qquad \forall t \in ((l-1)\Delta t, l\Delta t], \end{aligned}$$

cf. the corresponding inequality (3.4.24) in the incompressible setting. We integrate this estimate (discretely) over the interval $[0,\tau]$, which yields the inequality

$$\begin{aligned} & \frac{1}{2\mu} B_{\Delta t, \|\cdot\|}(\tau) + \Delta t \sum_{l=1}^{k-1} \int_\Omega \frac{1}{\sigma\mu^2} \left|\operatorname{curl} B^l_{\Delta t}\right|^2 + \frac{\epsilon}{\mu^3} \left|\operatorname{curl} B^l_{\Delta t}\right|^4 + \frac{\epsilon}{\mu} \left|\nabla \operatorname{curl} B^l_{\Delta t}\right|^2 \, dx \\ & + (\Delta t - s) \int_\Omega \frac{1}{\sigma\mu^2} \left|\operatorname{curl} B^k_{\Delta t}\right|^2 + \frac{\epsilon}{\mu^3} \left|\operatorname{curl} B^k_{\Delta t}\right|^4 + \frac{\epsilon}{\mu} \left|\nabla \operatorname{curl} B^k_{\Delta t}\right|^2 \, dx \\ \leqslant & \frac{1}{2\mu} B_{\Delta t, \|\cdot\|}(0) + \Delta t \sum_{l=1}^{k-1} \int_\Omega \frac{1}{\mu} \left(\tilde{u}^{l-1}_{\Delta t} \times B^{l-1}_{\Delta t}\right) \cdot \operatorname{curl} B^l_{\Delta t} + \frac{1}{\sigma\mu} J^l_{\Delta t} \cdot \operatorname{curl} B^l_{\Delta t} \, dx \\ & + (\Delta t - s) \int_\Omega \frac{1}{\mu} \left(\tilde{u}^{k-1}_{\Delta t} \times B^{k-1}_{\Delta t}\right) \cdot \operatorname{curl} B^k_{\Delta t} + \frac{1}{\sigma\mu} J^k_{\Delta t} \cdot \operatorname{curl} B^k_{\Delta t} \, dx \\ \leqslant & \frac{1}{2\mu} \int_\Omega \left|B_{0,\alpha}\right|^2 \, dx + \Delta t \sum_{l=1}^{k-1} \left[\frac{1}{\sqrt{\mu}\epsilon} \left\|B^{l-1}_{\Delta t}\right\|^2_{L^2(\Omega)} + \frac{\epsilon}{8} \left\|\tilde{u}^{l-1}_{\Delta t}\right\|^4_{L^4(\Omega)} + \frac{\epsilon}{8\mu^3} \left\|\operatorname{curl} B^l_{\Delta t}\right\|^4_{L^4(\Omega)} + \frac{1}{2\sigma} \left\|J^l_{\Delta t}\right\|^2_{L^2(\Omega)} \right. \\ & \left. + \frac{1}{2\sigma\mu^2} \left\|\operatorname{curl} B^l_{\Delta t}\right\|^2_{L^2(\Omega)} \right] + (\Delta t - s) \left[\frac{1}{\sqrt{\mu}\epsilon} \left\|B^{k-1}_{\Delta t}\right\|^2_{L^2(\Omega)} + \frac{\epsilon}{8} \left\|\tilde{u}^{k-1}_{\Delta t}\right\|^4_{L^4(\Omega)} + \frac{\epsilon}{8\mu^3} \left\|\operatorname{curl} B^k_{\Delta t}\right\|^4_{L^4(\Omega)} \right. \\ & \left. + \frac{1}{2\sigma} \left\|J^k_{\Delta t}\right\|^2_{L^2(\Omega)} + \frac{1}{2\sigma\mu^2} \left\|\operatorname{curl} B^k_{\Delta t}\right\|^2_{L^2(\Omega)} \right] \end{aligned} \tag{4.4.9}$$

under exploitation of Hölder's and Young's inequalities and in particular the estimate

$$\begin{aligned}\int_\Omega \frac{1}{\mu}\left(\tilde{u}_{\Delta t}^{l-1} \times B_{\Delta t}^{l-1}\right) \cdot \operatorname{curl} B_{\Delta t}^l \; dx &\leqslant \frac{1}{\mu}\left\|B_{\Delta t}^{l-1}\right\|_{L^2(\Omega)}\left\|\tilde{u}_{\Delta t}^{l-1}\right\|_{L^4(\Omega)}\left\|\operatorname{curl} B_{\Delta t}^l\right\|_{L^4(\Omega)} \\ &= \frac{\sqrt{2}}{\sqrt{\epsilon}\mu^{\frac14}}\left\|B_{\Delta t}^{l-1}\right\|_{L^2(\Omega)} \frac{\sqrt{\epsilon}}{\sqrt{2}\mu^{\frac34}}\left\|\tilde{u}_{\Delta t}^{l-1}\right\|_{L^4(\Omega)}\left\|\operatorname{curl} B_{\Delta t}^l\right\|_{L^4(\Omega)} \\ &\leqslant \frac{1}{\sqrt{\mu\epsilon}}\left\|B_{\Delta t}^{l-1}\right\|_{L^2(\Omega)}^2 + \frac{\sqrt{\epsilon}}{2}\left\|\tilde{u}_{\Delta t}^{l-1}\right\|_{L^4(\Omega)}^2 \frac{\sqrt{\epsilon}}{2\mu^{\frac32}}\left\|\operatorname{curl} B_{\Delta t}^l\right\|_{L^4(\Omega)}^2 \\ &\leqslant \frac{1}{\sqrt{\mu\epsilon}}\left\|B_{\Delta t}^{l-1}\right\|_{L^2(\Omega)}^2 + \frac{\epsilon}{8}\left\|\tilde{u}_{\Delta t}^{l-1}\right\|_{L^4(\Omega)}^4 + \frac{\epsilon}{8\mu^3}\left\|\operatorname{curl} B_{\Delta t}^l\right\|_{L^4(\Omega)}^4 .\end{aligned} \tag{4.4.10}$$

On the right-hand side of the inequality (4.4.9) we further estimate, due to the definition of $\tilde{u}_{\Delta t}^{l-1}$ in (4.2.20) and Jensen's inequality,

$$\left\|\tilde{u}_{\Delta t}^{l-1}\right\|_{L^4(\Omega)}^4 = \int_\Omega \left|\frac{1}{\Delta t}\int_{(l-2)\Delta t}^{(l-1)\Delta t} u_{\Delta t,l-1}(t,x)\; dt\right|^4 \; dx \leqslant \frac{1}{\Delta t}\int_\Omega \int_{(l-2)\Delta t}^{(l-1)\Delta t} |u_{\Delta t,l-1}(t)|^4 \; dtdx$$

for $l \geqslant 2$ and

$$\left\|\tilde{u}_{\Delta t}^{l-1}\right\|_{L^4(\Omega)}^4 = \int_\Omega |u_{0,\alpha}|^4 \; dx$$

for $l = 1$. Moreover, we calculate

$$\begin{aligned} &\Delta t \sum_{l=1}^{k-1} \frac{1}{\sqrt{\mu\epsilon}}\left\|B_{\Delta t}^{l-1}\right\|_{L^2(\Omega)}^2 + (\Delta t - s)\frac{1}{\sqrt{\mu\epsilon}}\left\|B_{\Delta t}^{k-1}\right\|_{L^2(\Omega)}^2 \\ \leqslant &\frac{2}{\sqrt{\mu\epsilon}}\sum_{l=1}^{k-1}\left(\frac{\Delta t}{2}\left\|B_{\Delta t}^{l}\right\|_{L^2(\Omega)}^2 + \frac{\Delta t}{2}\left\|B_{\Delta t}^{l-1}\right\|_{L^2(\Omega)}^2\right) \\ = &\frac{2}{\sqrt{\mu\epsilon}}\sum_{l=1}^{k-1}\left[\left(\frac{t^2}{2\Delta t} - (l-1)t\right)\left\|B_{\Delta t}^{l}\right\|_{L^2(\Omega)}^2 + \left(lt - \frac{t^2}{2\Delta t}\right)\left\|B_{\Delta t}^{l-1}\right\|_{L^2(\Omega)}^2\right]_{(l-1)\Delta t}^{l\Delta t} \\ = &\frac{2}{\sqrt{\mu\epsilon}}\sum_{l=1}^{k-1}\int_{(l-1)\Delta t}^{l\Delta t} B_{\Delta t,\|\cdot\|}(t)\; dt = \frac{2}{\sqrt{\mu\epsilon}}\int_0^{(k-1)\Delta t} B_{\Delta t,\|\cdot\|}(t)\; dt \leqslant \frac{2}{\sqrt{\mu\epsilon}}\int_0^\tau B_{\Delta t,\|\cdot\|}(t)\; dt. \end{aligned} \tag{4.4.11}$$

Hence, absorbing the terms depending on $\operatorname{curl} B_{\Delta t}^l$, $l = 1, ..., k$, in the inequality (4.4.9) into the left-hand side and expressing the sums as integrals, we end up with

$$\begin{aligned} &\frac{1}{2\mu}B_{\Delta t,\|\cdot\|}(\tau) + \int_0^\tau \int_\Omega \frac{1}{2\sigma\mu^2}\left|\operatorname{curl}\overline{B}_{\Delta t}\right|^2 + \frac{7\epsilon}{8\mu^3}\left|\operatorname{curl}\overline{B}_{\Delta t}\right|^4 + \frac{\epsilon}{\mu}\left|\nabla\operatorname{curl}\overline{B}_{\Delta t}\right|^2 \; dxdt \\ \leqslant &\frac{1}{2\mu}\int_\Omega |B_{0,\alpha}|^2 \; dx + \frac{\epsilon}{8}\Delta t \int_\Omega |u_{0,\alpha}|^4 \; dx + \int_0^\tau\int_\Omega \frac{1}{2\sigma}\left|\overline{J}_{\Delta t}\right|^2 + \frac{\epsilon}{8}|u_{\Delta t}|^4 \; dxdt + \frac{2}{\sqrt{\mu\epsilon}}\int_0^\tau B_{\Delta t,\|\cdot\|}(t)\; dt. \end{aligned} \tag{4.4.12}$$

For the mechanical part of the energy inequality we recall the identity (4.3.13), which $\rho_{\Delta t}$ and $u_{\Delta t}$ satisfy for the choice $s = 1$ and for all $t \in [0, T]$. Integrating this identity between 0 and τ we obtain the relation

$$\begin{aligned} &\int_\Omega \frac{1}{2}\rho_{\Delta t}(\tau)\left|u_{\Delta t}(\tau)\right|^2 + \frac{a}{\gamma-1}\rho_{\Delta t}^\gamma(\tau) + \frac{\alpha}{\beta-1}\rho_{\Delta t}^\beta(\tau)\; dx + \int_0^\tau\int_\Omega 2\nu\left(\overline{\chi}_{\Delta t}'\right)\left|\mathbb{D}\left(u_{\Delta t}\right)\right|^2 \\ &+ \lambda\left(\overline{\chi}_{\Delta t}'\right)\left|\operatorname{div} u_{\Delta t}\right|^2 + a\epsilon\gamma\rho_{\Delta t}^{\gamma-2}\left|\nabla\rho_{\Delta t}\right|^2 + \alpha\epsilon\beta\rho_{\Delta t}^{\beta-2}\left|\nabla\rho_{\Delta t}\right|^2 + \epsilon\left|u_{\Delta t}\right|^4 \; dxdt \\ = &\int_\Omega \frac{1}{2}\rho_{0,\alpha}\left|u_{0,\alpha}\right|^2 + \frac{a}{\gamma-1}\rho_{0,\alpha}^\gamma + \frac{\alpha}{\beta-1}\rho_{0,\alpha}^\beta\; dx + \int_0^\tau\int_\Omega \rho_{\Delta t} g\cdot u_{\Delta t} + \frac{1}{\mu}\left(\operatorname{curl}\overline{B}_{\Delta t}' \times \overline{B}_{\Delta t}'\right)\cdot u_{\Delta t}\; dxdt \\ \leqslant &\int_\Omega \frac{1}{2}\rho_{0,\alpha}\left|u_{0,\alpha}\right|^2 + \frac{a}{\gamma-1}\rho_{0,\alpha}^\gamma + \frac{\alpha}{\beta-1}\rho 0,\alpha 0^\beta\; dx + \int_0^\tau\int_\Omega \rho_{\Delta t} g\cdot u_{\Delta t} + \frac{\epsilon}{8}\left|u_{\Delta t}\right|^4 + \frac{\epsilon}{8\mu^3}\left|\operatorname{curl}\overline{B}_{\Delta t}'\right|^4 \; dxdt \\ &+ \frac{2}{\sqrt{\mu\epsilon}}\int_0^\tau B_{\Delta t,\|\cdot\|}(t)\; dt, \end{aligned} \tag{4.4.13}$$

where the last inequality uses the estimate

$$
\begin{aligned}
&\int_0^\tau \int_\Omega \frac{1}{\mu}\left(\operatorname{curl}\overline{B}'_{\Delta t} \times \overline{B}'_{\Delta t}\right) \cdot u_{\Delta t}\, dxdt \\
\leqslant &\int_0^\tau \frac{1}{\sqrt{\mu\epsilon}}\left\|\overline{B}'_{\Delta t}(t)\right\|^2_{L^2(\Omega)} + \frac{\epsilon}{8}\left\|u_{\Delta t}(t)\right\|^4_{L^4(\Omega)} + \frac{\epsilon}{8\mu^3}\left\|\operatorname{curl}\overline{B}'_{\Delta t}(t)\right\|^4_{L^4(\Omega)}\, dt \\
\leqslant &\frac{2}{\sqrt{\mu\epsilon}}\int_0^\tau B_{\Delta t,\|\cdot\|}(t)\, dt + \int_0^\tau \frac{\epsilon}{8}\left\|u_{\Delta t}(t)\right\|^4_{L^4(\Omega)} + \frac{\epsilon}{8\mu^3}\left\|\operatorname{curl}\overline{B}'_{\Delta t}(t)\right\|^4_{L^4(\Omega)}\, dt,
\end{aligned}
$$

which follows by the same estimates (4.4.10) and (4.4.11) as in the estimate of the corresponding term in the induction equation. Adding the inequality (4.4.13) to the inequality (4.4.12) and absorbing multiple terms from the right-hand side into the left-hand side, we finally get the energy inequality

$$
\begin{aligned}
&\int_\Omega \frac{1}{2}\rho_{\Delta t}(\tau)\left|u_{\Delta t}(\tau)\right|^2 + a\frac{\rho^\gamma_{\Delta t}(\tau)}{\gamma-1} + \frac{\alpha\rho^\beta_{\Delta t}(\tau)}{\beta-1}\, dx + \frac{1}{2\mu}B_{\Delta t,\|\cdot\|}(\tau) + \int_0^\tau\int_\Omega 2\nu\left(\overline{\chi}'_{\Delta t}\right)\left|\mathbb{D}\left(u_{\Delta t}\right)\right|^2 \\
&+\lambda\left(\overline{\chi}'_{\Delta t}\right)\left|\operatorname{div} u_{\Delta t}\right|^2 + a\epsilon\gamma\rho^{\gamma-2}_{\Delta t}\left|\nabla\rho_{\Delta t}\right|^2 + \alpha\epsilon\beta\rho^{\beta-2}_{\Delta t}\left|\nabla\rho_{\Delta t}\right|^2 + \frac{3\epsilon}{4}\left|u_{\Delta t}\right|^4 + \frac{1}{2\sigma\mu^2}\left|\operatorname{curl}\overline{B}_{\Delta t}\right|^2 \\
&+\frac{3\epsilon}{4\mu^3}\left|\operatorname{curl}\overline{B}_{\Delta t}\right|^4 + \frac{\epsilon}{\mu}\left|\nabla\operatorname{curl}\overline{B}_{\Delta t}\right|^2\, dxdt \\
\leqslant &\int_\Omega \frac{1}{2}\rho_{0,\alpha}\left|u_{0,\alpha}\right|^2 + \frac{a}{\gamma-1}\rho^\gamma_{0,\alpha} + \frac{\alpha}{\beta-1}\rho^\beta_{0,\alpha} + \frac{1}{2\mu}\left|B_0\right|^2 + \frac{\epsilon\Delta t}{8}\left|u_{0,\alpha}\right|^4 + \frac{\epsilon}{8\mu^3}\left|\operatorname{curl} B_{0,\alpha}\right|^4\, dx \\
&+\int_0^\tau\int_\Omega \frac{1}{2\sigma}\left|\overline{J}_{\Delta t}\right|^2 + \rho_{\Delta t} g\cdot u_{\Delta t}\, dxdt + \frac{4}{\sqrt{\mu\epsilon}}\int_0^\tau B_{\Delta t,\|\cdot\|}(t)\, dt \\
\leqslant &c + \int_0^\tau\int_\Omega \rho_{\Delta t} g\cdot u_{\Delta t}\, dxdt + \int_0^\tau B_{\Delta t,\|\cdot\|}(t)\, dt
\end{aligned} \tag{4.4.14}
$$

for all $\tau \in [0,T]$, where the constant $c = c\left(\rho_0, u_0, B_0, J, a, \sigma, \mu, \gamma, \beta, \alpha, \epsilon, T, \Omega\right) > 0$ is independent of Δt and τ. In particular, by use of the Gronwall Lemma and the estimates for the solution to the Neumann problem for the density, cf. Lemma A.6.1, we find a constant $c > 0$, independent of Δt, such that the following bounds hold true:

$$\left\|u_{\Delta t}\right\|_{C([0,T];V_n)} + \left\|\rho_{\Delta t}\right\|_{C([0,T];C^2(\overline{\Omega}))} + \left\|\partial_t\rho_{\Delta t}\right\|_{C(\overline{Q})} + (\alpha\epsilon)^{\frac{1}{2}}\left\|\rho^{\frac{\beta-2}{2}}_{\Delta t}\nabla\rho_{\Delta t}\right\|_{L^2(Q)} \leqslant c, \tag{4.4.15}$$

$$\left\|B_{\Delta t}\right\|_{L^\infty(0,T;L^2(\Omega))} + \left\|\overline{B}_{\Delta t}\right\|_{L^\infty(0,T;L^2(\Omega))} + \left\|\overline{B}'_{\Delta t}\right\|_{L^\infty(0,T;L^2(\Omega))} \leqslant c, \tag{4.4.16}$$

$$\left\|B_{\Delta t}\right\|_{L^2(0,T;H^1(\Omega))} + \left\|\overline{B}_{\Delta t}\right\|_{L^2(0,T;H^1(\Omega))} + \left\|\overline{B}'_{\Delta t}\right\|_{L^2(0,T;H^1(\Omega))} \leqslant c, \tag{4.4.17}$$

$$\epsilon^{\frac{1}{2}}\left\|\operatorname{curl} B_{\Delta t}\right\|_{L^2(0,T;H^1(\Omega))} + \epsilon^{\frac{1}{2}}\left\|\operatorname{curl}\overline{B}_{\Delta t}\right\|_{L^2(0,T;H^1(\Omega))} + \epsilon^{\frac{1}{2}}\left\|\operatorname{curl}\overline{B}'_{\Delta t}\right\|_{L^2(0,T;H^1(\Omega))} \leqslant c, \tag{4.4.18}$$

$$\epsilon^{\frac{1}{4}}\left\|\operatorname{curl} B_{\Delta t}\right\|_{L^4(Q)} + \epsilon^{\frac{1}{4}}\left\|\operatorname{curl}\overline{B}_{\Delta t}\right\|_{L^4(Q)} + \epsilon^{\frac{1}{4}}\left\|\operatorname{curl}\overline{B}'_{\Delta t}\right\|_{L^4(Q)} \leqslant c. \tag{4.4.19}$$

The bounds for the magnetic induction in $L^\infty(0,T;L^2(\Omega))$ in (4.4.16) follow from the choice $\tau = k\Delta t$, $k = 1, ..., \frac{T}{\Delta t}$ in the energy inequality (4.4.14), for which it holds $B_{\Delta t,\|\cdot\|}(\tau) = \|B^k_{\Delta t}\|^2_{L^2(\Omega)}$. Recalling the estimate (4.3.12) from the proof of the existence of the approximate velocity field, we also infer a bound of the time derivative of $\partial_t u_{\Delta t}$,

$$\left\|\partial_t u_{\Delta t}\right\|_{L^2(0,T;V_n)} \leqslant c, \tag{4.4.20}$$

where $c > 0$ is independent of Δt. Indeed, the function $u_{\Delta t}$ corresponds to the function u in the estimate (4.3.12) for the choice $w = u_{\Delta t}$. Consequently, the bound (4.4.20) follows by noting that the constant $c(n,w) > 0$ in (4.3.12) remains bounded for w bounded in $C([0,T];V_n)$, which is the case for $w = u_{\Delta t}$ according to the bound (4.4.15). The bounds (4.4.15)–(4.4.20) and the Aubin-Lions Lemma imply the existence of functions

$$0 \leqslant \rho \in \left\{\psi \in C\left([0,T]; H^1(\Omega)\right) \bigcap L^2\left(0,T; H^2(\Omega)\right) : \ \partial_t \psi \in L^2(Q), \ \nabla\psi \cdot \mathrm{n}|_{\partial\Omega} = 0\right\}, \tag{4.4.21}$$

$$u \in \left\{\phi \in C([0,T]; V_n) : \ \partial_t \phi \in L^2(0,T; V_n)\right\},$$

$$B \in \left\{b \in L^\infty\left(0,T; L^2(\Omega)\right) \bigcap L^2\left(0,T; H^1_{\mathrm{div}}(\Omega)\right) : \ \mathrm{curl}\, b \in L^2\left(0,T; H^1(\Omega)\right), \ b \cdot \mathrm{n}|_{\partial\Omega} = 0\right\} \tag{4.4.22}$$

such that, after the extraction of a subsequence,

$$\rho_{\Delta t} \rightharpoonup \rho \quad \text{in } L^2\left(0,T; H^2(\Omega)\right), \qquad \rho_{\Delta t} \to \rho \quad \text{in } C\left([0,T]; H^1(\Omega)\right), \tag{4.4.23}$$

$$\partial_t \rho_{\Delta t} \rightharpoonup \partial_t \rho \quad \text{in } L^2(Q), \qquad \rho_{\Delta t}^{\frac{\beta}{2}} \rightharpoonup \rho^{\frac{\beta}{2}} \quad \text{in } L^2\left(0,T; H^1(\Omega)\right), \tag{4.4.24}$$

$$u_{\Delta t} \to u \quad \text{in } C([0,T]; V_n), \qquad \partial_t u_{\Delta t} \rightharpoonup \partial_t u \quad \text{in } L^2(0,T; V_n), \tag{4.4.25}$$

$$B_{\Delta t}, \ \overline{B}_{\Delta t}, \ \overline{B}'_{\Delta t} \overset{*}{\rightharpoonup} B \quad \text{in } L^\infty\left(0,T; L^2(\Omega)\right), \qquad B_{\Delta t}, \ \overline{B}_{\Delta t}, \ \overline{B}'_{\Delta t} \rightharpoonup B \quad \text{in } L^2\left(0,T; H^1(\Omega)\right)$$

and

$$\mathrm{curl}\, B_{\Delta t}, \ \mathrm{curl}\, \overline{B}_{\Delta t}, \ \mathrm{curl}\, \overline{B}'_{\Delta t} \rightharpoonup B \qquad \text{in } L^2\left(0,T; H^1(\Omega)\right).$$

The fact that the weak limits of $B_{\Delta t}$, $\overline{B}_{\Delta t}$ and $\overline{B}'_{\Delta t}$ here coincide follows from Lemma A.3.1 in the appendix. Moreover, the boundary conditions of the limit functions ρ and B in (4.4.21) and (4.4.22) follow directly from the corresponding conditions on the Δt-level, cf. Proposition 4.3.1. Furthermore, the external force $\overline{J}_{\Delta t}$, discretized via (4.2.21), converges to its original time-dependent counterpart,

$$\overline{J}_{\Delta t} \to J \quad \text{in } L^p(Q) \qquad \forall 1 \leqslant p < \infty,$$

cf. Lemma A.3.2 (i) in the appendix. Finally, $\eta_{\Delta t}$, as the solution to the initial value problem (4.4.7), satisfies the conditions of Lemma A.4.3, which tells us that

$$\eta_{\Delta t} \to \eta \quad \text{in } C\left([0,T]; C_{\mathrm{loc}}\left(\mathbb{R}^3\right)\right), \tag{4.4.26}$$

$$\chi_{\Delta t} \to \chi := \mathbf{db}_{S(\cdot)}(\cdot) \quad \text{in } C\left([0,T]; C_{\mathrm{loc}}\left(\mathbb{R}^3\right)\right), \tag{4.4.27}$$

where $S(t) := (\eta(t,O))^\delta$ and η represents the solution to

$$\frac{d\eta(t,x)}{dt} = R_\delta[u](t, \eta(t,x)), \qquad \eta(0,x) = x$$

for all $x \in \mathbb{R}^3$, $t \in [0,T]$. In particular, as in the incompressible case (cf. the inclusions (3.4.66)), for any $\kappa > 0$ the uniform convergence (4.4.26) implies the existence of some value $\delta(\kappa) > 0$ such that

$$(S(t))_\kappa \subset \overline{\overline{S}}_{\Delta t}(t) \subset (S(t))^\kappa \qquad \forall t \in [0,T], \ \Delta t < \delta(\kappa). \tag{4.4.28}$$

4.4.2 Continuity equation

Due to the convergences (4.4.23)–(4.4.25) of the density and the velocity we can pass to the limit in the continuity equation (4.4.5) and infer that the limit functions ρ and u solve the initial value problem

$$\partial_t \rho + \nabla(\rho u) = \epsilon \Delta \rho \quad \text{a.e. in } Q, \qquad \rho(0,x) = \rho_{0,\alpha}(x) \quad \text{a.e. in } \Omega.$$

4.4.3 Induction equation

We first point out that, as in the incompressible case (cf. Section 3.4.4), the first inclusion in (4.4.28) shows that the magnetic induction B of the limit system is again curl-free in the solid domain,

$$\operatorname{curl} B = 0 \qquad \text{a.e. in } Q^s(S) \bigcap Q.$$

Next we show convergence of the quantity $\overline{\overline{u}}'_{\Delta t}$ from the mixed term in the discrete induction equation (4.4.8). We fix an arbitrary point $(t, x) \in Q$ and, for each sufficiently small $\Delta t > 0$, we choose $k_{\Delta t} \in \{2, ..., \frac{T}{\Delta t}\}$ such that $t \in [(k_{\Delta t} - 1)\Delta t, k_{\Delta t}\Delta t)$. It holds that

$$\begin{aligned}\left|\overline{\overline{u}}'_{\Delta t}(t, x) - u(t, x)\right| &= \left|\frac{1}{\Delta t}\int_{(k_{\Delta t}-2)\Delta t}^{(k_{\Delta t}-1)\Delta t} u_{\Delta t, k_{\Delta t}-1}(\tau, x) d\tau - u(t, x)\right| \\ &\leqslant \sup_{\tau \in [(k_{\Delta t}-2)\Delta t, (k_{\Delta t}-1)\Delta t]} |u_{\Delta t}(\tau - \Delta t, x) - u(t, x)| \to 0 \end{aligned} \tag{4.4.29}$$

due to the uniform convergence (4.4.25) of $u_{\Delta t}$. Moreover, the uniform bound of $u_{\Delta t}$ in (4.4.15) shows equiintegrability of $|\overline{\overline{u}}'_{\Delta t} - u_{\Delta t}|^p$ for any $1 \leqslant p < \infty$. This together with the pointwise convergence (4.4.29) gives us the conditions for the Vitali convergence theorem and we infer that

$$\overline{\overline{u}}'_{\Delta t} \to u \quad \text{in } L^p(Q) \quad \forall 1 \leqslant p < \infty. \tag{4.4.30}$$

Further, due to the uniform bound (4.4.19), we find a function $z \in L^{\frac{4}{3}}(Q)$ such that, possibly after the extraction of a suitable subsequence,

$$\epsilon \left|\operatorname{curl} \overline{B}_{\Delta t}\right|^2 \operatorname{curl} \overline{B}_{\Delta t} \rightharpoonup \epsilon z \quad \text{in } L^{\frac{4}{3}}(Q). \tag{4.4.31}$$

Since the quantity ϵz will vanish from the system in the limit passage with respect to $\epsilon \to 0$, there is no need to specify the form of the limit function z in (4.4.31). We now have all the necessary convergences for the limit passage in the induction equation at hand. As in the incompressible case (cf. the implication (3.4.76)), the inclusion (4.4.28) shows that any function $b \in \mathcal{Y}(S)$ constitutes an admissible test function in the discrete induction equation (4.4.8) for all sufficiently small $\Delta t > 0$. Therefore, passing to the limit with respect to $\Delta t \to 0$ under exploitation of the convergences (4.4.30) and (4.4.31), we infer the limit identity

$$\begin{aligned} &- \int_0^T \int_\Omega B \cdot \partial_t b \; dx dt - \int_\Omega B_{0,\alpha} \cdot b(0, x) \; dx \\ &= \int_0^T \int_\Omega \left[-\frac{1}{\sigma\mu} \operatorname{curl} B + u \times B - \frac{\epsilon}{\mu^2} z + \frac{1}{\sigma} J \right] \cdot \operatorname{curl} b - \epsilon \left(\nabla \operatorname{curl} B\right) \cdot \left(\nabla \operatorname{curl} b\right) \; dx dt \end{aligned}$$

for all $b \in \mathcal{Y}(S)$.

4.4.4 Momentum equation

In order to pass to the limit in the momentum equation, we need to show convergence of the piecewise constant Lorentz force. This is achieved by similar arguments as in the incompressible case, cf. Section 3.4.4: Indeed, the uniform bounds (4.4.16), (4.4.19) allow us to find functions $z_1,\ z_2 \in L^{\frac{4}{3}}(Q)$ and extract suitable subsequences such that

$$\operatorname{curl} \overline{B}'_{\Delta t} \times \overline{B}'_{\Delta t} \rightharpoonup z_1 \quad \text{in } L^{\frac{4}{3}}(Q), \qquad \operatorname{curl} \overline{B}_{\Delta t} \times \overline{B}'_{\Delta t} \rightharpoonup z_2 \quad \text{in } L^{\frac{4}{3}}(Q). \tag{4.4.32}$$

Since, according to (4.2.16), it holds that

$$|Q| = \left|Q^s(S) \bigcup Q^f(S)\right|,$$

it is sufficient to identify z_1 and z_2 in $Q^s(S)$ and $Q^f(S)$. In $Q^s(S)$ the desired identification follows under exploitation of the fact that the magnetic induction is curl-free in the solid domain, as was seen in the corresponding identity (3.4.74) in the incompressible case. In $Q^f(S)$ it can be shown by deducing a dual estimate for $\partial_t B_{\Delta t}$ from the discrete induction equation (4.4.8), which then, after an application of the discrete Aubin-Lions Lemma A.3.3 and Remark A.3.1 leads to strong convergence of the magnetic induction in a suitable dual space, cf. (3.4.78). Altogether we obtain

$$z_1 = z_2 = \operatorname{curl} B \times B \quad \text{a.e. in } Q. \tag{4.4.33}$$

We further exploit the uniform convergence (4.4.27) of the signed distance function together with the definition (4.2.19) of the variable viscosity coefficients as smooth functions of $\overline{\chi}'_{\Delta t}$ to infer that

$$\nu\left(\overline{\chi}'_{\Delta t}\right) \to \nu(\chi) \quad \text{in } C\left([0,T]; C_{\text{loc}}\left(\mathbb{R}^3\right)\right), \qquad \lambda\left(\overline{\chi}'_{\Delta t}\right) \to \lambda(\chi) \quad \text{in } C\left([0,T]; C_{\text{loc}}\left(\mathbb{R}^3\right)\right). \tag{4.4.34}$$

This, in combination with the convergence of the Lorentz force, cf. (4.4.32), (4.4.33), allows us to pass to the limit in the momentum equation (4.4.6) and infer the equation

$$\begin{aligned}
&\int_0^T \int_\Omega \partial_t(\rho u)\cdot \phi \, dxdt \\
&= \int_0^T \int_\Omega (\rho u \otimes u) : \mathbb{D}(\phi) + \left(a\rho^\gamma + \alpha\rho^\beta\right) \operatorname{div}\phi - 2\nu(\chi)\,\mathbb{D}(u) : \mathbb{D}(\phi) - \lambda(\chi) \operatorname{div} u \operatorname{div}\phi \\
&\quad + \rho g\cdot\phi + \frac{1}{\mu}(\operatorname{curl} B \times B)\cdot\phi - \epsilon |u|^2 u\cdot\phi - \epsilon(\nabla u \nabla\rho)\cdot\phi \, dxdt
\end{aligned}$$

for all $\phi \in C([0,T]; V_n)$.

4.4.5 Energy inequality

We choose an arbitrary time $\tau \in (0,T]$ and $k \in \{1, ..., \frac{T}{\Delta t}\}$, such that $\tau = k\Delta t - s$ for some $s \in [0, \Delta t)$. We evaluate the first inequality in (4.4.9) at the time $k\Delta t$ and add a zero of the form

$$\begin{aligned}
0 = &- s\int_\Omega \frac{1}{\mu}\left(\tilde{u}^{k-1}_{\Delta t} \times B^{k-1}_{\Delta t}\right)\cdot \operatorname{curl} B^k_{\Delta t} + \frac{1}{\sigma\mu} J^k_{\Delta t}\cdot \operatorname{curl} B^k_{\Delta t} \, dx \\
&+ s\int_\Omega \frac{1}{\mu}\left(\tilde{u}^{k-1}_{\Delta t} \times B^{k-1}_{\Delta t}\right)\cdot \operatorname{curl} B^k_{\Delta t} + \frac{1}{\sigma\mu} J^k_{\Delta t}\cdot \operatorname{curl} B^k_{\Delta t} \, dx \\
\leqslant &- s\int_\Omega \frac{1}{\mu}\left(\tilde{u}^{k-1}_{\Delta t} \times B^{k-1}_{\Delta t}\right)\cdot \operatorname{curl} B^k_{\Delta t} + \frac{1}{\sigma\mu} J^k_{\Delta t}\cdot \operatorname{curl} B^k_{\Delta t} \, dx + c\left[\Delta t + (\Delta t)^{\frac{1}{2}}\right],
\end{aligned}$$

producing the estimate

$$\begin{aligned}
&\int_\Omega \frac{1}{2\mu}\left|B^k_{\Delta t}\right|^2 dx + \Delta t \sum_{l=1}^{k-1} \int_\Omega \frac{1}{\sigma\mu^2}\left|\operatorname{curl} B^l_{\Delta t}\right|^2 + \frac{\epsilon}{\mu^3}\left|\operatorname{curl} B^l_{\Delta t}\right|^4 + \frac{\epsilon}{\mu}\left|\nabla \operatorname{curl} B^l_{\Delta t}\right|^2 dx \\
&+ (\Delta t - s)\int_\Omega \frac{1}{\sigma\mu^2}\left|\operatorname{curl} B^k_{\Delta t}\right|^2 + \frac{\epsilon}{\mu^3}\left|\operatorname{curl} B^k_{\Delta t}\right|^4 + \frac{\epsilon}{\mu}\left|\nabla \operatorname{curl} B^k_{\Delta t}\right|^2 dx \\
\leqslant &\int_\Omega \frac{1}{2\mu}|B_{0,\alpha}|^2 dx + \Delta t \sum_{l=1}^{k-1} \int_\Omega \frac{1}{\mu}\left(\tilde{u}^{l-1}_{\Delta t} \times B^{l-1}_{\Delta t}\right)\cdot \operatorname{curl} B^l_{\Delta t} + \frac{1}{\sigma\mu} J^l_{\Delta t}\cdot \operatorname{curl} B^l_{\Delta t} \, dx \\
&+ (\Delta t - s)\int_\Omega \frac{1}{\mu}\left(\tilde{u}^{k-1}_{\Delta t} \times B^{k-1}_{\Delta t}\right)\cdot \operatorname{curl} B^k_{\Delta t} + \frac{1}{\sigma\mu} J^k_{\Delta t}\cdot \operatorname{curl} B^k_{\Delta t} \, dx + c\left[\Delta t + (\Delta t)^{\frac{1}{2}}\right].
\end{aligned}$$

Subsequently, we add the first identity in (4.4.13) at the time τ to obtain

$$\begin{aligned}
&\int_\Omega \frac{1}{2}\rho_{\Delta t}(\tau)\,|u_{\Delta t}(\tau)|^2 + \frac{a}{\gamma-1}\rho^\gamma_{\Delta t}(\tau) + \frac{\alpha}{\beta-1}\rho^\beta_{\Delta t}(\tau) + \frac{1}{2\mu}\left|\overline{B}_{\Delta t}(\tau)\right|^2 dx \\
&+ \int_0^\tau \int_\Omega 2\nu\left(\overline{\chi}'_{\Delta t}\right)|\mathbb{D}(u_{\Delta t})|^2 + \lambda\left(\overline{\chi}'_{\Delta t}\right)|\operatorname{div} u_{\Delta t}|^2 + a\epsilon\gamma\rho^{\gamma-2}_{\Delta t}|\nabla\rho_{\Delta t}|^2 + \alpha\epsilon\beta\rho^{\beta-2}_{\Delta t}|\nabla\rho_{\Delta t}|^2 + \epsilon|u_{\Delta t}|^4 \\
&+ \frac{1}{\sigma\mu^2}\left|\operatorname{curl}\overline{B}_{\Delta t}\right|^2 + \frac{\epsilon}{\mu^3}\left|\operatorname{curl}\overline{B}_{\Delta t}\right|^4 + \frac{\epsilon}{\mu}\left|\nabla \operatorname{curl}\overline{B}_{\Delta t}\right|^2 dxdt \\
\leqslant &\int_\Omega \frac{1}{2}\rho_{0,\alpha}|u_{0,\alpha}|^2 + \frac{a}{\gamma-1}\rho^\gamma_{0,\alpha} + \frac{\alpha}{\beta-1}\rho^\beta_{0,\alpha} + \frac{1}{2\mu}|B_{0,\alpha}|^2 dx + \int_0^\tau \int_\Omega \rho_{\Delta t} g\cdot u_{\Delta t} \\
&+ \frac{1}{\mu}\left(\operatorname{curl}\overline{B}'_{\Delta t} \times \overline{B}'_{\Delta t}\right)\cdot u_{\Delta t} + \frac{1}{\mu}\left(\overline{\tilde{u}}'_{\Delta t} \times \overline{B}'_{\Delta t}\right)\cdot \operatorname{curl}\overline{B}_{\Delta t} + \frac{1}{\sigma\mu}\overline{J}_{\Delta t}\cdot \operatorname{curl}\overline{B}_{\Delta t} \, dxdt + c\left[\Delta t + (\Delta t)^{\frac{1}{2}}\right].
\end{aligned}$$

Here, on the left-hand side we drop the term $\rho_{\Delta t}^{\gamma-2} |\nabla \rho_{\Delta t}|^2$. Then we pass to the limit under exploitation of the weak lower semicontinuity of norms as well as in particular the convergence (4.4.30) of $\overline{\overline{u}}'_{\Delta t}$, the convergence (4.4.32), (4.4.33) of the Lorentz force and the convergence (4.4.34) of the variable viscosity coefficients. Altogether we obtain

$$\int_\Omega \frac{1}{2}\rho(\tau) |u(\tau)|^2 + \frac{a}{\gamma-1}\rho^\gamma(\tau) + \frac{\alpha}{\beta-1}\rho^\beta(\tau) + \frac{1}{2\mu} |B(\tau)|^2 \; dx + \int_0^\tau \int_\Omega 2\nu(\chi) |\mathbb{D}(u)|^2 + \lambda(\chi) |\operatorname{div} u|^2$$
$$+ \alpha\epsilon\beta\rho^{\beta-2} |\nabla\rho|^2 + \epsilon |u|^4 + \frac{1}{\sigma\mu^2} |\operatorname{curl} B|^2 + \frac{\epsilon}{\mu} |\nabla \operatorname{curl} B|^2 + \frac{\epsilon}{\mu^3} |z|^4 \; dxdt$$
$$\leqslant \int_\Omega \frac{1}{2}\rho_{0,\alpha} |u_{0,\alpha}|^2 + \frac{a}{\gamma-1}\rho_{0,\alpha}^\gamma + \frac{\alpha}{\beta-1}\rho_{0,\alpha}^\beta + \frac{1}{2\mu} |B_{0,\alpha}|^2 \; dx + \int_0^\tau \int_\Omega \rho g \cdot u + \frac{1}{\sigma\mu} J \cdot \operatorname{curl} B \; dxdt$$

for almost all $\tau \in [0,T]$. Hence we have proved the following result.

Proposition 4.4.1. *Let all the assumptions of Theorem 4.1.1 be satisfied, let* $n, m \in \mathbb{N}$, $\epsilon, \alpha > 0$, *let* $\beta > \max\{4,\gamma\}$ *be sufficiently large and let* $\delta > 0$ *be as in* (4.2.12)*. Assume in addition the regularized initial data* $\rho_{0,\alpha}$, $u_{0,\alpha}$, $B_{0,\alpha}$ *to satisfy the conditions* (4.2.22), (4.2.23)*. Then, there exists a function* $\eta_n : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ *and*

$$0 \leqslant \rho_n \in \left\{\psi \in C\left([0,T]; H^1(\Omega)\right) \bigcap L^2\left(0,T; H^2(\Omega)\right) : \; \partial_t\psi \in L^2(Q), \; \nabla\psi \cdot \mathrm{n}|_{\partial\Omega} = 0\right\}, \tag{4.4.35}$$
$$u_n \in \left\{\phi \in C([0,T]; V_n) : \; \partial_t \phi \in L^2(0,T; V_n)\right\},$$
$$B_n \in \Big\{b \in L^\infty\left(0,T; L^2(\Omega)\right) \bigcap L^2\left(0,T; H^1_{\operatorname{div}}(\Omega)\right) : \; \operatorname{curl} b \in L^2\left(0,T; H^1(\Omega)\right),$$
$$\operatorname{curl} b = 0 \;\; in \; Q^s(S_n), \;\; b \cdot \mathrm{n}|_{\partial\Omega} = 0\Big\}, \tag{4.4.36}$$
$$z_n \in L^{\frac{4}{3}}(Q)$$

for $S_n = S_n(\cdot) = (\eta_n(\cdot, O))^\delta$, *such that*

$$\frac{d\eta_n(t,x)}{dt} = R_\delta[u_n](t, \eta_n(t,x)), \qquad \eta_n(0,x) = x,$$
$$\partial_t\rho_n + \operatorname{div}(\rho_n u_n) = \epsilon\Delta\rho_n \quad a.e. \; in \; Q, \tag{4.4.37}$$
$$\int_0^T \int_\Omega \partial_t(\rho_n u_n) \cdot \phi \; dxdt = \int_0^T \int_\Omega (\rho_n u_n \otimes u_n) : \mathbb{D}(\phi) + \left(a\rho_n^\gamma + \alpha\rho_n^\beta\right) \operatorname{div}\phi$$
$$- 2\nu(\chi_n)\mathbb{D}(u_n) : \mathbb{D}(\phi) - \lambda(\chi_n) \operatorname{div} u_n \operatorname{div}\phi + \rho_n g \cdot \phi$$
$$+ \frac{1}{\mu}(\operatorname{curl} B_n \times B_n) \cdot \phi - \epsilon |u_n|^2 u_n \cdot \phi$$
$$- \epsilon(\nabla u_n \nabla\rho_n) \cdot \phi \; dxdt, \tag{4.4.38}$$
$$-\int_0^T \int_\Omega B_n \cdot \partial_t b \; dxdt - \int_\Omega B_{0,\alpha} \cdot b(0,x) \; dx = \int_0^T \int_\Omega \left[-\frac{1}{\sigma\mu}\operatorname{curl} B_n + u_n \times B_n + \frac{1}{\sigma} J - \frac{\epsilon}{\mu^2} z_n\right] \cdot \operatorname{curl} b$$
$$- \epsilon(\nabla \operatorname{curl} B_n) : (\nabla \operatorname{curl} b) \; dxdt,$$

where $\chi_n(t,x) := \boldsymbol{db}_{S_n(t)}(x)$, *for all* $\phi \in C([0,T]; V_n)$ *and all* $b \in \mathcal{Y}(S_n)$*. Moreover, these functions satisfy the initial conditions*

$$\rho_n(0) = \rho_{0,\alpha}, \qquad u_n(0) = u_{0,\alpha}$$

as well as the energy inequality

$$\int_\Omega \frac{1}{2}\rho_n(\tau) |u_n(\tau)|^2 + \frac{a}{\gamma-1}\rho_n^\gamma(\tau) + \frac{\alpha}{\beta-1}\rho_n^\beta(\tau) + \frac{1}{2\mu} |B_n(\tau)|^2 \; dx + \int_0^\tau \int_\Omega 2\nu(\chi_n) |\mathbb{D}(u_n)|^2$$
$$+ \lambda(\chi_n) |\operatorname{div} u_n|^2 + \alpha\epsilon\beta\rho_n^{\beta-2} |\nabla\rho_n|^2 + \epsilon |u_n|^4 + \frac{1}{\sigma\mu^2} |\operatorname{curl} B_n|^2 + \frac{\epsilon}{\mu} |\nabla \operatorname{curl} B_n|^2 + \frac{\epsilon}{\mu^3} |z_n|^{\frac{4}{3}} \; dxdt$$
$$\leqslant \int_\Omega \frac{1}{2}\rho_{0,\alpha} |u_{0,\alpha}|^2 + \frac{a}{\gamma-1}\rho_{0,\alpha}^\gamma + \frac{\alpha}{\beta-1}\rho_{0,\alpha}^\beta + \frac{1}{2\mu} |B_{0,\alpha}|^2 \; dx + \int_0^\tau \int_\Omega \rho_n g \cdot u_n + \frac{1}{\sigma\mu} J \cdot \operatorname{curl} B_n \; dxdt \tag{4.4.39}$$

for almost all $\tau \in [0,T]$.

4.5 Limit passage with respect to $n \to \infty$

Next, we let n tend to infinity, i.e. we pass to the limit in the Galerkin method. Applying the Gronwall Lemma to the energy inequality (4.4.39) we find a constant $c > 0$, independent of n, such that

$$\|\sqrt{\rho_n} u_n\|_{L^\infty(0,T;L^2(\Omega))} + \|\rho_n\|_{L^\infty(0,T;L^\beta(\Omega))} + \|u_n\|_{L^2(0,T;H^1(\Omega))} \leqslant c, \tag{4.5.1}$$

$$\|B_n\|_{L^\infty(0,T;L^2(\Omega))} + \|B_n\|_{L^2(0,T;H^1(\Omega))} \leqslant c, \tag{4.5.2}$$

$$(\alpha\epsilon)^{\frac{1}{2}} \left\| \rho_n^{\frac{\beta-2}{2}} \nabla \rho_n \right\|_{L^2(Q)} + \epsilon^{\frac{1}{4}} \|u_n\|_{L^4(Q)} + \epsilon^{\frac{1}{2}} \|\nabla \operatorname{curl} B_n\|_{L^2(0,T;H^1(\Omega))} + \epsilon^{\frac{3}{4}} \|z_n\|_{L^{\frac{4}{3}}(Q)} \leqslant c. \tag{4.5.3}$$

Using Lebesgue interpolation, we further infer from these bounds the existence of a constant $c(\epsilon, \alpha) > 0$, independent of n, such that

$$\begin{aligned} \|\rho_n\|_{L^{\frac{5}{3}\beta}(Q)} &\leqslant \|\rho_n\|^{\frac{2}{5}}_{L^\infty(0,T;L^\beta(\Omega))} \|\rho_n\|^{\frac{3}{5}}_{L^\beta(0,T;L^{3\beta}(\Omega))} \\ &\leqslant \|\rho_n\|^{\frac{2}{5}}_{L^\infty(0,T;L^\beta(\Omega))} \left\| \rho_n^{\frac{\beta}{2}} \right\|^{\frac{6}{5\beta}}_{L^2(0,T;H^1(\Omega))} \leqslant c(\epsilon, \alpha). \end{aligned} \tag{4.5.4}$$

Moreover, from the classical L^p-L^q regularity results for parabolic equations, cf. [94, Lemma 7.37, Lemma 7.38, Section 7.8.2], we infer that ρ_n as the solution to the regularized continuity equation (4.4.37) satisfies the estimates

$$\epsilon \|\nabla \rho_n\|_{L^r(Q)} + \epsilon \|\partial_t \rho_n\|_{L^{\tilde r}(Q)} + \epsilon^2 \left\|\nabla^2 \rho_n\right\|_{L^{\tilde r}(Q)} \leqslant c \tag{4.5.5}$$

for

$$r := \frac{10\beta - 6}{3\beta + 3} > 2, \quad \tilde r := \frac{5\beta - 3}{4\beta} > 1 \qquad \forall \beta > 6$$

and a constant $c > 0$ independent of n, m and ϵ. The uniform bounds (4.5.1)–(4.5.5), together with the Aubin-Lions Lemma, allow us to extract suitable subsequences and find functions $z \in L^{\frac{4}{3}}(Q)$ and

$$\begin{aligned} 0 \leqslant \rho \in &\Big\{ \psi \in L^\infty\left(0,T;L^\beta(\Omega)\right) \bigcap L^r\left(0,T;W^{1,r}(\Omega)\right) \bigcap L^{\tilde r}\left(0,T;W^{2,\tilde r}(\Omega)\right) : \\ &\quad \partial_t \psi \in L^{\tilde r}(Q), \ \nabla\psi \cdot \mathrm{n}|_{\partial\Omega} = 0 \Big\} \subset C\left([0,T];L^2(\Omega)\right), \end{aligned} \tag{4.5.6}$$

$$u \in L^2(0,T;H_0^1(\Omega)), \tag{4.5.7}$$

$$B \in \Big\{ b \in L^\infty\left(0,T;L^2(\Omega)\right) \bigcap L^2\left(0,T;H^1_{\mathrm{div}}(\Omega)\right) : \ \operatorname{curl} b \in L^2\left(0,T;H^1(\Omega)\right), \ b \cdot \mathrm{n}|_{\partial\Omega} = 0 \Big\} \tag{4.5.8}$$

such that

$$\begin{aligned} \rho_n &\to \rho \quad \text{in } L^\beta(Q), & \rho_n &\to \rho \quad \text{in } L^2\left(0,T;H^1(\Omega)\right) & (4.5.9)\\ \rho_n &\rightharpoonup \rho \quad \text{in } L^{\tilde r}\left(0,T;W^{2,\tilde r}(\Omega)\right), & \partial_t\rho_n &\rightharpoonup \partial_t\rho \quad \text{in } L^{\tilde r}(Q), & (4.5.10)\\ u_n &\rightharpoonup u \quad \text{in } L^2(0,T;H^1(\Omega)), & B_n &\overset{*}{\rightharpoonup} B \quad \text{in } L^\infty(0,T;L^2(\Omega)), & (4.5.11)\\ B_n &\rightharpoonup B \quad \text{in } L^2(0,T;H^1(\Omega)), & \operatorname{curl} B_n &\rightharpoonup \operatorname{curl} B \quad \text{in } L^2\left(0,T;H^1(\Omega)\right), & \\ z_n &\rightharpoonup z \quad \text{in } L^{\frac{4}{3}}(Q). \end{aligned}$$

The boundary conditions of the limit functions in (4.5.6)–(4.5.8) follow directly from the corresponding boundary conditions (4.4.35) and (4.4.36) of $\nabla\rho_n$ and B_n and the fact that $u_n \in V_n$ vanishes on $\partial\Omega$ for all $n \in \mathbb{N}$. Further, the initial value problem (4.4.7), solved by η_n, yields that the conditions of Lemma A.4.3 are satisfied. Hence

$$\eta_n \to \eta \quad \text{in } C\left([0,T];C_{\mathrm{loc}}\left(\mathbb{R}^3\right)\right), \tag{4.5.12}$$

$$\chi_n \to \chi := \mathbf{db}_{S(\cdot)}(\cdot) \quad \text{in } C\left([0,T];C_{\mathrm{loc}}\left(\mathbb{R}^3\right)\right), \tag{4.5.13}$$

where $S(t) := (\eta(t, O))^{\delta}$ and η denotes the unique solution to the initial value problem

$$\frac{d\eta(t,x)}{dt} = R_\delta \left[u\right](t, \eta(t,x)), \qquad \eta(0,x) = x$$

for all $x \in \mathbb{R}^3$ and almost all $t \in [0,T]$. Finally, for any $\kappa > 0$, the uniform convergence (4.5.12) implies the existence of some number $N(\kappa) \in \mathbb{N}$ such that

$$(S(t))_\kappa \subset S_n(t) \subset (S(t))^{\frac{\kappa}{2}} \subset (S(t))^{\kappa} \qquad \forall t \in [0,T], \; n > N(\kappa).$$

4.5.1 Continuity equation

The convergences (4.5.9)–(4.5.11) of ρ_n and u_n allow us to pass to the limit in the continuity equation (4.4.37). Consequently, the limit functions ρ and u satisfy the continuity equation

$$\partial_t \rho + \operatorname{div}(\rho u) = \epsilon \Delta \rho \quad \text{a.e. in } Q, \qquad \rho(0) = \rho_{0,\alpha}.$$

4.5.2 Induction equation

At this stage - as well as in the later sections - the limit passage in the induction equation does not differ from the incompressible case in Section 3.5.3. Thus we waive a repetition of the arguments therein and only present the final results. As in the identity (3.5.20) from the incompressible case it holds that the magnetic induction of the limit system is curl-free in the solid region,

$$\operatorname{curl} B = 0 \qquad \text{a.e. in } Q^s(S) \bigcap Q.$$

As in the corresponding relations (3.5.23), (3.5.24) and (3.5.28) in the incompressible case we find functions $z_3, z_4 \in L^{\frac{6}{5}}(Q)$ such that, for a suitable subsequence, it holds

$$\operatorname{curl} B_n \times B_n \rightharpoonup z_3 \quad \text{in } L^{\frac{6}{5}}(Q), \qquad u_n \times B_n \rightharpoonup z_4 \quad \text{in } L^{\frac{6}{5}}(Q) \tag{4.5.14}$$

and

$$z_3 = \operatorname{curl} B \times B, \quad z_4 \cdot \operatorname{curl} b = u \times B \cdot \operatorname{curl} b \qquad \text{a.e. in } Q \tag{4.5.15}$$

for any $b \in \mathcal{Y}(S)$. In particular, we may pass to the limit in the induction equation to observe that

$$\begin{aligned} &- \int_0^T \int_\Omega B \cdot \partial_t b \; dxdt - \int_\Omega B_{0,\alpha} \cdot b(0,x) \; dx \\ =& \int_0^T \int_\Omega \left[-\frac{1}{\sigma\mu} \operatorname{curl} B + u \times B + \frac{1}{\sigma} J - \frac{\epsilon}{\mu^2} z \right] \cdot \operatorname{curl} b - \epsilon \left(\nabla \operatorname{curl} B\right) \cdot \left(\nabla \operatorname{curl} b\right) \; dxdt \end{aligned}$$

for any $b \in \mathcal{Y}(S)$.

4.5.3 Momentum equation

By the same methods as for the (purely mechanical) compressible Navier-Stokes system, cf. [94, Section 7.8.2], we derive convergence of the function $\rho_n u_n \otimes u_n$: First of all, we estimate

$$\begin{aligned} &\|\rho_n u_n \otimes u_n\|_{L^2(0,T;L^{\frac{6\beta}{4\beta+3}}(\Omega))} \\ \leqslant& \|\rho_n u_n\|_{L^\infty(0,T;L^{\frac{2\beta}{\beta+1}}(\Omega))} \|u_n\|_{L^2(0,T;L^6(\Omega))} \\ \leqslant& c \|\sqrt{\rho_n}\|_{L^\infty(0,T;L^{2\beta}(\Omega))} \|\sqrt{\rho_n} u_n\|_{L^\infty(0,T;L^2(\Omega))} \|u_n\|_{L^2(0,T;H^1(\Omega))} \leqslant c \end{aligned}$$

for a constant $c > 0$, independent of n according to the uniform bounds (4.5.1). Hence, for a subsequence and some function $z_5 \in L^2(0,T;L^{\frac{6\beta}{4\beta+3}}(\Omega))$ we may assume that

$$\rho_n u_n \otimes u_n \rightharpoonup z_5 \qquad \text{in } L^2\left(0,T;L^{\frac{6\beta}{4\beta+3}}(\Omega)\right). \tag{4.5.16}$$

Then, in order to identify the limit function z_5, we denote by P_n the orthogonal projection of $L^2(\Omega)$ onto the Galerkin space V_n and test the momentum equation (4.4.38) — after a density argument — by $P_n(\phi)$ for an arbitrary function $\phi \in L^{\frac{5\beta-3}{\beta-3}}(0,T;H_0^2(\Omega))$. Doing so, we estimate

$$
\begin{aligned}
&\left|\int_0^T\int_\Omega \partial_t P_n(\rho_n u_n)\cdot\phi\,dxdt\right| = \left|\int_0^T\int_\Omega \partial_t(\rho_n u_n)\cdot P_n(\phi)\,\,dxdt\right| \\
=&\Bigg|\int_0^T\int_\Omega (\rho_n u_n\otimes u_n):\mathbb{D}(P_n(\phi)) + \left(a\rho_n^\gamma+\alpha\rho_n^\beta\right)\operatorname{div}P_n(\phi) - 2\nu(\chi_n)\mathbb{D}(u_n):\mathbb{D}(P_n(\phi)) \\
&-\lambda(\chi_n)\operatorname{div}u_n\operatorname{div}P_n(\phi) + \rho_n g\cdot P_n(\phi) + \frac{1}{\mu}(\mathrm{curl}B_n\times B_n)\cdot P_n(\phi) - \epsilon|u_n|^2u_n\cdot P_n(\phi) \\
&-\epsilon(\nabla u_n\nabla\rho_n)\cdot P_n(\phi)\,\,dxdt\Bigg| \\
\leqslant& \|\rho_n\|_{L^4(Q)}\|u_n\|_{L^4(Q)}^2\|\mathbb{D}(P_n(\phi))\|_{L^4(Q)} + a\|\rho_n\|_{L^{\frac{5}{3}\gamma}(Q)}^\gamma\|\operatorname{div}P_n(\phi)\|_{L^{\frac{5}{2}}(Q)} \\
&+\alpha\|\rho_n\|_{L^{\frac{5}{3}\beta}(Q)}^\beta\|\operatorname{div}P_n(\phi)\|_{L^{\frac{5}{2}}(Q)} + \|2\nu(\chi_n)\|_{L^\infty(Q)}\|\mathbb{D}(u_n)\|_{L^2(Q)}\|\mathbb{D}(P_n(\phi))\|_{L^2(Q)} \\
&+\|\lambda(\chi_n)\|_{L^\infty(Q)}\|\operatorname{div}u_n\|_{L^2(Q)}\|\operatorname{div}P_n(\phi)\|_{L^2(Q)} + \|g\|_{L^\infty(Q)}\|\rho_n\|_{L^2(Q)}\|P_n(\phi)\|_{L^2(Q)} \\
&+\frac{1}{\mu}\|\mathrm{curl}\,B_n\|_{L^2(0,T;L^6(\Omega))}\|B_n\|_{L^\infty(0,T;L^2(\Omega))}\|P_n(\phi)\|_{L^2(0,T;L^3(\Omega))} \\
&+\epsilon\|u_n\|_{L^4(Q)}^3\|P_n(\phi)\|_{L^4(Q)} + \epsilon\|\nabla u_n\|_{L^2(Q)}\|\nabla\rho_n\|_{L^{\frac{10\beta-6}{3\beta+3}}(Q)}\|P_n(\phi)\|_{L^{\frac{5\beta-3}{\beta-3}}(Q)} \\
\leqslant& c\|P_n(\phi)\|_{L^{\frac{5\beta-3}{\beta-3}}(0,T;H^2(\Omega))} \leqslant c\|\phi\|_{L^{\frac{5\beta-3}{\beta-3}}(0,T;H^2(\Omega))}
\end{aligned}
$$

for a constant $c > 0$ independent of n due to the uniform bounds (4.5.1)–(4.5.5). This yields the dual estimate

$$
\|\partial_t P_n(\rho_n u_n)\|_{L^{\tilde{r}}(0,T;H^{-2}(\Omega))} = \|\partial_t P_n(\rho_n u_n)\|_{L^{\frac{5\beta-3}{4\beta}}(0,T;H^{-2}(\Omega))} \leqslant c.
$$

Consequently, from the Aubin-Lions Lemma, we conclude that

$$
P_n(\rho_n u_n) \to \rho u \quad \text{in } L^2\left(0,T;H^{-1}(\Omega)\right).
$$

Writing

$$
\|\rho_n u_n - \rho u\|_{L^2(0,T;H^{-1}(\Omega))} \leqslant \|\rho_n u_n - P_n(\rho_n u_n)\|_{L^2(0,T;H^{-1}(\Omega))} + \|P_n(\rho_n u_n) - \rho u\|_{L^2(0,T;H^{-1}(\Omega))}
$$

and realizing that the first term on the right-hand side vanishes for $n \to \infty$ due to the general properties of the projection P_n (cf. [94, Section 7.4.3]), we thus infer that

$$
\rho_n u_n \to \rho u \quad \text{in } L^2\left(0,T;H^{-1}(\Omega)\right).
$$

Therefore, as u_n converges weakly in $L^2(0,T;H_0^1(\Omega))$, we infer that the limit function z_5 of the convergence (4.5.16) is given by

$$
z_5 = \rho u \otimes u \qquad \text{a.e. in } Q.
$$

Next, we notice that the bound (4.5.3) of u_n in $L^4(Q)$ implies the existence of some $\tilde{z} \in L^{\frac{4}{3}}(Q)$ such that, for a chosen subsequence,

$$
\epsilon|u_n|^2u_n \rightharpoonup \epsilon\tilde{z} \quad \text{in } L^{\frac{4}{3}}(Q).
$$

Using further the uniform convergence (4.5.13) of the signed distance function for passing to the limit in the variable viscosity coefficients and the convergence (4.5.14), (4.5.15) of the Lorentz force, we can now let n tend to infinity in the momentum equation (4.4.38). We infer that

$$
\begin{aligned}
&-\int_0^T\int_\Omega \rho u\cdot\partial_t\phi\; dxdt-\int_\Omega(\rho u)_{0,\alpha}\cdot\phi(0,x)\; dx\\
&=\int_0^T\int_\Omega(\rho u\otimes u):\mathbb{D}(\phi)+\left(a\rho^\gamma+\alpha\rho^\beta\right)\operatorname{div}\phi-2\nu(\chi)\,\mathbb{D}(u):\mathbb{D}(\phi)-\lambda(\chi)\operatorname{div}u\operatorname{div}\phi+\rho g\cdot\phi\\
&\quad+\frac{1}{\mu}(\operatorname{curl}B\times B)\cdot\phi-\epsilon\tilde{z}\cdot\phi-\epsilon(\nabla u\nabla\rho)\cdot\phi\; dxdt \qquad (4.5.17)
\end{aligned}
$$

for any $\phi\in C_c^1([0,T);V_{n_0})$ with fixed $n_0\in\mathbb{N}$. Since $\bigcup_{n=1}^\infty V_n$ is dense in $H_0^{1,2}(\Omega)$, we finally conclude that (4.5.17) also holds true for any $\phi\in\mathcal{D}([0,T)\times\Omega)$. Exploiting further the weak lower semicontinuity of norms to pass to the limit in both the energy inequality (4.4.39) and the uniform bounds (4.5.5) we have proved the following proposition:

Proposition 4.5.1. *Let all the assumptions of Theorem 4.1.1 be satisfied, let $m\in\mathbb{N}$, $\epsilon,\alpha>0$, let $\beta>\max\{4,\gamma\}$ be sufficiently large and let $\delta>0$ be as in (4.2.12). Assume in addition that*

$$
\begin{aligned}
&\rho_{0,\alpha}\in C^{2,\xi}\left(\overline{\Omega}\right), \qquad (\rho u)_{0,\alpha}\in C^2\left(\overline{\Omega}\right), \qquad B_{0,\alpha}\in H^2_{\operatorname{div}}(\Omega), && (4.5.18)\\
&0<\alpha\leqslant\rho_{0,\alpha}\leqslant\alpha^{-\frac{1}{2\beta}}, \qquad \nabla\rho_{0,\alpha}\cdot \mathrm{n}|_{\partial\Omega}=0, \qquad B_{0,\alpha}\cdot\mathrm{n}|_{\partial\Omega}=0. && (4.5.19)
\end{aligned}
$$

Then, there exists a function $\eta_m:[0,T]\times\mathbb{R}^3\to\mathbb{R}^3$ and

$$
\begin{aligned}
0\leqslant\rho_m\in&\Big\{\psi\in L^\infty\left(0,T;L^\beta(\Omega)\right)\bigcap L^r\left(0,T;W^{1,r}(\Omega)\right)\bigcap L^{\tilde{r}}\left(0,T;W^{2,\tilde{r}}(\Omega)\right):\\
&\quad\partial_t\psi\in L^{\tilde{r}}(Q),\;\;\nabla\psi\cdot\mathrm{n}|_{\partial\Omega}=0\Big\},\\
u_m\in&L^2\left(0,T;H_0^1(\Omega)\right),\\
B_m\in&\Big\{b\in L^\infty\left(0,T;L^2(\Omega)\right)\bigcap L^2\left(0,T;H^1_{\operatorname{div}}(\Omega)\right):\;\operatorname{curl}b\in L^2\left(0,T;H^1(\Omega)\right),\\
&\quad\operatorname{curl}b=0\ \text{in}\ Q^s(S_m),\;\;b\cdot\mathrm{n}\,|_{\partial\Omega}=0\Big\},\\
z_m,\tilde{z}_m\in&L^{\frac{4}{3}}(Q)
\end{aligned}
$$

for $S_m=S_m(\cdot)=(\eta_m(\cdot,O))^\delta$, such that

$$
\begin{aligned}
\frac{d\eta_m(t,x)}{dt}&=R_\delta\left[u_m\right](t,\eta_m(t,x)), \qquad \eta_m(0,x)=x,\\
\partial_t\rho_m+\operatorname{div}(\rho_m u_m)&=\epsilon\Delta\rho_m\quad a.e.\ in\ Q, \qquad \rho(0)=\rho_{0,\alpha}, \qquad (4.5.20)
\end{aligned}
$$

$$
\begin{aligned}
-\int_0^T\int_\Omega\rho_m u_m\cdot\partial_t\phi\; dxdt-\int_\Omega(\rho u)_{0,\alpha}\cdot\phi(0,x)\; dx=&\int_0^T\int_\Omega(\rho_m u_m\otimes u_m):\mathbb{D}(\phi)+\left(a\rho_m^\gamma+\alpha\rho_m^\beta\right)\operatorname{div}\phi\\
&-2\nu(\chi_m)\,\mathbb{D}(u_m):\mathbb{D}(\phi)-\lambda(\chi_m)\operatorname{div}u_m\operatorname{div}\phi\\
&+\rho_m g\cdot\phi+\frac{1}{\mu}(\operatorname{curl}B_m\times B_m)\cdot\phi-\epsilon\tilde{z}_m\cdot\phi\\
&-\epsilon(\nabla u_m\nabla\rho_m)\cdot\phi\; dxdt, \qquad (4.5.21)
\end{aligned}
$$

$$
\begin{aligned}
-\int_0^T\int_\Omega B_m\cdot\partial_t b\; dxdt-\int_\Omega B_{0,\alpha}\cdot b(0,x)\; dx=&\int_0^T\int_\Omega\Big[-\frac{1}{\sigma\mu}\operatorname{curl}B_m+u_m\times B_m+\frac{1}{\sigma}J\\
&-\frac{\epsilon}{\mu^2}z_m\Big]\cdot\operatorname{curl}b-\epsilon(\nabla\operatorname{curl}B_m):(\nabla\operatorname{curl}b)\; dxdt,
\end{aligned}
$$
(4.5.22)

where $\chi_m(t,x):=\boldsymbol{db}_{S_m(t)}(x)$, for all $\phi\in\mathcal{D}([0,T)\times\Omega)$ and all $b\in\mathcal{Y}(S_m)$. Moreover, these functions satisfy the energy inequality

$$\int_\Omega \frac{1}{2}\rho_m(\tau)\,|u_m(\tau)|^2 + \frac{a}{\gamma-1}\rho_m^\gamma(\tau) + \frac{\alpha}{\beta-1}\rho_m^\beta(\tau) + \frac{1}{2\mu}\,|B_m(\tau)|^2\;dx + \int_0^\tau\int_\Omega 2\nu\,(\chi_m)\,|\mathbb{D}\,(u_m)\,|^2$$
$$+\lambda\,(\chi_m)\,|\operatorname{div} u_m|^2 + \alpha\epsilon\beta\rho_m^{\beta-2}\,|\nabla\rho_m|^2 + \epsilon\,|\tilde z_m|^{\frac{4}{3}} + \frac{1}{\sigma\mu^2}\,|\operatorname{curl} B_m|^2 + \frac{\epsilon}{\mu}\,|\nabla\operatorname{curl} B_m|^2 + \frac{\epsilon}{\mu^3}\,|z_m|^{\frac{4}{3}}\;dxdt$$
$$\leqslant \int_\Omega \frac{1}{2}\frac{\left|(\rho u)_{0,\alpha}\right|^2}{\rho_{0,\alpha}} + \frac{a}{\gamma-1}\rho_{0,\alpha}^\gamma + \frac{\alpha}{\beta-1}\rho_{0,\alpha}^\beta + \frac{1}{2\mu}\,|B_{0,\alpha}|^2\;dx + \int_0^\tau\int_\Omega \rho_m g\cdot u_m + \frac{1}{\sigma\mu}J\cdot\operatorname{curl} B_m\;dxdt \tag{4.5.23}$$

for almost all $\tau \in [0,T]$ and the estimate

$$\epsilon\,\|\nabla\rho_m\|_{L^r(Q)} + \epsilon\,\|\partial_t\rho_m\|_{L^{\tilde r}(Q)} + \epsilon^2\,\|\nabla^2\rho_m\|_{L^{\tilde r}(Q)} \leqslant c \tag{4.5.24}$$

with a constant $c > 0$ independent of m and ϵ.

From this point on, the remainder of the proof of Theorem 4.1.1 is straight forward: In the mechanical part of the problem we can follow precisely the arguments from [43, Sections 7–9], the additional Lorentz force (cf. [104]) and regularization term in the momentum equation do not cause any essential further difficulties. In the induction equation, each limit passage from now on can be carried out as in the incompressible case, cf. Section 3.5.3. However, for the convenience of the reader, we will sketch the main arguments for the remaining three limit passages in the following sections.

4.6 Limit passage with respect to $m \to \infty$

We continue by passing to the limit in the penalization method, i.e. we pass to the limit with respect to $m \to \infty$. Due to the energy inequality (4.5.23) and the uniform bounds (4.5.24) we may extract suitable subsequences and find functions $z, \tilde z \in L^{\frac{4}{3}}(Q)$ and

$$0 \leqslant \rho \in \Big\{\psi \in L^\infty\left(0,T;L^\beta(\Omega)\right)\bigcap L^r\left(0,T;W^{1,r}(\Omega)\right)\bigcap L^{\tilde r}\left(0,T;W^{2,\tilde r}(\Omega)\right):$$
$$\partial_t\psi \in L^{\tilde r}(Q),\;\; \nabla\psi\cdot \mathrm{n}|_{\partial\Omega} = 0\Big\} \subset C\left([0,T];L^2(\Omega)\right), \tag{4.6.1}$$
$$u \in L^2(0,T;H_0^1(\Omega)), \tag{4.6.2}$$
$$B \in \Big\{b \in L^\infty\left(0,T;L^2(\Omega)\right)\bigcap L^2\left(0,T;H^1_{\operatorname{div}}(\Omega)\right):\;\operatorname{curl} b \in L^2\left(0,T;H^1(\Omega)\right)$$
$$\operatorname{curl} b = 0 \text{ in } Q^s(S),\;\; b\cdot\mathrm{n}|_{\partial\Omega} = 0\Big\}, \tag{4.6.3}$$

such that

$$\rho_m \to \rho \quad \text{in } L^\beta(Q), \qquad \rho_m \to \rho \quad \text{in } L^2\left(0,T;H^1(\Omega)\right), \tag{4.6.4}$$
$$\rho_m \rightharpoonup \rho \quad \text{in } L^{\tilde r}\left(0,T;W^{2,\tilde r}(\Omega)\right), \qquad \partial_t\rho_m \rightharpoonup \partial_t\rho \quad \text{in } L^{\tilde r}(Q), \tag{4.6.5}$$
$$u_m \rightharpoonup u \quad \text{in } L^2\left(0,T;H^1(\Omega)\right), \qquad B_m \overset{*}{\rightharpoonup} B \quad \text{in } L^\infty\left(0,T;L^2(\Omega)\right), \tag{4.6.6}$$
$$B_m \rightharpoonup B \quad \text{in } L^2\left(0,T;H^1(\Omega)\right), \qquad \operatorname{curl} B_m \rightharpoonup \operatorname{curl} B \quad \text{in } L^2\left(0,T;H^1(\Omega)\right),$$
$$z_m \rightharpoonup z \quad \text{in } L^{\frac{4}{3}}(Q) \qquad \tilde z_m \rightharpoonup \tilde z \quad \text{in } L^{\frac{4}{3}}(Q).$$

The boundary conditions of the limit functions in (4.6.1)–(4.6.3) follow directly from the corresponding boundary conditions on the m-level, see Proposition 4.5.1. The set-valued function S in (4.6.3) is defined by $S := S(\cdot) := (\eta(\cdot,O))^\delta$ where η, given by the first convergence in

$$\eta_m \to \eta \quad \text{in } C\left([0,T];C_{\mathrm{loc}}\left(\mathbb{R}^3\right)\right),$$
$$\chi_m \to \chi := \mathbf{db}_{S(\cdot)}(\cdot) \quad \text{in } C\left([0,T];C_{\mathrm{loc}}\left(\mathbb{R}^3\right)\right),$$

denotes the solution to the initial value problem

$$\frac{d\eta(t,x)}{dt} = R_\delta[u]\,(t,\eta(t,x))\,, \qquad \eta(0,x) = x \qquad \forall x \in \mathbb{R}^3 \tag{4.6.7}$$

for all $x \in \mathbb{R}^3$ and almost all $t \in [0,T]$, cf. Lemma A.4.3.

4.6.1 Continuity equation

Making use of the convergences (4.6.4)–(4.6.6) of ρ_m and u_m, we can pass to the limit in the continuity equation (4.5.20) and ensure that

$$\partial_t\rho + \operatorname{div}(\rho u) = \epsilon\Delta\rho \quad \text{a.e. in } Q, \qquad \rho(0) = \rho_{0,\alpha}. \tag{4.6.8}$$

Moreover, this pointwise identity can be renormalized by multiplying it by $\zeta'(\rho)$ for an arbitrary convex function $\zeta \in C^2([0,+\infty))$. Since $\zeta'' \geqslant 0$, this yields

$$\partial_t\zeta(\rho) + \operatorname{div}(\zeta(\rho)\,u) + \left[\zeta'(\rho)\,\rho - \zeta(\rho)\right]\operatorname{div} u - \epsilon\Delta\zeta(\rho) = -\epsilon\zeta''(\rho)\,|\nabla\rho|^2 \leqslant 0 \quad \text{a.e. in } Q. \tag{4.6.9}$$

This relation will turn out useful in the limit passage with respect to $\epsilon \to 0$ in Section 4.7.

4.6.2 Induction equation

For the limit passage in the induction equation we can argue exactly as in Section 3.5.3 in the incompressible case to show strong convergence of B_m in the fluid domain. Hence, we can pass to the limit in the induction equation (4.5.22) and obtain the identity

$$\begin{aligned} &-\int_0^T\int_\Omega B\cdot\partial_t b\;dxdt - \int_\Omega B_{0,\alpha}\cdot b(0,x)\;dx \\ &= \int_0^T\int_\Omega \left[-\frac{1}{\sigma\mu}\operatorname{curl} B + u\times B + \frac{1}{\sigma}J - \frac{\epsilon}{\mu^2}z\right]\cdot\operatorname{curl} b - \epsilon\,(\nabla\operatorname{curl} B)\cdot(\nabla\operatorname{curl} b)\;dxdt \end{aligned} \tag{4.6.10}$$

for all $b \in \mathcal{Y}(S)$.

4.6.3 Momentum equation and compatibility of the velocity field

From the uniform bounds given by the energy inequality (4.5.23) we further infer the existence of $z_6 \in L^2(0,T;L^{\frac{6\beta}{4\beta+3}}(\Omega))$ such that, for a chosen subsequence,

$$\rho_m u_m \otimes u_m \rightharpoonup z_6 \quad \text{in } L^2\left(0,T;L^{\frac{6\beta}{4\beta+3}}(\Omega)\right).$$

For the limit passage in the momentum equation we need to identify z_6. To this end we notice that, according to their definition in (4.2.18), (4.2.19), the variable viscosity coefficients satisfy

$$\nu\left(\chi_m(t,x)\right) = \nu + mH\left(\mathbf{db}_{S_m(t)}(x)\right) = \nu, \quad \lambda\left(\chi_m(t,x)\right) = \lambda + mH\left(\mathbf{db}_{S_m(t)}(x)\right) = \lambda \tag{4.6.11}$$

for $(t,x) \in Q$ with $\mathbf{db}_{S_m(t)}(x) < 0$, i.e. they remain constant and in particular bounded in the fluid domain $Q^f(S_m)$ on the m-level. This enables us to deduce strong convergence of the momentum function $\rho_m u_m$ in the fluid domain similarly to the strong convergence (3.5.27) of the magnetic induction in the incompressible case: We fix an arbitrary interval $I \subset (0,T)$ and an arbitrary ball $U \subset \Omega$ such that $\overline{I\times U} \subset Q^f(S)$. Then we deduce from the momentum equation (4.5.21), the relations (4.6.11) as well as the uniform bounds given by the energy inequality (4.5.23) the dual estimate

$$\left\|\partial_t\int_U \rho_m u_m\cdot\Phi dx\right\|_{L^{\min\left(\frac{6}{5},\frac{5\beta-3}{4\beta}\right)}(I)} \leqslant c$$

for any fixed $\Phi \in \mathcal{D}(U)$ with a constant $c > 0$ depending on Φ but not on m. From Lemma A.4.1 we infer that

$$\rho_m u_m \to \rho u \quad \text{in } C_{\text{weak}}\left(\overline{I}; L^{\frac{2\beta}{\beta+1}}(U)\right) \quad \text{and thus in } L^p\left(I; H^{-1,2}(U)\right) \qquad \forall 1 \leqslant p < \infty,$$

which implies that

$$z_6 : \mathbb{D}(\phi) = (\rho u \otimes u) : \mathbb{D}(\phi) \quad \text{a.e. in } Q$$

for all test functions $\phi \in \mathcal{Z}(S)$, which satisfy $\mathbb{D}(\phi) = 0$ in a neighborhood of $\overline{Q^s(S)}$. Letting m tend to infinity in (4.5.21) we thus obtain

$$\begin{aligned} &-\int_0^T \int_\Omega \rho u \cdot \partial_t \phi \, dxdt - \int_\Omega (\rho u)_{0,\alpha} \cdot \phi(0,x) \, dx \\ &= \int_0^T \int_\Omega (\rho u \otimes u) : \mathbb{D}(\phi) + \left(a\rho^\gamma + \alpha\rho^\beta\right) \operatorname{div} \phi - 2\nu \mathbb{D}(u) : \mathbb{D}(\phi) - \lambda \operatorname{div} u \operatorname{div} \phi + \rho g \cdot \phi \\ &+ \frac{1}{\mu} (\text{curl} B \times B) \cdot \phi - \epsilon \tilde{z} \cdot \phi - \epsilon (\nabla u \nabla \rho) \cdot \phi \, dxdt \end{aligned} \tag{4.6.12}$$

for all $\phi \in \mathcal{Z}(S)$. Moreover, since $\nu(\chi_m)$ and $\lambda(\chi_m)$ blow up in the solid part of the domain, the energy inequality (4.5.23) shows that

$$\mathbb{D}(u) = 0 \quad \text{a.e. in } Q^s(S).$$

Hence, there are rigid velocity fields u^{s^i} which coincide with u almost everywhere in the δ-neighborhoods $S^i(t) := (\eta(t, O^i))^\delta$ of the sets $\eta(t, O^i)$, $i = 1, ..., N$,

$$u(t,\cdot) = u^{s^i}(t,\cdot) \quad \text{a.e. in } (\eta(t, O^i))^\delta. \tag{4.6.13}$$

Consequently, due to the property (4.2.24) of the regularized velocity field $R_\delta[u]$, we can replace $R_\delta[u]$ in the initial value problem (4.6.7) by u^{s^i} for $x \in O^i$. In particular, we conclude the existence of isometries $\eta^i(t)$, coinciding with $\eta(t)$ in O^i, such that

$$\frac{d\eta^i(t,x)}{dt} = u^{s^i}\left(t, \eta^i(t,x)\right), \qquad \eta^i(0,x) = x \tag{4.6.14}$$

for all $x \in \mathbb{R}^3$ and almost all $t \in [0,T]$. The combination of the conditions (4.6.13) and (4.6.14) at first yields compatibility (cf. (4.1.1), (4.1.2)) of u with the system $\{O^i, \eta^i\}_{i=1}^N$. However, the fact that each $\eta^i(t)$ is an isometry implies that

$$S^i(t) = \left(\eta^i\left(t, O^i\right)\right)^\delta = \eta^i\left(t, S_0^i\right) \qquad \text{and thus} \qquad u(t,\cdot) = u^{s^i}(t,\cdot) \quad \text{a.e. in } \eta^i\left(t, S_0^i\right).$$

Consequently, we infer that u is even compatible with $\{S_0^i, \eta^i\}_{i=1}^N$.

4.6.4 Energy inequality

We drop, among other non-negative terms, the variable parts of the viscosity coefficients on the left-hand side of the energy inequality (4.5.23) and pass to the limit to see that

$$\begin{aligned} &\int_\Omega \frac{1}{2}\rho(\tau)\left|u(\tau)\right|^2 + \frac{a}{\gamma-1}\rho^\gamma(\tau) + \frac{\alpha}{\beta-1}\rho^\beta(\tau) + \frac{1}{2\mu}\left|B(\tau)\right|^2 \, dx + \int_0^\tau \int_\Omega 2\nu|\mathbb{D}(u)|^2 + \lambda\left|\operatorname{div} u\right|^2 + \epsilon\left|\tilde{z}\right|^{\frac{4}{3}} \\ &+ \frac{1}{\sigma\mu^2}\left|\operatorname{curl} B\right|^2 + \frac{\epsilon}{\mu}\left|\nabla \operatorname{curl} B\right|^2 + \frac{\epsilon}{\mu^3}\left|z\right|^{\frac{4}{3}} \, dxdt \\ &\leqslant \int_\Omega \frac{1}{2}\frac{\left|(\rho u)_{0,\alpha}\right|^2}{\rho_{0,\alpha}} + \frac{a}{\gamma-1}\rho_{0,\alpha}^\gamma + \frac{\alpha}{\beta-1}\rho_{0,\alpha}^\beta + \frac{1}{2\mu}\left|B_{0,\alpha}\right|^2 \, dx + \int_0^\tau \int_\Omega \rho g \cdot u + \frac{1}{\sigma\mu} J \cdot \operatorname{curl} B \, dxdt \end{aligned} \tag{4.6.15}$$

for almost all $\tau \in [0,T]$.

4.7 Limit passage with respect to $\epsilon \to 0$

The next step is the limit passage with respect to $\epsilon \to 0$. Testing the continuity equation (4.6.8) by ρ_ϵ, we see that

$$\epsilon^{\frac{1}{2}} \|\nabla \rho_\epsilon\|_{L^2(Q)} \leqslant c$$

for a constant $c > 0$ independent of ϵ. This, together with the energy inequality (4.6.15), yields the existence of functions

$$
\begin{aligned}
0 \leqslant \rho \in & L^\infty(0,T;L^\beta(\Omega)),\\
u \in & \left\{\phi \in L^2\left(0,T;H_0^1(\Omega)\right) : \ \mathbb{D}(\phi) = 0 \text{ in } Q^s(S)\right\}, \qquad (4.7.1)\\
B \in & \left\{b \in L^\infty\left(0,T;L^2(\Omega)\right) \bigcap L^2\left(0,T;H^1_{\text{div}}(\Omega)\right) : \ \operatorname{curl} b = 0 \text{ in } Q^s(S), \ b\cdot \mathrm{n}|_{\partial\Omega} = 0\right\} \qquad (4.7.2)
\end{aligned}
$$

such that for certain extracted subsequences it holds

$$
\begin{aligned}
\rho_\epsilon &\overset{*}{\rightharpoonup} \rho \quad \text{in } L^\infty\left(0,T;L^\beta(\Omega)\right), & u_\epsilon &\rightharpoonup u \quad \text{in } L^2\left(0,T;H^1(\Omega)\right),\\
B_\epsilon &\overset{*}{\rightharpoonup} B \quad \text{in } L^\infty\left(0,T;L^2(\Omega)\right), & B_\epsilon &\rightharpoonup B \quad \text{in } L^2\left(0,T;H^1(\Omega)\right),\\
\epsilon\nabla\rho_\epsilon,\ \epsilon\Delta B_\epsilon &\to 0 \quad \text{in } L^2(Q), & \epsilon z_\epsilon,\ \epsilon\tilde{z}_\epsilon &\to 0 \quad \text{in } L^{\frac{4}{3}}(Q). \qquad (4.7.3)
\end{aligned}
$$

The boundary conditions of the limit functions in (4.7.1) and (4.7.2) follow directly from the corresponding boundary conditions of the velocity field and the magnetic induction in (4.6.2) and (4.6.3) on the ϵ-level. The set-valued function S in the inclusions (4.7.1) and (4.7.2) is defined by $S = S(\cdot) := \eta(\cdot, S_0)$, where $\eta : [0,T] \times S_0 \to \mathbb{R}^3$, $\eta(t)|_{S_0^i} := \eta^i(t)$, $i = 1, ..., N$, and each $\eta^i(t) : \mathbb{R}^3 \to \mathbb{R}^3$ denotes an isometry given by

$$\eta^i_\epsilon \to \eta^i \quad \text{in } C\left([0,T]; C_{\text{loc}}\left(\mathbb{R}^3\right)\right),$$

cf. Corollary A.4.1 in the appendix. Moreover, Corollary A.4.1 implies that the velocity field u is compatible with the system $\{S_0^i, \eta^i\}_{i=1}^N$.

4.7.1 Continuity equation

Similarly as in Section 3.5.1 in the incompressible case, we deduce from the continuity equation (4.6.8) that ρ_ϵ even converges to ρ in $C_{\text{weak}}([0,T];L^\beta(\Omega))$. This, together with the vanishing artificial viscosity term, cf. (4.7.3), is sufficient to pass to the limit in the regularized continuity equation (4.6.8) and to obtain

$$-\int_0^T\int_\Omega \rho\partial_t\psi dxdt - \int_\Omega \rho_{0,\alpha}\psi(0,x)\ dx = \int_0^T\int_\Omega (\rho u)\cdot\nabla\psi\ dxdt \qquad (4.7.4)$$

for all $\psi \in \mathcal{D}([0,T) \times \Omega)$. In fact, as at this stage of the approximation it holds that $\rho \in L^2(Q)$, $u \in L^2(0,T;H_0^{1,2}(\Omega))$, we can use the regularization procedure by DiPerna and Lions, cf. [94, Lemma 6.8, Lemma 6.9], to see that ρ and u, extended by 0 outside of Ω, even satisfy the renormalized continuity equation (4.1.14), (4.1.15).

4.7.2 Induction equation

We argue as in Section 3.5.3 in the incompressible case to pass to the limit with respect to $\epsilon \to 0$ in the induction equation (4.6.10) and to infer that

$$-\int_0^T\int_\Omega B\cdot\partial_t b\ dxdt - \int_\Omega B_{0,\alpha}\cdot b(0,x)\ dx = \int_0^T\int_\Omega \left[-\frac{1}{\sigma\mu}\operatorname{curl} B + u\times B + \frac{1}{\sigma}J\right]\cdot\operatorname{curl} b\ dxdt \quad (4.7.5)$$

for all $b \in \mathcal{Y}(S)$.

4.7.3 Momentum equation

In order to pass to the limit in the pressure terms, we first consider an arbitrary compact set $K \subset Q^f(S)$. Denoting by $\mathcal{B}_\Omega$ the Bogovskii operator in Ω (cf. [94, Section 3.3.1.2]), we test the momentum equation (4.6.12) by

$$\phi_\epsilon(t,x) := \Phi(t,x)\mathcal{B}_\Omega\left[\rho_\epsilon(t,\cdot) - \frac{1}{|\Omega|}\int_\Omega \rho_\epsilon(t,y)\, dy\right](t,x), \tag{4.7.6}$$

where $\Phi \in \mathcal{D}(Q^f(S))$ denotes a cut-off function equal to 1 in K. This procedure leads to a bound of ρ_ϵ in $L^{\beta+1}(K)$ uniformly in ϵ, cf. [43, Lemma 8.1] and the references therein. These bounds in turn allow us to find $z_7 \in L^{\frac{\gamma+1}{\gamma}}(K)$, $z_8 \in L^{\frac{\beta+1}{\beta}}(K)$ such that

$$\rho_\epsilon^\gamma \rightharpoonup z_7 \quad \text{in } L^{\frac{\gamma+1}{\gamma}}(K), \qquad \rho_\epsilon^\beta \rightharpoonup z_8 \quad \text{in } L^{\frac{\beta+1}{\beta}}(K).$$

With the aim of identifying these limit functions we set, for arbitrary $\tilde{\Phi} \in \mathcal{D}(Q^f(S))$,

$$\tilde{\phi}_\epsilon(t,x) := \tilde{\Phi}(t,x)\left(\nabla\Delta^{-1}\right)[\rho_\epsilon(t,\cdot)](t,x), \qquad \tilde{\phi}(t,x) := \tilde{\Phi}(t,x)\left(\nabla\Delta^{-1}\right)[\rho(t,\cdot)](t,x), \tag{4.7.7}$$

where Δ^{-1} denotes the inverse Laplacian on $\mathbb{R}^3$, cf. [45, Section 10.16]. We compare the momentum equation (4.6.12) on the ϵ-level, tested by $\tilde{\phi}_\epsilon$, to a corresponding limit identity, tested by $\tilde{\phi}$. This enables us to deduce the effective viscous flux identity

$$(\lambda+2\nu)\lim_{\epsilon\to 0}\int_0^T\int_\Omega \tilde{\Phi}\left(\rho_\epsilon \operatorname{div} u_\epsilon - \rho \operatorname{div} u\right)\, dxdt = \lim_{\epsilon\to 0}\int_0^T\int_\Omega \tilde{\Phi}\left(\left[a\rho_\epsilon^\gamma + \alpha\rho_\epsilon^\beta\right]\rho_\epsilon - \left[a\rho_\epsilon^\gamma + \alpha\rho_\epsilon^\beta\right]\rho\right)\, dxdt \tag{4.7.8}$$

for all $\tilde{\Phi} \in \mathcal{D}(Q^f(S))$, cf. [43, Lemma 8.2] and the references therein. Moreover, after a density argument, we can consider the choice $\zeta(s) = s\ln(s)$ in both the renormalized continuity equations (4.6.9) on the ϵ-level and (4.1.14) in the limit. A comparison between the resulting identities then leads us to

$$\int_\Omega \rho(\tau)\ln(\rho(\tau))\, dx - \lim_{\epsilon\to 0}\int_\Omega \rho_\epsilon(\tau)\ln(\rho_\epsilon(\tau))\, dx \geqslant \lim_{\epsilon\to 0}\int_0^\tau\int_\Omega \rho_\epsilon \operatorname{div} u_\epsilon\, dxdt - \int_0^\tau\int_\Omega \rho \operatorname{div} u\, dxdt \geqslant 0 \tag{4.7.9}$$

for $\tau \in [0,T]$, where the last inequality follows from the effective viscous flux identity (4.7.8) and the monotonicity of the mapping $s \mapsto as^\gamma + \alpha s^\beta$, as well as the fact that $\operatorname{div} u = 0$ in $Q^s(S)$. Exactly as in the incompressible setting (cf. the derivation of the convergence (3.5.12)), this estimate together with the strict convexity of $z \mapsto z\ln(z)$ implies pointwise convergence of ρ_ϵ in Q. It follows that $z_7 = \rho^\gamma$, $z_8 = \rho^\beta$ almost everywhere in $Q^f(S)$. In the remaining terms of the momentum equation (4.6.12) we can pass to the limit as during the past limit passages. We end up with

$$\begin{aligned}
&-\int_0^T\int_\Omega \rho u\cdot\partial_t\phi\, dxdt - \int_\Omega (\rho u)_{0,\alpha}\cdot\phi(0,x)\, dx \\
&= \int_0^T\int_\Omega (\rho u\otimes u):\mathbb{D}(\phi) + \left(a\rho^\gamma + \alpha\rho^\beta\right)\operatorname{div}\phi - 2\nu\mathbb{D}(u):\mathbb{D}(\phi) - \lambda \operatorname{div} u \operatorname{div}\phi + \rho g\cdot\phi \\
&\quad + \frac{1}{\mu}(\operatorname{curl} B\times B)\cdot\phi\, dxdt
\end{aligned} \tag{4.7.10}$$

for all $\phi \in \mathcal{Z}(S)$.

4.7.4 Energy inequality

Neglecting the regularization terms on the left-hand side of the energy inequality (4.6.15) on the ϵ-level, we can pass to the limit with respect to $\epsilon \to 0$ and obtain

$$\int_\Omega \frac{1}{2}\rho(\tau)\,|u(\tau)|^2 + \frac{a}{\gamma-1}\rho^\gamma(\tau) + \frac{\alpha}{\beta-1}\rho^\beta(\tau) + \frac{1}{2\mu}\,|B(\tau)|^2 \; dx + \int_0^\tau \int_\Omega 2\nu|\mathbb{D}(u)|^2 + \lambda\,|\operatorname{div} u|^2$$
$$+ \frac{1}{\sigma\mu^2}\,|\operatorname{curl} B|^2 \; dxdt$$
$$\leqslant \int_\Omega \frac{1}{2}\frac{\left|(\rho u)_{0,\alpha}\right|^2}{\rho_{0,\alpha}} + \frac{a}{\gamma-1}\rho_{0,\alpha}^\gamma + \frac{\alpha}{\beta-1}\rho_{0,\alpha}^\beta + \frac{1}{2\mu}\,|B_{0,\alpha}|^2 \; dx + \int_0^\tau \int_\Omega \rho g\cdot u + \frac{1}{\sigma\mu} J\cdot \operatorname{curl} B \; dxdt \tag{4.7.11}$$

for almost all $t \in [0,T]$.

4.8 Limit passage with respect to $\alpha \to 0$

Finally it remains to pass to the limit with respect to $\alpha \to 0$. We now consider initial data ρ_0, $(\rho u)_0$ and B_0 as in Theorem 4.1.1 and construct - cf. [47, Section 4] - the initial data $\rho_{0,\alpha}$, $(\rho u)_{0,\alpha}$ and $B_{0,\alpha}$ on the α-level (cf. (4.5.18), (4.5.19)) in such a way that

$$\begin{aligned} &\rho_{0,\alpha} \to \rho_0 \quad \text{in } L^\gamma(\Omega), && \alpha\rho_{0,\alpha}^\beta \to 0 \quad \text{in } L^1(\Omega),\\ &(\rho u)_{0,\alpha} \to (\rho u)_0 \quad \text{in } L^1(\Omega), && \frac{\left|(\rho u)_{0,\alpha}\right|^2}{\rho_{0,\alpha}} \to \frac{|(\rho u)_0|^2}{\rho_0} \quad \text{in } L^1(\Omega),\\ &B_{0,\alpha} \to B_0 \quad \text{in } L^2(\Omega). \end{aligned}$$

Further, from the energy inequality (4.7.11) we obtain the existence of functions

$$0 \leqslant \rho \in L^\infty(0,T;L^\gamma(\Omega)), \tag{4.8.1}$$
$$u \in \left\{\phi \in L^2\left(0,T;H_0^1(\Omega)\right) : \; \mathbb{D}(\phi) = 0 \text{ in } Q^s(S)\right\} \tag{4.8.2}$$
$$B \in \left\{b \in L^\infty\left(0,T;L^2(\Omega)\right) \bigcap L^2\left(0,T;H^1_{\operatorname{div}}(\Omega)\right) : \; \operatorname{curl} b = 0 \text{ in } Q^s(S), \; b\cdot \mathrm{n}|_{\partial\Omega} = 0\right\} \tag{4.8.3}$$

such that, for suitable subsequences,

$$\begin{aligned} &\rho_\alpha \overset{*}{\rightharpoonup} \rho \quad \text{in } L^\infty\left(0,T;L^\gamma(\Omega)\right), && u_\alpha \rightharpoonup u \quad \text{in } L^2\left(0,T;H^1(\Omega)\right),\\ &B_\alpha \overset{*}{\rightharpoonup} B \quad \text{in } L^\infty\left(0,T;L^2(\Omega)\right), && B_\alpha \rightharpoonup B \quad \text{in } L^2\left(0,T;H^1(\Omega)\right). \end{aligned}$$

The boundary conditions of the limit functions in (4.8.2) and (4.8.3) follow directly from the corresponding boundary conditions of the velocity field and the magnetic induction in (4.7.1) and (4.7.2) on the α-level. The set-valued function S in the inclusions (4.8.2) and (4.8.3) is given by $S = S(\cdot) := \eta(\cdot, S_0)$, where

$$\eta : [0,T] \times S_0 \to \mathbb{R}^3, \quad \eta(t)|_{S_0^i} := \eta^i(t), \qquad i = 1,\dots,N, \tag{4.8.4}$$

and each $\eta^i(t) : \mathbb{R}^3 \to \mathbb{R}^3$ denotes an isometry given by

$$\eta_\alpha^i \to \eta^i \quad \text{in } C\left([0,T];C_{\mathrm{loc}}\left(\mathbb{R}^3\right)\right),$$

cf. Corollary A.4.1 in the appendix. Moreover, again due to Corollary A.4.1,

$$u \quad \text{is compatible with the family} \quad \{S_0^i, \eta^i\}_{i=1}^N. \tag{4.8.5}$$

4.8.1 Continuity equation

After using the continuity equation (4.7.4) to deduce convergence of ρ_α in $C_{\text{weak}}([0,T];L^\gamma(\Omega))$, we pass to the limit in (4.7.4) and obtain

$$-\int_0^T\int_\Omega \rho\partial_t\psi dxdt - \int_\Omega \rho_0\psi(0,x)\ dx = \int_0^T\int_\Omega (\rho u)\cdot\nabla\psi\ dxdt \tag{4.8.6}$$

for all $\psi \in \mathcal{D}([0,T)\times\Omega)$. The proof of the renormalized continuity equation however needs to be postponed to Section 4.8.3 below, since at this stage ρ does not have the $L^2(Q)$-regularity required for the regularization technique by DiPerna and Lions anymore.

4.8.2 Induction equation

For the limit passage in the induction equation (4.7.5) we argue exactly as in Section 3.5.3 in the incompressible case and end up with the relation

$$-\int_0^T\int_\Omega B\cdot\partial_t b\ dxdt - \int_\Omega B_0\cdot b(0,x)\ dx = \int_0^T\int_\Omega \left[-\frac{1}{\sigma\mu}\operatorname{curl} B + u\times B + \frac{1}{\sigma}J\right]\cdot\operatorname{curl} b\ dxdt \tag{4.8.7}$$

for all $b \in \mathcal{Y}(S)$.

4.8.3 Momentum equation

For the limit passage in the pressure terms the strategy used during the limit passage with respect to $\epsilon \to 0$ in Section 4.7.3 needs to be modified to make up for the lower integrability of the density; the main ideas however remain the same. First we test the momentum equation (4.7.10) by functions of the form (4.7.6) with the density replaced by (a cut-off and smoothened version of) ρ_α^θ, $\theta > 0$. Choosing $\theta > 0$ sufficiently small, we find that, for any compact set $K \subset Q^f(S)$, ρ_α and $\alpha^{\frac{1}{\beta+\theta}}\rho_\alpha$ are bounded uniformly in $L^{\gamma+\theta}(K)$ and $L^{\beta+\theta}(K)$, respectively, cf. [46, Proposition 2.3], [47, Section 4.1]. In particular, there exists $z_9 \in L^{\frac{\gamma+\theta}{\gamma}}(K)$ such that, after the extraction of a subsequence,

$$\rho_\alpha^\gamma \rightharpoonup z_9 \quad \text{in } L^{\frac{\gamma+\theta}{\gamma}}(K), \qquad \alpha\rho_\alpha^\beta \rightharpoonup 0 \quad \text{in } L^{\frac{\beta+\theta}{\beta}}(K).$$

In order to identify z_9, we again need to show strong convergence of ρ_α. To this end we use the notion of the oscillation defect measure

$$\mathbf{osc}_{\gamma+1}[\rho_\alpha \to \rho](O) := \sup_{k\geqslant 1}\left[\limsup_{\alpha\to 0}\int_O |T_k(\rho_\alpha) - T_k(\rho)|^{\gamma+1}\ dxdt\right]$$

for measurable sets $O \subset (0,T)\times\mathbb{R}^3$ and a concave cut-off function $T_k \in C^\infty([0,\infty))$, $k \in \mathbb{N}$, coinciding with the identity function on $[0,k]$ and with $2k$ on $[3k,\infty)$. The proof of the pointwise convergence of ρ_α can be divided into three main steps, each of which consists of showing one of the following three conditions, respectively:

(i) The effective viscous flux identity

$$\begin{aligned}(\lambda+2\nu)\lim_{\alpha\to 0}&\int_0^T\int_\Omega \phi\left(T_k(\rho_\alpha)\operatorname{div} u_\alpha - \underline{T_k(\rho)}\operatorname{div} u_\alpha\right)\ dxdt\\ &= \lim_{\alpha\to 0}\int_0^T\int_\Omega \phi\left(a\rho_\alpha^\gamma T_k(\rho_\alpha) - a\rho_\alpha^\gamma\underline{T_k(\rho)}\right)\ dxdt,\end{aligned} \tag{4.8.8}$$

where $\underline{T_k(\rho)}$ denotes a weak $L^1(Q)$-limit of $T_k(\rho_\alpha)$, holds true for any $\phi \in \mathcal{D}(Q^f(S))$.

(ii) The oscillation defect measure is bounded on $(0,T)\times\mathbb{R}^3$,

$$\mathbf{osc}_{\gamma+1}[\rho_\alpha \to \rho]\left((0,T)\times\mathbb{R}^3\right) < +\infty. \tag{4.8.9}$$

(iii) The renormalized continuity equation (4.1.14), (4.1.15) is satisfied by ρ and u.

The effective viscous flux identity (4.8.8) can be shown by a comparison between the momentum equation (4.7.10) on the α-level and a corresponding limit identity, tested by suitably modified variants of the functions $\tilde{\phi}_\epsilon$ and $\tilde{\phi}$ in (4.7.7) with the density replaced by $T_k(\rho_\alpha)$ and $\overline{T_k(\rho)}$, respectively. The details, in the case without rigid bodies, are given e.g. in [47, Section 4.3], the adjustment to the fluid-structure interaction case poses no further difficulties. The proof of the bound (4.8.9) of the oscillation defect measure is split up into an estimate on $Q^s(S)$ and an estimate on $Q^f(S)$. From the representation of the density in the solid region in Lemma 4.1.1 (ii) it follows that $\rho_\alpha \to \rho$ in $L^1(K)$ for compact sets $K \subset Q^s(S)$ and thus $\mathbf{osc}_{\gamma+1}[\rho_\alpha \to \rho](Q^s(S)) = 0$. In the fluid region the bound is achieved, under exploitation of the effective viscous flux identity (4.8.8), by the same arguments as in the all-fluid case, cf. [42, Proposition 6.1]. Finally, the renormalized continuity equation in the limit is also obtained exactly as in the all-fluid case, cf. [42, Proposition 7.1]: The idea is to pass to the limit in the renormalized continuity equation (4.1.14) on the α-level for the choice $\zeta = T_k$. Thanks to the boundedness of T_k, the regularization technique by DiPerna and Lions (cf. [94, Lemma 6.9]) can be applied to the limit identity. Letting $k \to \infty$ and exploiting the bound (4.8.9) of the oscillation defect measure, we then obtain the renormalized continuity equation (4.1.14), (4.1.15) also for ρ and u.

Having shown the relations (i)–(iii) we now obtain strong convergence of ρ_α. Indeed, similarly as in the corresponding relation (4.7.9) in the ϵ-limit and under exploitation of the concavity of T_k, we see that the left-hand side of the effective viscous flux identity (4.8.8) is non-negative. This, in combination with the bound (4.8.9) of the oscillation defect measure and a comparison between the renormalized continuity equations on the α-level and in the limit, yields, similarly to the inequality (4.7.9),

$$\int_\Omega \rho(\tau)\ln(\rho(\tau))\ dx - \lim_{\alpha\to 0}\int_\Omega \rho_\alpha(\tau)\ln(\rho_\alpha(\tau))\ dx \geqslant 0$$

for $\tau \in [0,T]$. As in the derivation of the corresponding convergence (3.5.12) in the incompressible case, this inequality implies pointwise convergence of ρ_α in Q and therefore $z_9 = \rho^\gamma$ almost everywhere in $Q^f(S)$. In the remaining terms of the momentum equations (4.7.10) we may pass to the limit as in the previous limit passages and obtain

$$\begin{aligned}
&-\int_0^T\int_\Omega \rho u\cdot\partial_t\phi\ dxdt - \int_\Omega (\rho u)_0\cdot\phi(0,x)\ dx \\
&= \int_0^T\int_\Omega (\rho u\otimes u) : \mathbb{D}(\phi) + a\rho^\gamma \operatorname{div}\phi - 2\nu\mathbb{D}(u):\mathbb{D}(\phi) - \lambda \operatorname{div} u \operatorname{div}\phi + \rho g\cdot\phi + \frac{1}{\mu}(\operatorname{curl} B\times B)\cdot\phi\ dxdt
\end{aligned} \tag{4.8.10}$$

for all $\phi \in \mathcal{Z}(S)$.

4.8.4 Proof of the main result

We are now in the position to conclude the proof of Theorem 4.1.1. The function η in (4.1.9) is defined in (4.8.4). The fact that the associated isometries $\eta^i(t,\cdot)$ are orientation preserving follows from the relation $\eta^i(0,\cdot) = \mathrm{id}$ and the continuity of η^i. The properties of ρ, u and B in (4.1.10)–(4.1.12), except for the continuity of ρ in time, are shown in (4.8.1)–(4.8.3). The continuity equation (4.1.13) and its renormalization (4.1.14), (4.1.15) are derived in (4.8.6) and the relation (iii) in Section 4.8.3, respectively. In particular, ρ as a renormalized solution to the continuity equation satisfies $\rho \in C([0,T];L^1(\Omega))$, cf. [41, Proposition 4.3], which concludes the proof of (4.1.10). The momentum equation (4.1.16) and the induction equation (4.1.17) hold true according to (4.8.10) and (4.8.7). The compatibility of u with $\{S_0^i,\eta^i\}_{i=1}^N$ is shown in (4.8.5). Finally, in the energy inequality (4.7.11) on the α-level we can pass to the limit using the weak lower semicontinuity of norms to infer the energy inequality (4.1.19). This finishes the proof of Theorem 4.1.1.

Chapter 5

Evolution of a magnetoelastic material

In this chapter we perform a thematic switch in comparison to the previous chapters: Instead of the interaction between a fluid and a solid we now study the evolution of only one (solid) magnetoelastic material. Such - ferromagnetic and deformable - materials are characterized through the interaction between their magnetization and their deformation. More specifically, when magnetoelastic materials find themselves in the sphere of influence of a magnetic field they undergo a deformation and, the other way around, when they are subjected to a mechanical stress they encounter a change in their magnetization. As a mathematical model for the description of the evolution of such a material we use the system (1.3.30)–(1.3.32) of partial differential equations presented in Section 1.3.3, cf. also [6, 48]. The main result in this chapter is the proof of the local-in-time existence of weak solutions to this model.

This proof is based on De Giorgi's minimizing movements scheme. The problem is discretized with respect to the time and a minimization problem is solved at each discrete time, the associated Euler-Lagrange equations of which form a discrete approximation of the original equation of motion and magnetic force balance. The striking advantage of the minimizing movements scheme as opposed to other techniques lies in its compatibility with the non-convex energy functional. Indeed, for example the application of fixed point arguments, which constitutes a classical method for solving coupled systems of PDEs, usually relies on the energy functional being convex. But also an application of Rothe's method as in Chapter 3 and Chapter 4 in order to decouple the system and solve the equations successively but still directly is hindered by the non-convexity of the energy functional in the present case. This is because a discrete form of the chain rule would be required in order to obtain uniform a priori estimates for the discrete solution, which in turn again presupposes convexity of the energy. In De Giorgi's method instead an energy inequality is obtained directly by estimating the minimized functional in its minimizer against the same functional in the minimizer from the previous discrete time. For more details we refer to Section 5.2.1 below.

The greatest difficulty in our application of De Giorgi's scheme is to choose the functional for each discrete minimization problem in such a way that its variation with respect to the deformation and the magnetization gives rise to a suitable approximation of the equation of motion and the magnetic force balance, respectively. To this end it is important to realize that already on the continuous level these equations can be expressed with the help of the same energy and dissipation potentials (cf. the equations (1.3.30), (1.3.33) and (1.3.43)), thanks to the fact that the transport terms (1.3.42) in the magnetic force balance can be written by means of a dissipation potential which has no contribution to the equation of motion, cf. (1.3.37), (1.3.41). The functional to be minimized in the discrete problem can then be constructed on the basis of these energy and dissipation potentials.

Applications of magnetoelastic materials can for example be found in magnetic actuators, which transform changes in magnetic fields into mechanical energy (see [16, 110]) and in sensors with the ability of measuring mechanical stress in terms of changes in their magnetic fields (see [10, 11, 16, 62]). As a specific example we mention the use of such actuators as an alternative propulsion method for microrobots used in capsule endoscopy and remote drug delivery (see [114]), which builds a bridge between magnetoelasticity and the fluid-structure interaction problems studied in Chapter 3 and Chapter 4. For more details we refer to Section 1.1.

5.1 Weak solutions and main result

5.1.1 Notation

In the setting of this chapter we deal with a domain which is entirely occupied by a magnetoelastic material. In particular, the domain is deformable and thus movable. We introduce notation for the description of the motion of such a domain corresponding to the notation used for the description of the moving solid domain in the fluid-rigid body interaction problems in Chapter 3 and Chapter 4. We consider a time $T \in (0, \infty]$ (which we will specify later) and assume the reference configuration of the material to be given as a bounded domain $\Omega_0 \subset \mathbb{R}^3$. The deformation of the material is described by a mapping $\eta : [0,T] \times \Omega_0 \to \mathbb{R}^3$, i.e. the deformed configuration $\Omega(t)$ at any time $t \in [0,T]$ is given via the set-valued function

$$\Omega : [0,T] \to 2^{\mathbb{R}^3}, \qquad \Omega(t) := \eta\left(t, \Omega_0\right).$$

By $Q(\Omega,T)$ we denote the corresponding time-space domain

$$Q\left(\Omega, T\right) := \left\{(t,x) \in (0,T) \times \mathbb{R}^3 : \ x \in \Omega(t)\right\}.$$

Moreover, we partition the boundary $\partial\Omega_0$ into two parts

$$N \subset \partial\Omega_0, \qquad P := \partial\Omega_0 \backslash N,$$

where N is a free part while on P a boundary condition is prescribed for the deformation via a given function $\gamma : P \to \mathbb{R}^3$. In order to avoid self-penetration of the magnetoelastic body we restrict ourselves to internally injective deformations, or, more precisely, deformations from the set

$$\mathcal{E} := \left\{\eta \in W^{2,q}\left(\Omega_0; \mathbb{R}^3\right) : \ \tilde{E}_{\mathrm{el}}\left(\eta\right) < \infty, \ \left|\eta\left(\Omega_0\right)\right| = \int_{\Omega_0} \det\left(\nabla_X \eta\right) \, dX, \ \eta|_P = \gamma\right\},$$

wherein $\tilde{E}_{\mathrm{el}}$ denotes the elastic energy defined in (1.3.35). The identity

$$\left|\eta\left(\Omega_0\right)\right| = \int_{\Omega_0} \det\left(\nabla_X \eta\right) \, dX \tag{5.1.1}$$

in the definition of the set $\mathcal{E}$ is called the Ciarlet-Nečas condition. Provided that Ω_0 is of class $C^{0,1}$, any local homeomorphisms $\eta \in C^1(\overline{\Omega_0})$ satisfying this condition is in fact a global homeomorphism in Ω_0, i.e. injective except for possibly on the boundary $\partial\Omega_0$, cf. [24]. In particular, this holds true for any $\eta \in \mathcal{E}$: Indeed, from the Morrey embedding we know that $\mathcal{E} \subset C^1(\overline{\Omega_0})$. Further, as $\eta \in \mathcal{E}$ satisfies $\tilde{E}_{\mathrm{el}}(\eta) < \infty$, Lemma A.7.1 in the appendix implies that $\det(\nabla_X \eta) > 0$ in Ω_0. Consequently, by the inverse function theorem, η constitutes a local and hence a global homeomorphism.

Remark 5.1.1. *The set $\mathcal{E}$ constitutes a closed subset of the affine function space*

$$\left\{\eta \in W^{2,q}\left(\Omega_0; \mathbb{R}^3\right) : \ \eta|_P = \gamma\right\}.$$

The interior and the boundary of $\mathcal{E}$ can be characterized in the following way: For any $\eta \in \mathcal{E}$ it holds that

$$\eta \in \operatorname{int}\left(\mathcal{E}\right) \qquad \Leftrightarrow \qquad \eta|_N \text{ is injective.} \tag{5.1.2}$$

Remark 5.1.1 comes in handy when we construct a sequence of approximate solutions in the proof of the main result Theorem 5.1.1 of Chapter 5, cf. Section 5.2. Indeed, these approximate solutions are constructed by discretizing the problem with respect to the time and minimizing, at each discrete time, a suitable functional over $\mathcal{E} \times H^1(\Omega_0)$. We then obtain the approximate equation of motion satisfied by such a solution by taking the variation of this functional in the minimizer with respect to the deformation. To this end, however, we need to be able to test the functional in all directions, i.e. we need the minimizing deformation to be an interior point of $\mathcal{E}$. This requirement can be checked via the condition (5.1.2) in Remark 5.1.1. More precisely, we use Lemma A.7.2 in the appendix to guarantee the existence of a time interval on which the approximate deformations are injective on $\overline{\Omega_0}$ and thus remain in $\text{int}(\mathcal{E})$, provided that the initial deformation η_0 is contained in $\text{int}(\mathcal{E})$. After the construction of the approximate solution we pass to the limit in the approximation in order to obtain a solution to the original system. In this procedure it is important to know that the limit of the approximate deformations again lies in the set $\mathcal{E}$:

Remark 5.1.2. *The set $\mathcal{E}$ is weakly closed in $W^{2,q}(\Omega_0;\mathbb{R}^3)$, cf. [7, Lemma 2.4].*

In order to be able to also work in the current configuration, we further require a generalization of the classical Bochner spaces to the setting of the moving domain $\Omega(\cdot)$. For the definition of such a generalization we assume the deformation η to satisfy the conditions

$$\eta \in L^\infty(0,T;\mathcal{E})\bigcap C\left([0,T];C^1\left(\overline{\Omega_0}\right)\right), \quad \eta(t)\in \text{int}\left(\mathcal{E}\right) \text{ and } \tilde{E}_{\text{el}}\left(\eta(t)\right) \leqslant c \text{ for a.a. } t\in[0,T], \tag{5.1.3}$$

where $c > 0$ denotes a constant independent of $t \in [0,T]$. In particular, by the Ciarlet-Nečas condition (5.1.1) the mapping $X \mapsto \eta(t,X)$, $t \in [0,T]$, is injective in Ω_0 and thus possesses an inverse

$$\eta^{-1}(t,\cdot) : \Omega\left(t\right) \to \Omega_0.$$

Then, for values $1 \leqslant p < \infty$, $1 \leqslant r \leqslant \infty$ and $k = 0,1$, we define the generalized Bochner space

$$\begin{aligned} &L^r\left(0,T;W^{k,p}\left(\Omega\left(\cdot\right)\right)\right) \\ &:= \left\{ m:[0,T]\to \bigcup_{t\in[0,T]} W^{k,p}\left(\Omega(t)\right):\ m\left(\cdot,\eta^{-1}(\cdot,\cdot)\right)\in L^r\left(0,T;W^{k,p}\left(\Omega_0\right)\right)\right\}. \end{aligned} \tag{5.1.4}$$

where the union is taken over uncountably many sets. Under the assumptions (5.1.3) this space turns out to be a Banach space with the norm

$$\|m\|_{L^r(0,T;W^{k,p}(\Omega(\cdot)))} := \begin{cases} \left(\int_0^T \|m(t)\|^r_{W^{k,p}(\Omega(t))}\right)^{\frac{1}{r}} & \text{if } 1\leqslant r<\infty, \\ \operatorname{esssup}_{t\in[0,T]} \|m(t)\|_{W^{k,p}(\Omega(t))} & \text{if } r=\infty, \end{cases}$$

as can be seen via a transformation to the reference configuration, cf. Lemma A.7.3 in the Appendix. Finally, for the weak formulation of the system (1.3.30)–(1.3.32), presented in the Section 5.1.2 below, we define the stray field associated to the magnetization via the variational formulation of the Poisson problem (1.3.39). To this end we introduce the notation

$$\dot{H}^1\left(\mathbb{R}^3\right) := \tilde{H}^1\left(\mathbb{R}^3\right)/\mathbb{R}, \tag{5.1.5}$$

where

$$\tilde{H}^1\left(\mathbb{R}^3\right) := \left\{\psi\in H^1_{\text{loc}}\left(\mathbb{R}^3\right):\ \nabla\psi\in H^1\left(\mathbb{R}^3\right)\right\}$$

represents the space of local H^1-functions the gradient of which is square integrable over the whole space $\mathbb{R}^3$, cf. [98, Section 3]. The quotient space $\dot{H}^1(\mathbb{R}^3)$, in which the constant functions from the space $\tilde{H}^1(\mathbb{R}^3)$ are factored out, constitutes a Hilbert space with the bilinear product

$$\langle\phi,\psi\rangle_{\dot{H}^1(\mathbb{R}^3)\times\dot{H}^1(\mathbb{R}^3)} := \int_{\mathbb{R}^3}\nabla\phi\cdot\nabla\psi\ dx,$$

see [98, Lemma 3.2]. Then, for a given magnetization $\tilde{M}\in H^1(\Omega_0)$ in the reference configuration and a given deformation $\eta\in\mathcal{E}$ we denote the associated stray field by $H[\tilde{M},\eta] = -\nabla\phi[\tilde{M},\eta]\in L^2(\mathbb{R}^3)$, where $\phi[\tilde{M},\eta]\in\dot{H}^1(\mathbb{R}^3)\bigcap H^2_{\text{loc}}(\eta(\Omega_0))$ satisfies the Poisson equation

$$\int_{\mathbb{R}^3}\nabla\phi\left[\tilde{M},\eta\right]\cdot\nabla\psi\ dx = \int_{\eta(\Omega_0)} M\cdot\nabla\psi\ dx \qquad \forall\psi\in\dot{H}^1\left(\mathbb{R}^3\right), \tag{5.1.6}$$

cf. Lemma A.2.3 in the appendix, and

$$M := M_\eta\left[\tilde{M}\right] := \frac{1}{\det\left(\left[\nabla_X\eta\right]\left(\eta^{-1}\right)\right)}\tilde{M}\left(\eta^{-1}\right) \tag{5.1.7}$$

represents the magnetization in the current configuration.

5.1.2 Weak solutions

In this section we introduce our definition of weak solutions to the system (1.3.30)–(1.3.32), which describes the interplay between the deformation and the magnetization of a magnetoelastic material. First, however, we impose some additional conditions on the anisotropy energy density $\tilde{\Psi}$ and the elastic energy density W. More specifically, we assume that

$$\tilde{\Psi} \in C^1\left(\mathbb{R}^{3\times 3} \times \mathbb{R}^3; \mathbb{R}_0^+\right), \quad W \in C^1\left(\mathbb{R}^{3\times 3}; \mathbb{R}_0^+\right), \tag{5.1.8}$$

$$W(\theta) \geqslant c\left(|\theta|^{p_1} - 1\right) \qquad \forall \theta \in \mathbb{R}^{3\times 3}, \tag{5.1.9}$$

$$|W(\theta)| + |W'(\theta)| \leqslant c\left(1 + |\theta|^{p_2}\right) \qquad \forall \theta \in \mathbb{R}^{3\times 3}, \tag{5.1.10}$$

$$\left|\tilde{\Psi}(\theta,\xi)\right| + \left|\tilde{\Psi}_F(\theta,\xi)\right| \leqslant c\left(1 + |\theta|^{p_2} + |\xi|^{p_3}\right) \qquad \forall \theta \in \mathbb{R}^{3\times 3},\ \xi \in \mathbb{R}^3, \tag{5.1.11}$$

$$\left|\tilde{\Psi}_M(\theta,\xi)\right| \leqslant c\left(1 + |\theta|^{p_2} + |\xi|^{p_4}\right) \qquad \forall \theta \in \mathbb{R}^{3\times 3},\ \xi \in \mathbb{R}^3 \tag{5.1.12}$$

for some coefficients $p_1, p_2, p_3, p_4 \in \mathbb{R}$ satisfying $2 \leqslant p_1 < \infty$, $1 \leqslant p_2 < \infty$, $1 \leqslant p_3 < 6$, $1 \leqslant p_4 < 5$. Here $\tilde{\Psi}_F$, $\tilde{\Psi}_M$ denote the derivatives of $\tilde{\Psi}$ with respect to the first and the second variable respectively and $c > 0$ denotes a constant independent of θ and ξ. Our definition of weak solutions to the system (1.3.30)–(1.3.32) reads as follows.

Definition 5.1.1. *Let $\Omega_0 \subset \mathbb{R}^3$ be a bounded domain of class $C^{0,1}$. Let $N \subset \partial\Omega_0$, assume $P := \partial\Omega_0 \backslash N$ to have positive 2-dimensional Hausdorff measure $\mathcal{H}^2(P) > 0$ and let $\gamma : P \to \mathbb{R}^3$ be a given injective boundary deformation such that $\eta^\gamma|_P = \gamma$ for some deformation $\eta^\gamma \in W^{2,q}(\Omega_0)$ with $\tilde{E}_{\mathrm{el}}(\eta^\gamma) < \infty$. Let $\rho, A, \beta, \nu, \mu > 0$, $q > 3$ and $a > \frac{3q}{q-3}$ be some positive coefficients and consider some data $f \in L^\infty((0,\infty) \times \mathbb{R}^3)$, $H_{\mathrm{ext}} \in W^{1,\frac{4}{3}}(0,\infty; W^{1,\frac{4}{3}}(\mathbb{R}^3))$, $\eta_0 \in \mathrm{int}(\mathcal{E})$ and $\tilde{M}_0 \in H^1(\Omega_0)$ such that $\tilde{E}(\eta_0, \tilde{M}_0) < \infty$, where $\tilde{E}$ is defined in (1.3.34), and η_0 is injective on $\partial\Omega_0$. Assume further the data $\tilde{\Psi} \in C^1(\mathbb{R}^{3\times 3} \times \mathbb{R}^3)$, $W \in C^1(\mathbb{R}^{3\times 3})$ to satisfy the conditions (5.1.8)–(5.1.12). Then the system (1.3.30)–(1.3.32) is said to admit a weak solution on $[0,T)$ for some $T > 0$ if there exist functions*

$$\eta \in L^\infty(0,T;\mathcal{E}), \qquad \tilde{M} \in L^\infty\left(0,T;H^1(\Omega_0)\right) \tag{5.1.13}$$

with

$$\partial_t \eta \in L^2\left(0,T;H^1(\Omega_0)\right), \qquad \partial_t \tilde{M} \in L^2\left((0,T) \times \Omega_0\right), \tag{5.1.14}$$

such that the pair $(\eta, \tilde{M})$ satisfies

$$\begin{aligned} &\int_0^T \left\langle \tilde{E}_\eta\left(\eta, \tilde{M}\right), \chi \right\rangle_{(W^{2,q}(\Omega_0))^* \times W^{2,q}(\Omega_0)} + \left\langle \tilde{R}_{\partial_t \eta}\left(\eta, \partial_t \eta, \partial_t \tilde{M}\right), \chi \right\rangle_{(H^1(\Omega_0))^* \times H^1(\Omega_0)} dt \\ &- \int_0^T \int_{\Omega_0} \rho f(\eta) \cdot \chi + \mu \left[\left(\nabla_X \left(H_{\mathrm{ext}}(\eta)\right)\left(\nabla_X \eta\right)^{-1}\right)^T \tilde{M}\right] \cdot \chi \, dX dt = 0 \end{aligned} \tag{5.1.15}$$

for all $\chi \in \mathcal{D}((0,T) \times \Omega_0)$ as well as the initial conditions

$$\eta(0) = \eta_0, \qquad \tilde{M}(0) = \tilde{M}_0 \tag{5.1.16}$$

in the sense that

$$\lim_{t \to 0+} \|\eta(t) - \eta_0\|_{C^1(\overline{\Omega_0})} = 0, \qquad \lim_{t \to 0+} \left\|\tilde{M}(t) - \tilde{M}_0\right\|_{L^2(\Omega_0)} = 0 \tag{5.1.17}$$

and the pair

$$(\eta, M), \qquad M := M_\eta\left[\tilde{M}\right] = \frac{1}{\det\left(\left[\nabla_X \eta\right]\left(\eta^{-1}\right)\right)} \tilde{M}\left(\eta^{-1}\right) \in L^\infty\left(0,T;H^1(\Omega(\cdot))\right), \tag{5.1.18}$$

where $\Omega(t) := \eta(t, \Omega_0)$, *satisfies*

$$\int_0^T \left\langle E_M(\eta, M), \hat{M} \right\rangle_{(H^1(\Omega(t)))^* \times H^1(\Omega(t))} + \left\langle R_{D_t M}(\eta, v, D_t M), \hat{M} \right\rangle_{L^2(\Omega(t)) \times L^2(\Omega(t))} dt - \int_0^T \int_{\Omega(t)} \mu H_{\text{ext}} \cdot \hat{M} \; dx dt = 0 \tag{5.1.19}$$

for any test function $\hat{M} \in L^\infty(0, T; H^1(\Omega(\cdot)))$, *with the velocity field* v *defined in the current configuration via the relation*

$$v(t, \eta(t, X)) = \partial_t \eta(t, X) \quad \forall (t, X) \in [0, T] \times \Omega_0.$$

Remark 5.1.3. *The Fréchet derivatives in the equation of motion* (5.1.15) *can be calculated explicitly. Under exploitation of the identities*

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \det(\nabla_X(\eta + \epsilon\chi)) = \det(\nabla_X \eta) \operatorname{tr}\left(\nabla_X \chi (\nabla_X \eta)^{-1}\right),$$
$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} (\nabla_X(\eta + \epsilon\chi))^{-1} = -(\nabla_X \eta)^{-1} \nabla_X \chi (\nabla_X \eta)^{-1},$$

this calculation is straight forward, except for in the stray field part of the micromagnetic energy due to the implicit definition of the stray field via the Poisson equation (5.1.6)*. For this part, a lengthy calculation, the details of which are presented in Section A.8 in the appendix, shows that*

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \int_{\Omega_0} \frac{\mu}{2} \tilde{M}(t) \cdot H\left[\tilde{M}(t), \eta(t) + \epsilon\chi(t)\right](\eta(t) + \epsilon\chi(t)) \; dX = \int_{\Omega_0} \mu \left[\left(\left(\nabla_X H\left[\tilde{M}(t), \eta(t)\right](\eta(t))\right)(\nabla_X \eta(t))^{-1}\right)^T \tilde{M}(t)\right] \cdot \chi(t) \; dX \tag{5.1.20}$$

for $t \in [0, T]$ *and* $\chi \in \mathcal{D}((0, T) \times \Omega_0)$. *Altogether, the Fréchet derivatives in the equation of motion* (5.1.15) *may then be expressed as*

$$\begin{aligned}
&\int_0^T \left\langle \tilde{E}_\eta\left(\eta, \tilde{M}\right), \chi \right\rangle_{(W^{2,q}(\Omega_0))^* \times W^{2,q}(\Omega_0)} + \left\langle \tilde{R}_{\partial_t \eta}\left(\eta, \partial_t \eta, \partial_t \tilde{M}\right), \chi \right\rangle_{(H^1(\Omega_0))^* \times H^1(\Omega_0)} dt \\
&= \int_0^T \int_{\Omega_0} \left[W'(\nabla_X \eta) - a \frac{\operatorname{cof}(\nabla_X \eta)}{(\det(\nabla_X \eta))^a}\right] : \nabla_X \chi + \left|\nabla_X^2 \eta\right|^{q-2} \nabla_X^2 \eta \mathrel{\vdots} \nabla_X^2 \chi \; dX dt \\
&+ \int_0^T \int_{\Omega_0} \tilde{\Psi}_F(\nabla_X \eta, M) : \nabla_X \chi - \mu \left[\left(\left(\nabla_X H\left[\tilde{M}, \eta\right](\eta)\right)(\nabla_X \eta)^{-1}\right)^T \tilde{M}\right] \cdot \chi \\
&+ A \det(\nabla_X \eta) \left|\nabla_X\left(\frac{1}{\det(\nabla_X \eta)} \tilde{M}\right)(\nabla_X \eta)^{-1}\right|^2 \left((\nabla_X \eta)^{-1}\right)^T : \nabla_X \chi \\
&- 2A \det(\nabla_X \eta) \left[\nabla_X\left(\frac{1}{\det(\nabla_X \eta)} \tilde{M}\right)(\nabla_X \eta)^{-1}\right] : \left[\nabla_X\left(\frac{\operatorname{tr}\left(\nabla_X \chi (\nabla_X \eta)^{-1}\right)}{\det(\nabla_X \eta)} \tilde{M}\right)(\nabla_X \eta)^{-1}\right. \\
&\left. + \nabla_X\left(\frac{1}{\det(\nabla_X \eta)} \tilde{M}\right)(\nabla_X \eta)^{-1} \nabla_X \chi (\nabla_X \eta)^{-1}\right] \\
&+ \frac{1}{4\beta^2} \det(\nabla_X \eta) \left(\left|\frac{1}{\det(\nabla_X \eta)} \tilde{M}\right|^2 - 1\right)^2 \left((\nabla_X \eta)^{-1}\right)^T : \nabla_X \chi \\
&- \frac{1}{\beta^2 \det(\nabla_X \eta)} \left(\left|\frac{1}{\det(\nabla_X \eta)} \tilde{M}\right|^2 - 1\right) \left|\tilde{M}\right|^2 \left((\nabla_X \eta)^{-1}\right)^T : \nabla_X \chi \; dX dt \\
&+ \int_0^T \int_{\Omega_0} 2\nu \det(\nabla_X \eta) \left[\nabla_X \partial_t \eta (\nabla_X \eta)^{-1} \left((\nabla_X \eta)^{-1}\right)^T\right] : \nabla_X \chi \; dX dt.
\end{aligned} \tag{5.1.21}$$

Here, the first integral on the right-hand side represents the Fréchet derivative of the elastic energy (1.3.35)*, while the second and the third integral correspond to the micromagnetic energy* (1.3.36) *and the dissipation potential* (1.3.37)*, respectively. We may also calculate the Fréchet derivatives in the magnetic force balance* (5.1.19)*: For the stray field part we obtain, cf. Section A.8 in the appendix,*

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \int_{\Omega(t)} -\frac{\mu}{2}\left[M(t)+\epsilon\hat{M}(t)\right]\cdot H\left[\det\left(\nabla_X\eta(t)\right)\left(M(t)+\epsilon\hat{M}(t)\right),\eta(t)\right]\,dx$$
$$= \int_{\Omega(t)} -\mu H\left[\tilde{M}(t),\eta(t)\right]\cdot\hat{M}(t)\,dX \tag{5.1.22}$$

for $t\in[0,T]$ *and* $\hat{M}\in L^\infty(0,T;H^1(\Omega(\cdot)))$*. The calculation in the remaining terms is straight-forward and altogether the Fréchet derivatives in the magnetic force balance* (5.1.19) *may be written as*

$$\int_0^T \left\langle E_M(\eta,M),\hat{M}\right\rangle_{(H^1(\Omega(t)))^*\times H^1(\Omega(t))} + \left\langle R_{D_tM}(\eta,v,D_tM),\hat{M}\right\rangle_{L^2(\Omega(t))\times L^2(\Omega(t))}\,dt$$
$$= \int_0^T\int_{\Omega(t)} \tilde{\Psi}_M\left(\left[\nabla_X\eta\right]\left(\eta^{-1}\right),\det\left(\left[\nabla_X\eta\right]\left(\eta^{-1}\right)\right)M\right)\cdot\hat{M} - \mu H\left[\tilde{M},\eta\right]\cdot\hat{M} + 2A\nabla M:\nabla\hat{M}$$
$$+\frac{1}{\beta^2}\left(|M|^2-1\right)M\cdot\hat{M} + \left[\partial_t M + (v\cdot\nabla)M + (\nabla\cdot v)M\right]\cdot\hat{M}\,dxdt. \tag{5.1.23}$$

Remark 5.1.4. *The initial condition for the magnetization in Definition 5.1.1, which is formulated in the reference configuration in* (5.1.16) *and* (5.1.17)*, can be expressed equivalently in the current configuration. More specifically, for*

$$M_0 := M_{\eta_0}\left[\tilde{M}_0\right] = \frac{1}{\det\left(\left[\nabla_X\eta_0\right]\left(\eta_0^{-1}\right)\right)}\tilde{M}_0\left(\eta_0^{-1}\right),$$

a transformation between the reference configuration and the current configuration shows that the initial condition for $\tilde{M}$ *in* (5.1.16) *and* (5.1.17) *is equivalent to the relation*

$$\lim_{t\to 0+}\left\|M(t)-\tilde{M}_0\left(\eta_0\left(\eta^{-1}(t)\right)\right)\right\|_{L^2(\Omega(t))} = 0.$$

As it is more convenient, however, we choose the formulation in the reference configuration in Definition 5.1.

5.1.3 Main result

The main result of this chapter, which proves the existence of weak solutions to the system (1.3.30)–(1.3.32) as introduced in Definition 5.1.1, reads as follows.

Theorem 5.1.1. *Assume* $\Omega_0\subset\mathbb{R}^3$ *to be a bounded domain of class* $C^{0,1}$*. Let* $N\subset\partial\Omega_0$*, assume* $P:=\partial\Omega_0\backslash N$ *to have positive* 2*-dimensional Hausdorff measure* $\mathcal{H}^2(P)>0$ *and let* $\gamma:P\to\mathbb{R}^3$ *be a given boundary deformation such that* $\eta^\gamma|_P=\gamma$ *for some deformation* $\eta^\gamma\in W^{2,q}(\Omega_0)$ *with* $\tilde{E}_{\mathrm{el}}(\eta^\gamma)<\infty$*. Let* $\rho,A,\beta,\nu,\mu>0$*,* $q>3$ *and* $a>\frac{3q}{q-3}$ *be some positive coefficients and consider some data* $f\in L^\infty((0,\infty)\times\mathbb{R}^3)$*,* $H_{\mathrm{ext}}\in W^{1,\frac{4}{3}}(0,\infty;W^{1,\frac{4}{3}}(\mathbb{R}^3))$*,* $\eta_0\in\mathrm{int}(\mathcal{E})$ *and* $\tilde{M}_0\in H^1(\Omega_0)$ *such that* $\tilde{E}(\eta_0,\tilde{M}_0)<\infty$*, where* $\tilde{E}$ *is defined in* (1.3.34)*, and* η_0 *is injective on* $\partial\Omega_0$*. Assume further the data* $\tilde{\Psi}\in C^1(\mathbb{R}^{3\times3}\times\mathbb{R}^3)$*,* $W\in C^1(\mathbb{R}^{3\times3})$ *to satisfy the conditions* (5.1.8)*–*(5.1.12)*. Then there exists a time* $T'>0$ *such that the system* (1.3.30)*–*(1.3.32) *admits a weak solution* $(\eta,\tilde{M})$ *on* $[0,T')$ *in the sense of Definition 5.1.1. Moreover, the time* T' *can be chosen such that either* $T'=\infty$*, or* $\liminf_{t\to T'}\tilde{E}(\eta(t),\tilde{M}(t))=\infty$*, or* $\eta(T')\in\partial\mathcal{E}$*.*

Remark 5.1.5. *Theorem 5.1.1 needs to be understood as a local result in the sense that the test functions* χ *in the weak formulation* (5.1.15) *of the equation of motion are compactly supported. The reason for this lies in the regularity of the stray field: Indeed, the gradient of* $H[\tilde{M},\eta]$ *seems to be at best locally integrable in space, cf. Lemma A.2.3 in the appendix. For global integrability at least*

C^2-regularity of the current configuration $\Omega(t)$, $t \in [0,T]$, appears to be required. However, even if we assumed Ω_0 to be of class C^2, such a regularity could not be expected as the deformation is only known to possess $C^{1,\alpha}$-regularity, $\alpha = 1 - \frac{3}{q}$, in the spatial variable. A generalization of Theorem 5.1.1 to non-compactly supported test functions in the equation of motion remains an open problem for future research.

The proof of Theorem 5.1.1 will be achieved via an approximation method, which we present in the following Section and which is carried out in the Sections 5.3–5.4.

5.2 Approximate system

As in the fluid-structure interaction problems in Section 3.2 and Section 4.2 we introduce an approximate problem based on a time discretization. To this end, we again choose a parameter $\Delta t > 0$ and split up the interval $[0, \infty)$ into the discrete times $k\Delta t$, $k \in \mathbb{N}$. In contrast to our proofs for the fluid-structure interaction problems, however, we do not solve an approximate discrete system of equations directly. Instead we use De Giorgi's minimizing movements scheme (see [30]), i.e. at each discrete time $k\Delta t$ we solve a minimization problem and obtain discrete approximations of the equation of motion (5.1.15) and the magnetic force balance (5.1.19) as the corresponding Euler-Lagrange equations. Subsequently, passing to the limit in these equations, we obtain a solution to the original system.

5.2.1 Minimization problem

Before giving a detailed explanation of our approach we first present the full approximate problem: We fix an arbitrary discrete time $k\Delta t$, $k \in \mathbb{N}$. Given the solution $(\eta_{\Delta t}^{k-1}, \tilde{M}_{\Delta t}^{k-1}) \in \mathcal{E} \times H^1(\Omega_0)$ to the approximate problem at the time $(k-1)\Delta t$, we consider the minimization problem

$$\text{Find a minimizer} \quad \left(\eta_{\Delta t}^k, \tilde{M}_{\Delta t}^k\right) \in \mathcal{E} \times H^1(\Omega_0) \quad \text{of} \quad \tilde{F}_{\Delta t}^k(\cdot,\cdot) \quad \text{over} \quad \mathcal{E} \times H^1(\Omega_0), \tag{5.2.1}$$

where the functional $\tilde{F}_{\Delta t}^k : \mathcal{E} \times H^1(\Omega_0) \to \mathbb{R}$ is defined by

$$\begin{aligned} &\tilde{F}_{\Delta t}^k\left(\eta, \tilde{M}\right) \\ :=&\tilde{E}\left(\eta, \tilde{M}\right) + \Delta t \tilde{R}_{\Delta t}^k\left(\eta, \tilde{M}\right) - \int_{\Omega_0} \Delta t \rho f_{\Delta t}^k\left(\eta_{\Delta t}^{k-1}\right) \cdot \left(\frac{\eta - \eta_{\Delta t}^{k-1}}{\Delta t}\right) + \mu \tilde{M} \cdot (H_{\text{ext}})_{\Delta t}^k(\eta)\, dX \end{aligned} \tag{5.2.2}$$

with the energy potential $\tilde{E}$ defined in (1.3.34) and the discrete dissipation potential

$$\begin{aligned} &\tilde{R}_{\Delta t}^k\left(\eta, \tilde{M}\right) \\ :=& \int_{\Omega_0} \nu \left| \nabla_X \left(\frac{\eta - \eta_{\Delta t}^{k-1}}{\Delta t}\right) \left(\nabla_X \eta_{\Delta t}^{k-1}\right)^{-1} \right|^2 \det\left(\nabla_X \eta_{\Delta t}^{k-1}\right) \\ &+ \det\left(\nabla_X \eta_{\Delta t}^{k-1}\right) \frac{1}{2} \left| \frac{\frac{1}{\det(\nabla_X \eta)} \tilde{M} - \frac{1}{\det\left(\nabla_X \eta_{\Delta t}^{k-1}\right)} \tilde{M}_{\Delta t}^{k-1}}{\Delta t} + \frac{\operatorname{tr}\left(\nabla_X \left(\frac{\eta - \eta_{\Delta t}^{k-1}}{\Delta t}\right) \left(\nabla_X \eta_{\Delta t}^{k-1}\right)^{-1}\right)}{\det\left(\nabla_X \eta_{\Delta t}^{k-1}\right)} \tilde{M} \right|^2 dX. \end{aligned} \tag{5.2.3}$$

The discrete approximation $f_{\Delta t}^k$ of the given function f at the time $k\Delta t$ in the formula (5.2.2) is defined, in correspondence with the discretization of the forcing terms in the approximations of the fluid-structure interaction problems in Section 3.2 and Section 4.2, as

$$f_{\Delta t}^k := f_\gamma(k\Delta t), \quad f_\gamma(t) := \int_0^T \theta_\gamma\left(t + \xi_\gamma(t)) - s\right) f(s)\, ds, \quad \xi_\gamma(t) := \gamma \frac{T - 2t}{T}, \tag{5.2.4}$$

for a mollifier $\theta_\gamma : \mathbb{R} \to \mathbb{R}$ and a suitable choice of $\gamma = \gamma(\Delta t)$, $\gamma(\Delta t) \to 0$ for $\Delta t \to 0$. The discrete approximation $(H_{\text{ext}})^k_{\Delta t}$ of the given function H_{ext} at the time $k\Delta t$ instead is defined as the zero-order Clément quasi-interpolant

$$(H_{\text{ext}})^k_{\Delta t} := \frac{1}{\Delta t} \int_{(k-1)\Delta t}^{k\Delta t} H_{\text{ext}}(s)\ ds, \tag{5.2.5}$$

cf. [99, Remark 8.15]. Let us now discuss the ideas behind the above approximate problem: The general procedure in our proof follows closely the implementation of De Giorgi's minimizing movements scheme (cf. [30]) as used in [7, Section 2]. This approach which allows us to handle the coupling between the equation of motion (5.1.15) and the magnetic force balance (5.1.19). Indeed, at each fixed discrete time an approximate deformation and an approximate magnetization are determined simultaneously by solving only one single minimization problem, namely the problem (5.2.1). Both a discrete approximation of the equation of motion and a discrete approximation of the magnetic force balance are obtained subsequently as the Euler-Lagrange equations associated to this minimization problem, see Section 5.2.2 below. In particular, this method allows us to evade the utilization of fixed point arguments, which are usually used to solve coupled systems of partial differential equations and which typically presuppose convexity of the energy functional. It moreover helps us to deal with the non-convexity of the energy in another way: If we solved a discrete approximation of the equation of motion directly instead of via minimization, the classical way to obtain uniform (with respect to Δt) estimates for the deformation would be to test the equation at each time $k\Delta t$ by $\eta^k_{\Delta t} - \eta^{k-1}_{\Delta t}$. Then the desired bounds could be concluded provided that a discrete chain rule of the form

$$\tilde{E}\left(\eta^k_{\Delta t}, \tilde{M}\right) - \tilde{E}\left(\eta^{k-1}_{\Delta t}, \tilde{M}\right) \leqslant \int_{\Omega_0} \tilde{E}_\eta\left(\eta^k_{\Delta t}, \tilde{M}\right) \cdot \left(\eta^k_{\Delta t} - \eta^{k-1}_{\Delta t}\right)\ dX \qquad \forall \tilde{M} \in H^1(\Omega_0)$$

holds true. However, due to the non-convexity of $\tilde{E}$ in the first argument (due to the term (1.3.38)), such an estimate cannot be guaranteed. Since $\tilde{E}$ is further non-convex in the second argument (cf. the term (1.3.40)), a corresponding problem also arises for the magnetization. Nonetheless, the De Giorgi method provides the necessary bounds for the discrete solution in a different way. More precisely, we obtain a uniform estimate for both the energy and the (discrete) dissipation potential by comparing the value of the functional $\tilde{F}^k_{\Delta t}$ in its minimizer $(\eta^k_{\Delta t}, \tilde{M}^k_{\Delta t})$ to its value in the pair $(\eta^{k-1}_{\Delta t}, \tilde{M}^{k-1}_{\Delta t})$, cf. Section 5.4.1 below.

The main challenge in the application of De Giorgi's method to our problem is to choose the functional $\tilde{F}^k_{\Delta t}$ to the minimization problem (5.2.1) in such a way that its variation with respect to the deformation yields a suitable approximation of the equation of motion while its variation with respect to the magnetization leads to a suitable approximation of the magnetic force balance. This is difficult since for the continuous problem written in the form (1.3.30)–(1.3.32) it is not obvious that both equations can be expressed via the same energy and dissipation potentials. Indeed, for this to be possible, the extended material derivative (1.3.42) in the magnetic force balance (1.3.31) needs to appear in this dissipation potential without giving a contribution to the equation of motion. This difficulty, however, can be tackled by the term

$$\frac{1}{2}\left|\frac{1}{\det(\nabla_X \eta)} \partial_t \tilde{M}\right|^2 \det(\nabla_X \eta)$$

in the dissipation potential $\tilde{R}$ in (1.3.37). This term, being independent of $\partial_t \eta$, vanishes when the variation of $\tilde{R}$ is taken with respect to $\partial_t \eta$. Therefore, it plays no role in the equation of motion (1.3.30). Yet, when $\tilde{R}$ is transformed to the current configuration (cf. (1.3.41)), the term turns into $\frac{1}{2}|D_t M|^2$. This allows us to express the magnetic force balance in the desired form (1.3.43), in which the energy potential E and the dissipation potential R coincide with the corresponding potentials $\tilde{E}$ and $\tilde{R}$ from the equation of motion (1.3.30) formulated in the reference configuration. This knowledge offers us the opportunity to build the functional $\tilde{F}^k_{\Delta t}$ for our discrete minimization problem on the basis of $\tilde{E}$ and $\tilde{R}$. We do so in (5.2.2), (5.2.3), wherein the functional $\tilde{R}^k_{\Delta t}$ is chosen as a discretization of $\tilde{R}$: The first term in $\tilde{R}^k_{\Delta t}$ clearly represents a discretization of the first term in the definition (1.3.37) of $\tilde{R}$. The quantity

$$\frac{\frac{1}{\det(\nabla_X \eta)}\tilde{M} - \frac{1}{\det\left(\nabla_X \eta^{k-1}_{\Delta t}\right)}\tilde{M}^{k-1}_{\Delta t}}{\Delta t} + \frac{\operatorname{tr}\left(\nabla_X\left(\frac{\eta-\eta^{k-1}_{\Delta t}}{\Delta t}\right)\left(\nabla_X \eta^{k-1}_{\Delta t}\right)^{-1}\right)}{\det\left(\nabla_X \eta^{k-1}_{\Delta t}\right)}\tilde{M} \tag{5.2.6}$$

instead constitutes a discrete approximation of the extended material derivative $D_t M$ (cf. (1.3.42)) transformed to the reference configuration. Indeed, the first term in (5.2.6) corresponds to the classical material derivative whereas the second term corresponds to the additional quantity $(\nabla \cdot v)M$ in $D_t M$. In the limit passage with respect to $\Delta t \to 0$ (cf. the convergence (5.4.53) below) this quantity reduces to the expression $\frac{1}{\det(\nabla_X \eta)}\partial_t \tilde{M}$ appearing in the second term of $\tilde{R}$ in (1.3.37). Consequently, the functional $\tilde{R}^k_{\Delta t}$ makes up a suitable approximation of the dissipation potential $\tilde{R}$ defined in (1.3.37) and in particular the Euler-Lagrange equation (5.2.8) below, obtained by taking the variation of $\tilde{F}^k_{\Delta t}$ with respect to the magnetization, yields a suitable approximation of the weak form (5.1.19) of the magnetic force balance formulated in the reference configuration. We remark that formulating the approximate magnetic force balance in the reference configuration is in fact favorable, as the limit passage with respect to $\Delta t \to 0$ is more convenient in Lagrangian coordinates. The final magnetic force balance (5.1.19) is obtained afterwards by a simple transformation of the resulting limit identity to the current configuration. We further remark that, as opposed to its continuous counterpart $\frac{1}{\det(\nabla_X \eta)}\partial_t \tilde{M}$, the quantity (5.2.6) also gives a contribution to the discrete equation of motion (5.2.7) below. Indeed, the latter identity is obtained by taking the variation of $\tilde{F}^k_{\Delta t}$ with respect to the deformation itself instead of its (discrete) time derivative and hence also contains a contribution from the term (5.2.6). This contribution, however, can be seen to vanish as Δt tends to zero (cf. the convergence (5.4.67) below), leaving behind the desired continuous equation (5.1.15) after the limit passage.

Finally, we point out that the reason for discretizing H_{ext} via the zero-order Clément quasi interpolant (5.2.5) lies in the derivation of the energy estimate for the discretized system in Section 5.4.1 below: In order to deduce a bound independent of Δt we need to control the difference quotient of the chosen discretization of H_{ext} through the classical time derivative of H_{ext}. For the choice (5.2.5) of the discretization of H_{ext} this is possible via Lemma A.3.4 in the appendix.

In order to deal with the stray field, it is sometimes necessary to work in the current configuration instead of the reference configuration.Thus, it is further convenient to introduce some additional notation for the magnetization in the current configuration. To this end we denote the current configuration at the time $k\Delta t$ by

$$\Omega^k_{\Delta t} := \eta^k_{\Delta t}(\Omega_0)\,.$$

Then we define the magnetization in the current configuration via the formula

$$M^k_{\Delta t} := M_{\eta^k_{\Delta t}}\left[\tilde{M}^k_{\Delta t}\right] = \frac{1}{\det\left(\left[\nabla_X \eta^k_{\Delta t}\right]\left(\left(\eta^k_{\Delta t}\right)^{-1}\right)\right)}\tilde{M}^k_{\Delta t}\left(\left(\eta^k_{\Delta t}\right)^{-1}\right) \in H^1\left(\Omega^k_{\Delta t}\right).$$

5.2.2 Euler-Lagrange equations

The Euler-Lagrange equations associated to the functional $\tilde{F}^k_{\Delta t}$ constitute suitable discrete approximations of the equation of motion (5.1.15) and the magnetic force balance (5.1.19). If the minimizer $(\eta^k_{\Delta t}, \tilde{M}^k_{\Delta t}) \in \mathcal{E} \times H^1(\Omega_0)$ of $\tilde{F}^k_{\Delta t}$ is such that $\eta^k_{\Delta t} \notin \partial\mathcal{E}$, it holds that $\eta^k_{\Delta t} + \epsilon\chi \in \mathcal{E}$ for any $\chi \in \mathcal{D}(\Omega_0)$ and all sufficiently small $\epsilon = \epsilon(\chi) > 0$. This allows us to take the variation of $\tilde{F}^k_{\Delta t}$ at $(\eta^k_{\Delta t}, \tilde{M}^k_{\Delta t})$ with respect to the deformation. The variation of the stray field part can be calculated in the same way as its Fréchet derivative on the continuous level, cf. the identity (5.1.20) and its derivation in Section

A.8 in the appendix:

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \int_{\Omega_0} \frac{\mu}{2} \tilde{M}^k_{\Delta t} \cdot H\left[\tilde{M}^k_{\Delta t}, \eta^k_{\Delta t} + \epsilon\chi\right]\left(\eta^k_{\Delta t} + \epsilon\chi\right)\, dX$$
$$= \int_{\Omega_0} \mu \left[\left(\left(\nabla_X H\left[\tilde{M}^k_{\Delta t}, \eta^k_{\Delta t}\right]\left(\eta^k_{\Delta t}\right)\right)\left(\nabla_X \eta^k_{\Delta t}\right)^{-1}\right)^T \tilde{M}^k_{\Delta t}\right] \cdot \chi\, dX.$$

For the remaining parts of $\tilde{F}^k_{\Delta t}$ the calculation of the variation is straight forward and altogether we obtain the approximate equation of motion

$$\int_{\Omega_0} \left[W'\left(\nabla_X \eta^k_{\Delta t}\right) - a\frac{\operatorname{cof}\left(\nabla_X \eta^k_{\Delta t}\right)}{\left(\det\left(\nabla_X \eta^k_{\Delta t}\right)\right)^a}\right] : \nabla_X \chi + \left|\nabla^2_X \eta^k_{\Delta t}\right|^{q-2} \nabla^2_X \eta^k_{\Delta t} \vdots \nabla^2_X \chi$$
$$+ \tilde{\Psi}_F\left(\nabla_X \eta^k_{\Delta t}, \tilde{M}^k_{\Delta t}\right) : \nabla_X \chi - \mu\left[\left(\left(\nabla_X H\left[\tilde{M}^k_{\Delta t}, \eta^k_{\Delta t}\right]\left(\eta^k_{\Delta t}\right)\right)\left(\nabla_X \eta^k_{\Delta t}\right)^{-1}\right)^T \tilde{M}^k_{\Delta t}\right] \cdot \chi$$
$$+ A\det\left(\nabla_X \eta^k_{\Delta t}\right)\left|\nabla_X\left(\frac{1}{\det\left(\nabla_X \eta^k_{\Delta t}\right)}\tilde{M}^k_{\Delta t}\right)\left(\nabla_X \eta^k_{\Delta t}\right)^{-1}\right|^2 \left(\left(\nabla_X \eta^k_{\Delta t}\right)^{-1}\right)^T : \nabla_X \chi$$
$$- 2A\det\left(\nabla_X \eta^k_{\Delta t}\right)\left[\nabla_X\left(\frac{1}{\det\left(\nabla_X \eta^k_{\Delta t}\right)}\tilde{M}^k_{\Delta t}\right)\left(\nabla_X \eta^k_{\Delta t}\right)^{-1}\right] : \left[\nabla_X\left(\frac{\operatorname{tr}\left(\nabla_X\chi\left(\nabla_X \eta^k_{\Delta t}\right)^{-1}\right)}{\det\left(\nabla_X \eta^k_{\Delta t}\right)}\tilde{M}^k_{\Delta t}\right)\right.$$
$$\left.\left(\nabla_X \eta^k_{\Delta t}\right)^{-1} + \nabla_X\left(\frac{1}{\det\left(\nabla_X \eta^k_{\Delta t}\right)}\tilde{M}^k_{\Delta t}\right)\left(\nabla_X \eta^k_{\Delta t}\right)^{-1}\nabla_X\chi\left(\nabla_X \eta^k_{\Delta t}\right)^{-1}\right]$$
$$+ \frac{1}{4\beta^2}\det\left(\nabla_X \eta^k_{\Delta t}\right)\left(\left|\frac{1}{\det\left(\nabla_X \eta^k_{\Delta t}\right)}\tilde{M}^k_{\Delta t}\right|^2 - 1\right)^2 \left(\left(\nabla_X \eta^k_{\Delta t}\right)^{-1}\right)^T : \nabla_X \chi$$
$$- \frac{1}{\beta^2\det\left(\nabla_X \eta^k_{\Delta t}\right)}\left(\left|\frac{1}{\det\left(\nabla_X \eta^k_{\Delta t}\right)}\tilde{M}^k_{\Delta t}\right|^2 - 1\right)\left|\tilde{M}^k_{\Delta t}\right|^2 \left(\left(\nabla_X \eta^k_{\Delta t}\right)^{-1}\right)^T : \nabla_X \chi$$
$$+ 2\nu\det\left(\nabla_X \eta^{k-1}_{\Delta t}\right)\left[\nabla_X\left(\frac{\eta^k_{\Delta t} - \eta^{k-1}_{\Delta t}}{\Delta t}\right)\left(\nabla_X \eta^{k-1}_{\Delta t}\right)^{-1}\left(\left(\nabla_X \eta^{k-1}_{\Delta t}\right)^{-1}\right)^T\right] : \nabla_X \chi$$
$$+ \det\left(\nabla_X \eta^{k-1}_{\Delta t}\right)\left[\frac{\frac{1}{\det\left(\nabla_X \eta^k_{\Delta t}\right)}\tilde{M}^k_{\Delta t} - \frac{1}{\det\left(\nabla_X \eta^{k-1}_{\Delta t}\right)}\tilde{M}^{k-1}_{\Delta t}}{\Delta t} + \frac{\operatorname{tr}\left(\nabla_X\left(\frac{\eta^k_{\Delta t} - \eta^{k-1}_{\Delta t}}{\Delta t}\right)\left(\nabla_X \eta^{k-1}_{\Delta t}\right)^{-1}\right)}{\det\left(\nabla_X \eta^{k-1}_{\Delta t}\right)}\tilde{M}^k_{\Delta t}\right]$$
$$\cdot\left[-\frac{\operatorname{tr}\left(\nabla_X\chi\left(\nabla_X \eta^k_{\Delta t}\right)^{-1}\right)}{\det\left(\nabla_X \eta^k_{\Delta t}\right)}\tilde{M}^k_{\Delta t} + \frac{\operatorname{tr}\left(\nabla_X\chi\left(\nabla_X \eta^{k-1}_{\Delta t}\right)^{-1}\right)}{\det\left(\nabla_X \eta^{k-1}_{\Delta t}\right)}\tilde{M}^k_{\Delta t}\right]$$
$$- \rho f^k_{\Delta t}\left(\eta^{k-1}_{\Delta t}\right)\cdot\chi - \mu\left[\left(\nabla_X\left(\left(H_{\text{ext}}\right)^k_{\Delta t}\left(\eta^k_{\Delta t}\right)\right)\left(\nabla_X \eta^k_{\Delta t}\right)^{-1}\right)^T \tilde{M}^k_{\Delta t}\right]\cdot\chi\, dX = 0 \tag{5.2.7}$$

for all $\chi \in \mathcal{D}(\Omega_0)$. For the derivation of the approximate magnetic force balance we take the variation of $\tilde{F}^k_{\Delta t}$ at $(\eta^k_{\Delta t}, \tilde{M}^k_{\Delta t})$ with respect to the magnetization. Again, the variation of the stray field part is calculated similarly as on the continuous level, cf. the formula (5.1.22) and its derivation in Section A.8 in the appendix:

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \int_{\Omega_0} -\frac{\mu}{2}\left[\tilde{M}^k_{\Delta t} + \epsilon\tilde{\hat{M}}\right]\cdot H\left[\tilde{M}^k_{\Delta t} + \epsilon\tilde{\hat{M}}, \eta^k_{\Delta t}\right]\left(\eta^k_{\Delta t}\right)\, dX = \int_{\Omega_0} -\mu H\left[\tilde{M}^k_{\Delta t}, \eta^k_{\Delta t}\right]\left(\eta^k_{\Delta t}\right)\cdot\tilde{\hat{M}}\, dX$$

for all $\tilde{\hat{M}} \in H^1(\Omega_0)$. In the remaining parts of the functional the variation can be calculated in a straight forward way. Altogether we obtain the approximate magnetic force balance

$$\begin{aligned}
&\int_{\Omega_0} \tilde{\Psi}_M\left(\nabla_X \eta_{\Delta t}^k, \tilde{M}_{\Delta t}^k\right)\cdot \tilde{\hat{M}} - \mu H\left[\tilde{M}_{\Delta t}^k, \eta_{\Delta t}^k\right]\left(\eta_{\Delta t}^k\right)\cdot \tilde{\hat{M}} \\
&+ 2A\det\left(\nabla_X \eta_{\Delta t}^k\right)\left[\nabla_X\left(\frac{1}{\det\left(\nabla_X \eta_{\Delta t}^k\right)}\tilde{M}_{\Delta t}^k\right)\left(\nabla_X \eta_{\Delta t}^k\right)^{-1}\right] : \left[\nabla_X\left(\frac{1}{\det\left(\nabla_X \eta_{\Delta t}^k\right)}\tilde{\hat{M}}\right)\left(\nabla_X \eta_{\Delta t}^k\right)^{-1}\right] \\
&+ \frac{1}{\beta^2}\det\left(\nabla_X \eta_{\Delta t}^k\right)\left(\left|\frac{1}{\det\left(\nabla_X \eta_{\Delta t}^k\right)}\tilde{M}_{\Delta t}^k\right|^2 - 1\right)\left(\frac{1}{\det\left(\nabla_X \eta_{\Delta t}^k\right)}\tilde{M}_{\Delta t}^k\right)\cdot\left(\frac{1}{\det\left(\nabla_X \eta_{\Delta t}^k\right)}\tilde{\hat{M}}\right) \\
&+ \det\left(\nabla_X \eta_{\Delta t}^{k-1}\right)\left[\frac{\frac{1}{\det\left(\nabla_X \eta_{\Delta t}^k\right)}\tilde{M}_{\Delta t}^k - \frac{1}{\det\left(\nabla_X \eta_{\Delta t}^{k-1}\right)}\tilde{M}_{\Delta t}^{k-1}}{\Delta t} + \frac{\operatorname{tr}\left(\nabla_X\left(\frac{\eta_{\Delta t}^k - \eta_{\Delta t}^{k-1}}{\Delta t}\right)\left(\nabla_X \eta_{\Delta t}^{k-1}\right)^{-1}\right)}{\det\left(\nabla_X \eta_{\Delta t}^{k-1}\right)}\tilde{M}_{\Delta t}^k\right] \\
&\cdot\left[\frac{1}{\det\left(\nabla_X \eta_{\Delta t}^k\right)}\tilde{\hat{M}} + \frac{\operatorname{tr}\left(\nabla_X\left(\eta_{\Delta t}^k - \eta_{\Delta t}^{k-1}\right)\left(\nabla_X \eta_{\Delta t}^{k-1}\right)^{-1}\right)}{\det\left(\nabla_X \eta_{\Delta t}^{k-1}\right)}\tilde{\hat{M}}\right] - \mu\left(H_{\text{ext}}\right)_{\Delta t}^k\left(\eta_{\Delta t}^k\right)\cdot\tilde{\hat{M}}\,dX = 0
\end{aligned} \tag{5.2.8}$$

for all $\tilde{\hat{M}} \in H^1(\Omega_0)$.

5.3 Existence of the approximate solution

We fix some arbitrary discrete time index $k \in \mathbb{N}$ and assume that the minimization problem (5.2.1) has been solved for each time index $l = 1, ..., k-1$. By means of the direct method we show that the minimization problem (5.2.1) also possesses a solution at the discrete time $k\Delta t$. To this end we first check that the functional $\tilde{F}_{\Delta t}^k$ is bounded from below on $\mathcal{E} \times H^1(\Omega_0)$: We note that, by the definition of $H[\tilde{M}, \eta]$ via the Poisson equation (5.1.6),

$$\int_{\Omega_0} -\frac{\mu}{2}\tilde{M}\cdot H\left[\tilde{M},\eta\right](\eta)\,dX = \int_{\eta(\Omega_0)} -\frac{\mu}{2}M_\eta\left[\tilde{M}\right]\cdot H\left[\tilde{M},\eta\right]\,dx = \int_{\mathbb{R}^3}\frac{\mu}{2}\left|H\left[\tilde{M},\eta\right]\right|^2\,dx \geqslant 0. \tag{5.3.1}$$

Next we aim at controlling the f-dependent term in $\tilde{F}_{\Delta t}^k$. To this end we remark that, by Lemma A.7.1 in the appendix, the quantity $\det(\nabla_X \eta_{\Delta t}^{k-1})$ is bounded away from zero in dependence of only the value $\tilde{E}_{\text{el}}(\eta_{\Delta t}^{k-1})$. This allows us to estimate

$$\int_{\Omega_0}\nu\left|\nabla_X\left(\frac{\eta-\eta_{\Delta t}^{k-1}}{\Delta t}\right)\left(\nabla_X\eta_{\Delta t}^{k-1}\right)^{-1}\right|^2\det\left(\nabla_X\eta_{\Delta t}^{k-1}\right)\,dX \geqslant \nu c\left(\eta_{\Delta t}^{k-1}\right)\int_{\Omega_0}\left|\nabla_X\left(\frac{\eta-\eta_{\Delta t}^{k-1}}{\Delta t}\right)\right|^2\,dX$$

for a constant $c(\eta_{\Delta t}^{k-1}) > 0$ independent of η and $\tilde{M}$ which remains bounded away from 0 for bounded values of $\tilde{E}_{\text{el}}(\eta_{\Delta t}^{k-1})$. At the same time we exploit Young's inequality to estimate

$$\begin{aligned}
&\left|\int_{\Omega_0}\rho f_{\Delta t}^k\left(\eta_{\Delta t}^{k-1}\right)\cdot\left(\frac{\eta-\eta_{\Delta t}^{k-1}}{\Delta t}\right)\,dX\right| \\
\leqslant&\frac{\nu c\left(\eta_{\Delta t}^{k-1}\right)}{2\tilde{c}^2}\int_{\Omega_0}\left|\frac{\eta-\eta_{\Delta t}^{k-1}}{\Delta t}\right|^2\,dX + \frac{|\Omega_0|\left(\tilde{c}\rho\,\|f\|_{L^\infty((0,\infty)\times\mathbb{R}^3)}\right)^2}{2\nu c\left(\eta_{\Delta t}^{k-1}\right)} \\
\leqslant&\frac{\nu c\left(\eta_{\Delta t}^{k-1}\right)}{2}\int_{\Omega_0}\left|\nabla_X\left(\frac{\eta-\eta_{\Delta t}^{k-1}}{\Delta t}\right)\right|^2\,dX + \frac{|\Omega_0|\left(\tilde{c}\rho\,\|f\|_{L^\infty((0,\infty)\times\mathbb{R}^3)}\right)^2}{2\nu c\left(\eta_{\Delta t}^{k-1}\right)}
\end{aligned}$$

for the constant $\tilde{c} > 0$ from the Poincaré inequality on Ω_0. Combining the last two estimates we infer that

$$\begin{aligned}
&\int_{\Omega_0} \Delta t\nu \left|\nabla_X \left(\frac{\eta - \eta_{\Delta t}^{k-1}}{\Delta t}\right)\left(\nabla_X \eta_{\Delta t}^{k-1}\right)^{-1}\right|^2 \det\left(\nabla_X \eta_{\Delta t}^{k-1}\right) - \Delta t \rho f_{\Delta t}^k\left(\eta_{\Delta t}^{k-1}\right)\cdot\left(\frac{\eta - \eta_{\Delta t}^{k-1}}{\Delta t}\right)\, dX \\
\geqslant& \Delta t\left[\frac{\nu c\left(\eta_{\Delta t}^{k-1}\right)}{2}\int_{\Omega_0}\left|\nabla_X\left(\frac{\eta - \eta_{\Delta t}^{k-1}}{\Delta t}\right)\right|^2 dX - \frac{|\Omega_0|\left(\tilde{c}\rho\, \|f\|_{L^\infty((0,\infty)\times\mathbb{R}^3)}\right)^2}{2\nu c\left(\eta_{\Delta t}^{k-1}\right)}\right] \\
\geqslant& - c\left(\Delta t, \eta_{\Delta t}^{k-1}, \|f\|_{L^\infty((0,\infty)\times\mathbb{R}^3)}, \nu, \rho, \Omega_0\right)
\end{aligned} \tag{5.3.2}$$

for a constant $c(\Delta t, \eta_{\Delta t}^{k-1}, \|f\|_{L^\infty((0,\infty)\times\mathbb{R}^3)}, \nu, \rho, \Omega_0) > 0$ independent of η and $\tilde{M}$. Finally, in order to control the H_{ext}-dependent term in $\tilde{F}_{\Delta t}^k$, we first apply the Gagliardo-Nirenberg inequality and the Young inequality to estimate

$$\|\nabla_X\eta\|_{L^q(\Omega_0)} \leqslant c\left\|\nabla_X^2\eta\right\|_{L^q(\Omega_0)}^{\frac{5q-6}{7q-6}} \|\eta\|_{L^2(\Omega_0)}^{\frac{2}{7-\frac{6}{q}}} + c\,\|\eta\|_{L^2(\Omega_0)} \leqslant c\left\|\nabla_X^2\eta\right\|_{L^q(\Omega_0)} + c\,\|\eta\|_{L^2(\Omega_0)}$$

for a constant $c = c(\Omega_0, q) > 0$. From the Poincaré inequality it thus follows that

$$\|\eta\|_{W^{2,q}(\Omega_0)} \leqslant c\left\|\nabla_X^2\eta\right\|_{L^q(\Omega_0)} + c\,\|\nabla_X\eta\|_{L^{p_1}(\Omega_0)} + c(\gamma), \tag{5.3.3}$$

where $2 \leqslant p_1 < \infty$ is chosen as in the coercivity estimate (5.1.9) for W and the constant $c(\gamma)$ is due to the boundary condition $\eta|_P = \gamma$. Due to the Morrey embedding $W^{2,q}(\Omega_0) \subset W^{1,\infty}(\Omega_0)$ we infer the estimate

$$c_1\,\|\eta\|_{W^{1,\infty}(\Omega_0)}^2 \leqslant \int_{\Omega_0}\frac{1}{2}W\left(\nabla_X\eta\right) + \frac{1}{2q}\left|\nabla_X^2\eta\right|^q\, dX + 1 \tag{5.3.4}$$

for a constant $c_1 = c_1(\Omega_0, q, p_1, \gamma) > 0$. This, together with a transformation to the current configuration leads to the inequality

$$\begin{aligned}
&\int_{\Omega_0}\frac{1}{2}W\left(\nabla_X\eta\right) + \frac{1}{2q}\left|\nabla_X^2\eta\right|^q + \frac{1}{8\beta^2}\left(\left|\frac{1}{\det\left(\nabla_X\eta\right)}\tilde{M}\right|^2 - 1\right)^2\det\left(\nabla_X\eta\right) - \mu\tilde{M}\cdot\left(H_{\text{ext}}\right)_{\Delta t}^k(\eta)\, dX \\
\geqslant& c_1\,\|\eta\|_{W^{1,\infty}(\Omega_0)}^2 - 1 + \int_{\eta(\Omega_0)}\frac{1}{8\beta^2}\left(\left|M_\eta\left[\tilde{M}\right]\right|^2 - 1\right)^2 - \mu M_\eta\left[\tilde{M}\right]\cdot\left(H_{\text{ext}}\right)_{\Delta t}^k\, dx.
\end{aligned} \tag{5.3.5}$$

In order to further estimate the right-hand side of this inequality we make use of Young's inequality to see that

$$\frac{1}{8\beta^2}\left(\left|M_\eta\left[\tilde{M}\right]\right|^2 - 1\right)^2 \geqslant \frac{1}{16\beta^2}\left|M_\eta\left[\tilde{M}\right]\right|^4 - \frac{1}{8\beta^2} \tag{5.3.6}$$

as well as

$$\begin{aligned}
\left|\mu M_\eta\left[\tilde{M}\right]\cdot\left(H_{\text{ext}}\right)_{\Delta t}^k\right| =& \left|\left(\frac{1}{(2\beta)^{\frac{1}{2}}}M_\eta\left[\tilde{M}\right]\right)\cdot\left((2\beta)^{\frac{1}{2}}\mu\left(H_{\text{ext}}\right)_{\Delta t}^k\right)\right| \\
\leqslant& \frac{1}{16\beta^2}\left|M_\eta\left[\tilde{M}\right]\right|^4 + \frac{3\left((2\beta)^{\frac{1}{2}}\mu\left|\left(H_{\text{ext}}\right)_{\Delta t}^k\right|\right)^{\frac{4}{3}}}{4}.
\end{aligned} \tag{5.3.7}$$

Moreover, a transformation back to the reference configuration yields the estimate

$$\begin{aligned}
\int_{\eta(\Omega_0)}\frac{1}{8\beta^2}\, dx = \int_{\Omega_0}\frac{\det\left(\nabla_X\eta\right)}{8\beta^2}\, dX \leqslant& c_2\,\|\eta\|_{W^{1,\infty}(\Omega_0)} \\
=& \sqrt{2c_1}\,\|\eta\|_{W^{1,\infty}(\Omega_0)}\frac{c_2}{\sqrt{2c_1}} \leqslant c_1\,\|\eta\|_{W^{1,\infty}(\Omega_0)}^2 + \frac{c_2^2}{4c_1}
\end{aligned} \tag{5.3.8}$$

for the constant c_1 from the estimate (5.3.4) and another constant $c_2 = c_2(\Omega_0, \beta) > 0$. Applying the estimates (5.3.6)–(5.3.8) to the right-hand side of the inequality (5.3.5) we infer that

$$\begin{aligned}
&\int_{\Omega_0} \frac{1}{2} W\left(\nabla_X \eta\right) + \frac{1}{2q}\left|\nabla_X^2 \eta\right|^q + \frac{1}{8\beta^2}\left(\left|\frac{1}{\det\left(\nabla_X \eta\right)} \tilde{M}\right|^2 - 1\right)^2 \det\left(\nabla_X \eta\right) - \mu \tilde{M} \cdot \left(H_{\mathrm{ext}}\right)_{\Delta t}^k(\eta)\ dX \\
\geqslant & c_1 \|\eta\|_{W^{1,\infty}(\Omega_0)}^2 - 1 + \int_{\eta(\Omega_0)} \frac{1}{16\beta^2}\left|M_\eta\left[\tilde{M}\right]\right|^4 - \frac{1}{8\beta^2} - \frac{1}{16\beta^2}\left|M_\eta\left[\tilde{M}\right]\right|^4 - \frac{3\left(4\beta^2\mu^4\right)^{\frac{1}{3}}\left|\left(H_{\mathrm{ext}}\right)_{\Delta t}^k\right|^{\frac{4}{3}}}{4}\ dx \\
\geqslant & c_1 \|\eta\|_{W^{1,\infty}(\Omega_0)}^2 - 1 - c_1 \|\eta\|_{W^{1,\infty}(\Omega_0)}^2 - \frac{c_2^2}{4c_1} - \int_{\mathbb{R}^3} \frac{3\left(\beta^2\mu^4\right)^{\frac{1}{3}}\left|\left(H_{\mathrm{ext}}\right)_{\Delta t}^k\right|^{\frac{4}{3}}}{4}\ dx \\
\geqslant & -c\left(H_{\mathrm{ext}}, q, \beta, \mu\right)
\end{aligned} \tag{5.3.9}$$

for a constant $c\left(H_{\mathrm{ext}}, q, \beta, \mu\right) > 0$ independent of η and $\tilde{M}$. The estimate (5.3.1) shows that $\tilde{E}(\eta, \tilde{M})$ is non-negative. Hence, from the estimates (5.3.2) and (5.3.9) it follows that

$$\tilde{F}_{\Delta t}^k\left(\eta, \tilde{M}\right) \geqslant \frac{1}{2}\tilde{E}\left(\eta, \tilde{M}\right) - c \geqslant -c > -\infty \tag{5.3.10}$$

for all $(\eta, \tilde{M}) \in \mathcal{E} \times H^1(\Omega_0)$ and a constant $c > 0$ independent of η and $\tilde{M}$. Consequently, there exists a minimizing sequence $(\eta_j, \tilde{M}_j)_{j\in\mathbb{N}}$ for $\tilde{F}_{\Delta t}^k$. The inequality (5.3.10) further shows that this sequence satisfies the uniform bounds

$$\|\eta_j\|_{W^{2,q}(\Omega_0)} + \left\|\frac{1}{\det\left(\nabla_X \eta_j\right)}\right\|_{L^\infty(\Omega_0)} + \left\|\frac{\tilde{M}_j}{\det\left(\nabla_X \eta_j\right)}\right\|_{H^1(\Omega_0)} + \left\|\tilde{M}_j\right\|_{L^6(\Omega_0)} \leqslant c \tag{5.3.11}$$

for a constant $c > 0$ independent of j. Indeed, the bound of η_j in $W^{2,q}(\Omega_0)$ follows under exploitation of the inequality (5.3.3) and the coercivity (5.1.9) of W. The bound of $\det(\nabla_X \eta_j)$ away from zero follows from the boundedness of the quantity $\frac{1}{(\det(\nabla_X \eta_j))^a}$, cf. Lemma A.7.1 in the appendix, and in turn, in combination with the embedding $H^1(\Omega) \subset L^6(\Omega)$, implies the bound of $\tilde{M}_j$ in $L^6(\Omega)$. These bounds allow us to extract a subsequence such that, for some functions $\eta \in \mathcal{E}$, $\tilde{M} \in H^1(\Omega_0)$,

$$\eta_j \rightharpoonup \eta \quad \text{in } W^{2,q}\left(\Omega_0\right), \qquad \eta_j \to \eta \quad \text{in } C^1\left(\overline{\Omega_0}\right), \tag{5.3.12}$$

$$\frac{1}{\det\left(\nabla_X \eta_j\right)}\tilde{M}_j \rightharpoonup \frac{1}{\det\left(\nabla_X \eta\right)}\tilde{M} \quad \text{in } H^1\left(\Omega_0\right), \qquad \tilde{M}_j \to \tilde{M} \quad \text{in } L^p\left(\Omega_0\right) \tag{5.3.13}$$

for all $1 \leqslant p < 6$. In order to prove that the pair $(\eta, \tilde{M})$ is the desired minimizer of $\tilde{F}_{\Delta t}^k$, it remains to show that

$$\tilde{F}_{\Delta t}^k\left(\eta, \tilde{M}\right) \leqslant \liminf_{j\to\infty} \tilde{F}_{\Delta t}^k\left(\eta_j, \tilde{M}_j\right). \tag{5.3.14}$$

To this end, we first focus on the quantity $\tilde{M} \cdot H[\tilde{M}, \eta]$. From the estimate (A.2.9) for the solution to the Poisson equation (5.1.6) given by Lemma A.2.3 in the appendix we see that

$$\left\|\phi\left[\tilde{M}_j, \eta_j\right]\right\|_{\dot{H}^1(\mathbb{R}^3)} = \left\|\nabla\phi\left[\tilde{M}_j, \eta_j\right]\right\|_{L^2(\mathbb{R}^3)} \leqslant c$$

for a constant $c > 0$ independent of j. In particular we may assume that, for another subsequence and some function $\phi \in \dot{H}^1(\mathbb{R}^3)$,

$$\phi\left[\tilde{M}_j, \eta_j\right] \rightharpoonup \phi \quad \text{in } \dot{H}^1\left(\mathbb{R}^3\right).$$

In order to identify the limit function ϕ we see that by the Poisson equation (5.1.6), for any $\psi \in \mathcal{D}(\mathbb{R}^3)$,

$$\begin{aligned}
\int_{\mathbb{R}^3} \nabla\phi \cdot \nabla\psi\ dx \leftarrow & \int_{\mathbb{R}^3} \nabla\phi\left[\tilde{M}_j, \eta_j\right] \cdot \nabla\psi\ dx \\
= & \int_{\Omega_0} \tilde{M}_j \cdot \nabla\psi\left(\eta_j\right)\ dX \to \int_{\Omega_0} \tilde{M} \cdot \nabla\psi(\eta)\ dX = \int_{\eta(\Omega_0)} M_\eta\left[\tilde{M}\right] \cdot \nabla\psi\ dx
\end{aligned}$$

and due to the density of $\mathcal{D}(\mathbb{R}^3)$ in $\dot{H}^1(\mathbb{R}^3)$ (cf. [98, Lemma 3.3]) this identity in fact holds true for any $\psi \in \dot{H}^1(\mathbb{R}^3)$. Consequently, ϕ can be identified as the solution $\phi = \phi[\tilde{M}, \eta]$ to the Poisson equation with the right-hand side $M_\eta[\tilde{M}]$ and in particular it holds that

$$H\left[\tilde{M}_j, \eta_j\right] = -\nabla\phi\left[\tilde{M}_j, \eta_j\right] \rightharpoonup -\nabla\phi\left[\tilde{M}, \eta\right] = H\left[\tilde{M}, \eta\right] \qquad \text{in } L^2\left(\mathbb{R}^3\right).$$

Now the Poisson equation (5.1.6) and the weak lower semicontinuity of the $L^2(\mathbb{R}^3)$-norm imply that

$$\begin{aligned}\int_{\Omega_0} -\frac{\mu}{2}\tilde{M}\cdot H\left[\tilde{M},\eta\right](\eta)\;dX &= \int_{\mathbb{R}^3}\frac{\mu}{2}\left|H\left[\tilde{M},\eta\right]\right|^2\;dx\\ &\leqslant \liminf_{j\to\infty}\int_{\mathbb{R}^3}\frac{\mu}{2}\left|H\left[\tilde{M}_j,\eta_j\right]\right|^2\;dx\\ &= \liminf_{j\to\infty}\int_{\Omega_0}-\frac{\mu}{2}\tilde{M}_j\cdot H\left[\tilde{M}_j,\eta_j\right](\eta_j)\;dX. \end{aligned} \tag{5.3.15}$$

In order to deal with the quantity $\tilde{M}\cdot(H_{\text{ext}})^k_{\Delta t}(\eta)$ we notice that, by Corollary A.7.1 for the convergence of compositions with the deformation,

$$(H_{\text{ext}})^k_{\Delta t}(\eta_j) \to (H_{\text{ext}})^k_{\Delta t}(\eta) \quad \text{in } L^{\frac{4}{3}}(\Omega_0).$$

Combining this with the convergence (5.3.13) of $\tilde{M}_j$ we conclude that

$$\int_{\Omega_0} -\mu\tilde{M}_j\cdot(H_{\text{ext}})^k_{\Delta t}(\eta_j)\;dX \to \int_{\Omega_0} -\mu\tilde{M}\cdot(H_{\text{ext}})^k_{\Delta t}(\eta)\;dX. \tag{5.3.16}$$

Moreover, due to the boundedness assumptions (5.1.10), (5.1.11) on W and $\tilde{\Psi}$ it holds that

$$\left|W(\nabla_X\eta_j) + \tilde{\Psi}\left(\nabla_X\eta_j, \tilde{M}_j\right)\right| \leqslant c\left(1 + |\nabla_X\eta_j|^{p_2} + \left|\tilde{M}_j\right|^{p_3}\right),$$

where $1 \leqslant p_2 < \infty$ and $1 \leqslant p_3 < 6$. Together with the uniform bounds (5.3.11) this implies that

$$\left\|W(\nabla_X\eta_j) + \tilde{\Psi}\left(\nabla_X\eta_j, \tilde{M}_j\right)\right\|_{L^p(\Omega_0)} \leqslant c$$

for some $p > 1$. Thus, due to the convergences (5.3.12), (5.3.13), the continuity (5.1.8) of W and $\tilde{\Psi}$ and the Vitali convergence theorem, we may assume that

$$\int_{\Omega_0} W(\nabla_X\eta_j) + \tilde{\Psi}\left(\nabla_X\eta_j, \tilde{M}_j\right)\;dX \to \int_{\Omega_0} W(\nabla_X\eta) + \tilde{\Psi}\left(\nabla_X\eta, \tilde{M}\right)\;dX. \tag{5.3.17}$$

Finally the weak lower semicontinuity of $\tilde{F}^k_{\Delta t}$ in the remaining terms follows directly from the convergences (5.3.12), (5.3.13) and the weak lower semicontinuity of norms,

$$\begin{aligned}&\int_{\Omega_0}\frac{1}{(\det(\nabla_X\eta))^a} + \frac{1}{q}\left|\nabla^2_X\eta\right|^q + A\left|\nabla_X\left(\frac{1}{\det(\nabla_X\eta)}\tilde{M}\right)(\nabla_X\eta)^{-1}\right|^2\det(\nabla_X\eta)\\ &+\frac{1}{4\beta^2}\left(\left|\frac{1}{\det(\nabla_X\eta)}\tilde{M}\right|^2 - 1\right)^2\det(\nabla_X\eta)\;dX + \Delta t R^k_{\Delta t}\left(\eta,\tilde{M}\right)\\ &-\int_{\Omega_0}\Delta t\rho f^k_{\Delta t}\left(\eta^{k-1}_{\Delta t}\right)\cdot\left(\frac{\eta-\eta^{k-1}_{\Delta t}}{\Delta t}\right)\;dX\\ \leqslant \liminf_{j\to\infty}&\left[\int_{\Omega_0}\frac{1}{(\det(\nabla_X\eta_j))^a} + \frac{1}{q}\left|\nabla^2_X\eta_j\right|^q + A\left|\nabla_X\left(\frac{1}{\det(\nabla_X\eta_j)}\tilde{M}_j\right)(\nabla_X\eta_j)^{-1}\right|^2\det(\nabla_X\eta_j)\right.\\ &+\frac{1}{4\beta^2}\left(\left|\frac{1}{\det(\nabla_X\eta_j)}\tilde{M}_j\right|^2 - 1\right)^2\det(\nabla_X\eta_j)\;dX + \Delta t R^k_{\Delta t}\left(\eta_j,\tilde{M}_j\right)\\ &\left.-\int_{\Omega_0}\Delta t\rho f^k_{\Delta t}\left(\eta^{k-1}_{\Delta t}\right)\cdot\left(\frac{\eta_j-\eta^{k-1}_{\Delta t}}{\Delta t}\right)\;dX\right].\end{aligned} \tag{5.3.18}$$

From the relations (5.3.15), (5.3.16), (5.3.17) and (5.3.18) we infer the desired weak lower semicontinuity (5.3.14) of $\tilde{F}^k_{\Delta t}$. This proves the existence of the desired solution $(\eta^k_{\Delta t}, \tilde{M}^k_{\Delta t}) = (\eta, \tilde{M})$ to the minimization problem (5.2.1). Thus, recalling the Euler-Lagrange equations (5.2.7) and (5.2.8) associated to the functional $\tilde{F}^k_{\Delta t}$, we have proved the following result.

Proposition 5.3.1. *Let all the assumptions of Theorem 5.1.1 be satisfied and let $\Delta t > 0$. Let further $f^k_{\Delta t}$ be given by* (5.2.4) *and $(H_{ext})^k_{\Delta t}$ be given by* (5.2.5) *for any $k \in \mathbb{N}_0$. Then, for all $k \in \mathbb{N}$, there exists a solution*

$$\left(\eta^k_{\Delta t}, \tilde{M}^k_{\Delta t}\right) \in \mathcal{E} \times H^1(\Omega_0)$$

to the minimization problem (5.2.1)*. Further, for any $k \in \mathbb{N}$ such that $\eta^k_{\Delta t} \notin \partial\mathcal{E}$, the pair $(\eta^k_{\Delta t}, \tilde{M}^k_{\Delta t})$ satisfies the equation of motion* (5.2.7) *for all $\chi \in \mathcal{D}(\Omega_0)$. Moreover, for any $k \in \mathbb{N}$, the pair $(\eta^k_{\Delta t}, \tilde{M}^k_{\Delta t})$ satisfies the magnetic force balance* (5.2.8) *for all $\hat{\tilde{M}} \in H^1(\Omega_0)$.*

5.4 Limit passage with respect to $\Delta t \to 0$

In order to return from the discrete system to the original continuous system we pass to the limit with respect to $\Delta t \to 0$. As in the fluid-structure interaction problems in Section 3.4 and Section 4.4 we define piecewise affine and piecewise constant interpolants of the discrete quantities: For all time-independent functions $h^k_{\Delta t}$, $k \in \mathbb{N}_0$ we set

$$\begin{aligned}
h_{\Delta t}(t) &:= \left(\frac{t}{\Delta t} - (k-1)\right) h^k_{\Delta t} + \left(k - \frac{t}{\Delta t}\right) h^{k-1}_{\Delta t} && \forall t \in ((k-1)\Delta t, k\Delta t], \quad k \in \mathbb{N}, && (5.4.1)\\
\overline{h}_{\Delta t}(t) &:= h^k_{\Delta t} && \forall t \in ((k-1)\Delta t, k\Delta t], \quad k \in \mathbb{N}_0, && (5.4.2)\\
\overline{h}'_{\Delta t}(t) &:= h^{k-1}_{\Delta t} && \forall t \in ((k-1)\Delta t, k\Delta t], \quad k \in \mathbb{N}.
\end{aligned}$$

Similarly we assemble the discrete deformed configurations and the discrete dissipation potentials,

$$\begin{aligned}
\overline{\overline{\Omega}}_{\Delta t}(t) :=& \Omega^k_{\Delta t} = \eta^k_{\Delta t}(\Omega_0) && \forall t \in ((k-1)\Delta t, k\Delta t], \quad k \in \mathbb{N}, && (5.4.3)\\
\overline{\tilde{R}}_{\Delta t}(t) :=& \tilde{R}^k_{\Delta t} && \forall t \in ((k-1)\Delta t, k\Delta t], \quad k \in \mathbb{N}.
\end{aligned}$$

We remark that we use the notation $\overline{\overline{\Omega}}_{\Delta t}(t)$ instead of $\overline{\Omega}_{\Delta t}(t)$ for the piecewise constant interpolant of the deformed configuration in order to avoid confusion with the notation for the closure of sets. The interpolated functions allow us to express the discrete equation of motion (5.2.7) as well as the discrete magnetic force balance (5.2.8) as time-dependent equations. Indeed, in case of the equation of motion, we choose $T > 0$ such that $\frac{T}{\Delta t} \in \mathbb{N}$ and $\eta^k_{\Delta t} \notin \partial\mathcal{E}$ for all $k = 1, ..., \frac{T}{\Delta t}$. The existence of such a time T independent of Δt is shown below in Lemma 5.4.2. Further we choose some arbitrary test function $\chi \in \mathcal{D}((0,T) \times \Omega_0)$. Then, for all $t \in [(k-1)\Delta t, k\Delta t]$, $k = 1, ..., \frac{T}{\Delta t}$ we may test the discrete equation of motion (5.2.7) at the time $k\Delta t$ by $\chi(t)$. We integrate the resulting identity over

$[(k-1)\Delta t, k\Delta t]$, sum over k and obtain the identity

$$\int_0^T \int_{\Omega_0} \left[W'(\nabla_X \overline{\eta}_{\Delta t}) - a \frac{\operatorname{cof}(\nabla_X \overline{\eta}_{\Delta t})}{(\det(\nabla_X \overline{\eta}_{\Delta t}))^a} \right] : \nabla_X \chi + \left|\nabla_X^2 \overline{\eta}_{\Delta t}\right|^{q-2} \nabla_X^2 \overline{\eta}_{\Delta t} \,\vdots\, \nabla_X^2 \chi$$
$$+ \tilde{\Psi}_F \left(\nabla_X \overline{\eta}_{\Delta t}, \overline{\tilde{M}}_{\Delta t}\right) : \nabla_X \chi - \mu \left[\left(\left(\nabla_X H \left[\overline{\tilde{M}}_{\Delta t}, \overline{\eta}_{\Delta t}\right](\overline{\eta}_{\Delta t}) \right) (\nabla_X \overline{\eta}_{\Delta t})^{-1} \right)^T \overline{\tilde{M}}_{\Delta t} \right] \cdot \chi$$
$$+ A \det(\nabla_X \overline{\eta}_{\Delta t}) \left| \nabla_X \left(\frac{1}{\det(\nabla_X \overline{\eta}_{\Delta t})} \overline{\tilde{M}}_{\Delta t} \right) (\nabla_X \overline{\eta}_{\Delta t})^{-1} \right|^2 \left((\nabla_X \overline{\eta}_{\Delta t})^{-1} \right)^T : \nabla_X \chi$$
$$- 2A \det(\nabla_X \overline{\eta}_{\Delta t}) \left[\nabla_X \left(\frac{1}{\det(\nabla_X \overline{\eta}_{\Delta t})} \overline{\tilde{M}}_{\Delta t} \right) (\nabla_X \overline{\eta}_{\Delta t})^{-1} \right] : \left[\nabla_X \left(\frac{\operatorname{tr}\left(\nabla_X \chi (\nabla_X \overline{\eta}_{\Delta t})^{-1}\right)}{\det(\nabla_X \overline{\eta}_{\Delta t})} \overline{\tilde{M}}_{\Delta t} \right) \right.$$
$$\left. (\nabla_X \overline{\eta}_{\Delta t})^{-1} + \nabla_X \left(\frac{1}{\det(\nabla_X \overline{\eta}_{\Delta t})} \overline{\tilde{M}}_{\Delta t} \right) (\nabla_X \overline{\eta}_{\Delta t})^{-1} \nabla_X \chi (\nabla_X \overline{\eta}_{\Delta t})^{-1} \right]$$
$$+ \frac{1}{4\beta^2} \det(\nabla_X \overline{\eta}_{\Delta t}) \left(\left| \frac{1}{\det(\nabla_X \overline{\eta}_{\Delta t})} \overline{\tilde{M}}_{\Delta t} \right|^2 - 1 \right)^2 \left((\nabla_X \overline{\eta}_{\Delta t})^{-1} \right)^T : \nabla_X \chi$$
$$- \frac{1}{\beta^2 \det(\nabla_X \overline{\eta}_{\Delta t})} \left(\left| \frac{1}{\det(\nabla_X \overline{\eta}_{\Delta t})} \overline{\tilde{M}}_{\Delta t} \right|^2 - 1 \right) \left| \overline{\tilde{M}}_{\Delta t} \right|^2 \left((\nabla_X \overline{\eta}_{\Delta t})^{-1} \right)^T : \nabla_X \chi$$
$$+ 2\nu \det\left(\nabla_X \overline{\eta}'_{\Delta t}\right) \left[\nabla_X \left(\frac{\overline{\eta}_{\Delta t} - \overline{\eta}'_{\Delta t}}{\Delta t} \right) (\nabla_X \overline{\eta}'_{\Delta t})^{-1} \left((\nabla_X \overline{\eta}'_{\Delta t})^{-1} \right)^T \right] : \nabla_X \chi$$
$$+ \det\left(\nabla_X \overline{\eta}'_{\Delta t}\right) \left[\frac{\frac{1}{\det(\nabla_X \overline{\eta}_{\Delta t})} \overline{\tilde{M}}_{\Delta t} - \frac{1}{\det\left(\nabla_X \overline{\eta}'_{\Delta t}\right)} \overline{\tilde{M}}'_{\Delta t}}{\Delta t} + \frac{\operatorname{tr}\left(\nabla_X \left(\frac{\overline{\eta}_{\Delta t} - \overline{\eta}'_{\Delta t}}{\Delta t} \right) (\nabla_X \overline{\eta}'_{\Delta t})^{-1} \right)}{\det\left(\nabla_X \overline{\eta}'_{\Delta t}\right)} \overline{\tilde{M}}_{\Delta t} \right]$$
$$\cdot \left[- \frac{\operatorname{tr}\left(\nabla_X \chi (\nabla_X \overline{\eta}_{\Delta t})^{-1}\right)}{\det(\nabla_X \overline{\eta}_{\Delta t})} \overline{\tilde{M}}_{\Delta t} + \frac{\operatorname{tr}\left(\nabla_X \chi (\nabla_X \overline{\eta}'_{\Delta t})^{-1}\right)}{\det\left(\nabla_X \overline{\eta}'_{\Delta t}\right)} \overline{\tilde{M}}_{\Delta t} \right]$$
$$- \rho \overline{f}_{\Delta t}\left(\overline{\eta}'_{\Delta t}\right) \cdot \chi - \mu \left[\left(\nabla_X \left(\overline{(H_{\text{ext}})}_{\Delta t} (\overline{\eta}_{\Delta t}) \right) (\nabla_X \overline{\eta}_{\Delta t})^{-1} \right)^T \overline{\tilde{M}}_{\Delta t} \right] \cdot \chi \, dX dt = 0 \tag{5.4.4}$$

for all $\chi \in \mathcal{D}((0,T) \times \Omega_0)$. Similarly, for the magnetic force balance, we consider an arbitrary function $\hat{\tilde{M}} \in L^\infty(0,T; H^1(\Omega_0))$. Then, for all $k = 1, \ldots, \frac{T}{\Delta t}$ and almost all $t \in [(k-1)\Delta t, k\Delta t]$ we may test the discrete magnetic force balance (5.2.8) by $\hat{\tilde{M}}(t,\cdot)$. Integrating the resulting identity over $[(k-1)\Delta t, k\Delta t]$ and summing over k we deduce the equation

$$\int_0^T \int_{\Omega_0} \tilde{\Psi}_M \left(\nabla_X \overline{\eta}_{\Delta t}, \overline{\tilde{M}}_{\Delta t}\right) \cdot \hat{\tilde{M}} - \mu H \left[\overline{\tilde{M}}_{\Delta t}, \overline{\eta}_{\Delta t}\right](\overline{\eta}_{\Delta t}) \cdot \hat{\tilde{M}}$$
$$+ 2A \det(\nabla_X \overline{\eta}_{\Delta t}) \left[\nabla_X \left(\frac{1}{\det(\nabla_X \overline{\eta}_{\Delta t})} \overline{\tilde{M}}_{\Delta t} \right) (\nabla_X \overline{\eta}_{\Delta t})^{-1} \right] : \left[\nabla_X \left(\frac{1}{\det(\nabla_X \overline{\eta}_{\Delta t})} \hat{\tilde{M}} \right) (\nabla_X \overline{\eta}_{\Delta t})^{-1} \right]$$
$$+ \frac{1}{\beta^2} \det(\nabla_X \overline{\eta}_{\Delta t}) \left(\left| \frac{1}{\det(\nabla_X \overline{\eta}_{\Delta t})} \overline{\tilde{M}}_{\Delta t} \right|^2 - 1 \right) \left(\frac{1}{\det(\nabla_X \overline{\eta}_{\Delta t})} \overline{\tilde{M}}_{\Delta t} \right) \cdot \left(\frac{1}{\det(\nabla_X \overline{\eta}_{\Delta t})} \hat{\tilde{M}} \right)$$
$$+ \det\left(\nabla_X \overline{\eta}'_{\Delta t}\right) \left[\frac{\frac{1}{\det(\nabla_X \overline{\eta}_{\Delta t})} \overline{\tilde{M}}_{\Delta t} - \frac{1}{\det\left(\nabla_X \overline{\eta}'_{\Delta t}\right)} \overline{\tilde{M}}'_{\Delta t}}{\Delta t} + \frac{\operatorname{tr}\left(\nabla_X \left(\frac{\overline{\eta}_{\Delta t} - \overline{\eta}'_{\Delta t}}{\Delta t} \right) (\nabla_X \overline{\eta}'_{\Delta t})^{-1} \right)}{\det\left(\nabla_X \overline{\eta}'_{\Delta t}\right)} \overline{\tilde{M}}_{\Delta t} \right]$$
$$\cdot \left[\frac{1}{\det(\nabla_X \overline{\eta}_{\Delta t})} \hat{\tilde{M}} + \frac{\operatorname{tr}\left((\nabla_X \overline{\eta}_{\Delta t} - \nabla_X \overline{\eta}'_{\Delta t}) (\nabla_X \overline{\eta}'_{\Delta t})^{-1} \right)}{\det\left(\nabla_X \overline{\eta}'_{\Delta t}\right)} \hat{\tilde{M}} \right] - \mu \overline{(H_{\text{ext}})}_{\Delta t} (\overline{\eta}_{\Delta t}) \cdot \hat{\tilde{M}} \, dX dt = 0 \tag{5.4.5}$$

for all $\hat{\tilde{M}} \in L^\infty(0,T; H^1(\Omega_0))$.

5.4.1 Energy inequality on the Δt-level

Our next step is to establish an interval $[0,T]$ with some $T > 0$ independent of Δt on which we are able to find an energy estimate for the discrete solution, uniform with respect to $\Delta t > 0$. More precisely, we show the following lemma.

Lemma 5.4.1. *There exists a time $T > 0$, independent of Δt, and a constant $c > 0$, independent of Δt and $k = 1, ..., \frac{T}{\Delta t}$, such that*

$$\tilde{E}\left(\eta_{\Delta t}^k, \tilde{M}_{\Delta t}^k\right) + \sum_{l=1}^{k} \Delta t \tilde{R}_{\Delta t}^l \left(\eta_{\Delta t}^l, \tilde{M}_{\Delta t}^l\right) \leqslant c \qquad \forall k = 1, ..., \frac{T}{\Delta t}. \tag{5.4.6}$$

The proof of this lemma is mostly standard with the only difficulties being caused by the term depending on the external forcing term f in the discrete functional $\tilde{F}_{\Delta t}^k$. Indeed, in order to control this term during the derivation of the estimate (5.4.6) for some fixed k, we already need to know a uniform bound of $\det(\nabla_X \eta_{\Delta t}^{k-1})$ away from zero, cf. the deduction of the estimate (5.3.2). We achieve this via an induction argument, allowing us to assume $\tilde{E}_{\text{el}}(\eta_{\Delta t}^{k-1})$ to be bounded.

Proof of Lemma 5.4.1

We choose some number $E_0 > 0$ such that

$$\tilde{E}(\eta_0, \tilde{M}_0) < c_4 < E_0,$$

for the constant $c_4 > 0$ chosen below in (5.4.15), dependent on the data but independent of $\Delta t > 0$. We further choose a time $T = T(E_0) > 0$ sufficiently small such that

$$\left[c_4 + T\left(c_3 + \frac{|\Omega_0| \left(\tilde{c}\rho \, \|f\|_{L^\infty((0,\infty)\times\mathbb{R}^3)}\right)^2}{2\nu c_5}\right)\right] e^{\max\{1,\ 8\beta^2\}T} \leqslant E_0, \tag{5.4.7}$$

where $\tilde{c} > 0$ denotes the constant from the Poincaré inequality on Ω_0 and $c_3, c_5 > 0$ denote the constants chosen below in (5.4.12) and (5.4.16), respectively, dependent on E_0 and the data but independent of $\Delta t > 0$ and $k = 1, ..., \frac{T}{\Delta t}$. We argue via induction: We choose an arbitrary discrete time index $k = 1, ..., \frac{T}{\Delta t}$ and assume that

$$\frac{1}{2}\tilde{E}\left(\eta_{\Delta t}^l, \tilde{M}_{\Delta t}^l\right) \leqslant E_0 \qquad \forall l = 1, ..., k-1. \tag{5.4.8}$$

Each pair $(\eta_{\Delta t}^l, \tilde{M}_{\Delta t}^l)$, $l = 1, ..., k$, as a minimizer of the functional $\tilde{F}_{\Delta t}^l$, satisfies

$$\begin{aligned}
&\tilde{E}\left(\eta_{\Delta t}^l, \tilde{M}_{\Delta t}^l\right) + \Delta t \tilde{R}_{\Delta t}^l\left(\eta_{\Delta t}^l, \tilde{M}_{\Delta t}^l\right) - \int_{\Omega_0} \Delta t \rho f_{\Delta t}^l\left(\eta_{\Delta t}^{l-1}\right) \cdot \left(\frac{\eta_{\Delta t}^l - \eta_{\Delta t}^{l-1}}{\Delta t}\right) \\
&+ \mu \tilde{M}_{\Delta t}^l \cdot (H_{\text{ext}})_{\Delta t}^l\left(\eta_{\Delta t}^l\right) \, dX \\
\leqslant &\tilde{E}\left(\eta_{\Delta t}^{l-1}, \tilde{M}_{\Delta t}^{l-1}\right) - \int_{\Omega_0} \mu \tilde{M}_{\Delta t}^{l-1} \cdot (H_{\text{ext}})_{\Delta t}^l\left(\eta_{\Delta t}^{l-1}\right) \, dX.
\end{aligned}$$

We sum this inequality over all indices $l = 1, ..., k$ to infer that

$$\begin{aligned}
&\tilde{E}\left(\eta_{\Delta t}^k, \tilde{M}_{\Delta t}^k\right) + \Delta t \sum_{l=1}^{k} \tilde{R}_{\Delta t}^l\left(\eta_{\Delta t}^l, \tilde{M}_{\Delta t}^l\right) \\
\leqslant &\tilde{E}\left(\eta_0, \tilde{M}_0\right) + \sum_{l=1}^{k}\left[\int_{\Omega_0} \mu \tilde{M}_{\Delta t}^l \cdot (H_{\text{ext}})_{\Delta t}^l\left(\eta_{\Delta t}^l\right) \, dX - \int_{\Omega_0} \mu \tilde{M}_{\Delta t}^{l-1} \cdot (H_{\text{ext}})_{\Delta t}^l\left(\eta_{\Delta t}^{l-1}\right) \, dX\right] \\
&+ \Delta t \sum_{l=1}^{k} \int_{\Omega_0} \rho f_{\Delta t}^l\left(\eta_{\Delta t}^{l-1}\right) \cdot \left(\frac{\eta_{\Delta t}^l - \eta_{\Delta t}^{l-1}}{\Delta t}\right) \, dX.
\end{aligned} \tag{5.4.9}$$

In order to control the first sum on the right-hand side we denote by $\chi_{\Omega^l_{\Delta t}}$ the characteristic function of $\Omega^l_{\Delta t}$. Then we use a transformation to the current configuration and sum by parts to see that

$$\begin{aligned}
&\sum_{l=1}^{k}\left[\int_{\Omega_0}\mu\tilde{M}^l_{\Delta t}\cdot(H_{\text{ext}})^l_{\Delta t}\left(\eta^l_{\Delta t}\right)\,dX-\int_{\Omega_0}\mu\tilde{M}^{l-1}_{\Delta t}\cdot(H_{\text{ext}})^l_{\Delta t}\left(\eta^{l-1}_{\Delta t}\right)\,dX\right]\\
&=\sum_{l=1}^{k}\int_{\mathbb{R}^3}\mu\left[\chi_{\Omega^l_{\Delta t}}M^l_{\Delta t}-\chi_{\Omega^{l-1}_{\Delta t}}M^{l-1}_{\Delta t}\right]\cdot(H_{\text{ext}})^l_{\Delta t}\,dX\\
&=\int_{\mathbb{R}^3}\mu\chi_{\Omega^k_{\Delta t}}M^k_{\Delta t}\cdot(H_{\text{ext}})^k_{\Delta t}\,dx-\int_{\mathbb{R}^3}\mu\chi_{\Omega^0_{\Delta t}}M^0_{\Delta t}\cdot(H_{\text{ext}})^1_{\Delta t}\,dx\\
&\quad-\Delta t\sum_{l=2}^{k}\int_{\mathbb{R}^3}\mu\chi_{\Omega^{l-1}_{\Delta t}}M^{l-1}_{\Delta t}\cdot\left[\frac{(H_{\text{ext}})^l_{\Delta t}-(H_{\text{ext}})^{l-1}_{\Delta t}}{\Delta t}\right]\,dx. \qquad (5.4.10)
\end{aligned}$$

The first integral on the right-hand side of this equation can be controlled by the left-hand side of the inequality (5.4.9) due to the estimate

$$\begin{aligned}
&\frac{1}{2}\tilde{E}\left(\eta^k_{\Delta t},\tilde{M}^k_{\Delta t}\right)-\int_{\mathbb{R}^3}\mu\chi_{\Omega^k_{\Delta t}}M^k_{\Delta t}\cdot(H_{\text{ext}})^k_{\Delta t}\,dx\\
&\geqslant\int_{\Omega_0}\frac{1}{2}W\left(\nabla_X\eta^k_{\Delta t}\right)+\frac{1}{2q}\left|\nabla^2_X\eta^k_{\Delta t}\right|^q+\frac{1}{8\beta^2}\left(\left|\frac{1}{\det\left(\nabla_X\eta^k_{\Delta t}\right)}\tilde{M}^k_{\Delta t}\right|^2-1\right)^2\det\left(\nabla_X\eta^k_{\Delta t}\right)\\
&\quad-\mu\tilde{M}^k_{\Delta t}\cdot(H_{\text{ext}})^k_{\Delta t}\left(\eta^k_{\Delta t}\right)\,dX\\
&\geqslant-1-\frac{c_2^2}{4c_1}-\int_{\mathbb{R}^3}\frac{3\left(\beta^2\mu^4\right)^{\frac{1}{3}}\left|(H_{\text{ext}})^k_{\Delta t}\right|^{\frac{4}{3}}}{4}\,dx, \qquad (5.4.11)
\end{aligned}$$

cf. the second inequality in (5.3.9). The sum on the right-hand side of the equation (5.4.10) can be estimated, under exploitation of Hölder's and Young's inequalities, by

$$\begin{aligned}
&\Delta t\sum_{l=2}^{k}\int_{\mathbb{R}^3}\mu\chi_{\Omega^{l-1}_{\Delta t}}M^{l-1}_{\Delta t}\cdot\left[\frac{(H_{\text{ext}})^l_{\Delta t}-(H_{\text{ext}})^{l-1}_{\Delta t}}{\Delta t}\right]\,dx\\
&\leqslant\Delta t\sum_{l=2}^{k}\left[\frac{3\mu^{\frac{4}{3}}}{4}\left\|\frac{(H_{\text{ext}})^l_{\Delta t}-(H_{\text{ext}})^{l-1}_{\Delta t}}{\Delta t}\right\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(\mathbb{R}^3)}+\frac{1}{4}\int_{\Omega^{l-1}_{\Delta t}}\left|M^{l-1}_{\Delta t}\right|^4\,dx\right]\\
&\leqslant\mu^{\frac{4}{3}}\left\|\partial_t H_{\text{ext}}\right\|^{\frac{4}{3}}_{L^{\frac{4}{3}}((0,\infty)\times\mathbb{R}^3)}+\Delta t\sum_{l=1}^{k-1}\int_{\Omega^l_{\Delta t}}2\left(\left|M^l_{\Delta t}\right|^2-1\right)^2+2\,dx\\
&=\mu^{\frac{4}{3}}\left\|\partial_t H_{\text{ext}}\right\|^{\frac{4}{3}}_{L^{\frac{4}{3}}((0,\infty)\times\mathbb{R}^3)}+\Delta t\sum_{l=1}^{k-1}\int_{\Omega_0}2\left(\left|\frac{1}{\det\left(\nabla_X\eta^l_{\Delta t}\right)}\tilde{M}^l_{\Delta t}\right|^2-1\right)^2\det\left(\nabla_X\eta^l_{\Delta t}\right)\,dX\\
&\quad+\Delta t\sum_{l=1}^{k-1}\int_{\Omega_0}2\det\left(\nabla_X\eta^l_{\Delta t}\right)\,dX,
\end{aligned}$$

where in the second inequality we used Lemma A.3.4 in the appendix to control the discrete difference quotient of H_{ext} via $\partial_t H_{\text{ext}}$. Using the inequality (5.3.4) together with Young's inequality, we thus find a constant

$$c_3=c_3\left(\Omega_0,q,p_1,\gamma\right)>0 \qquad (5.4.12)$$

such that

$$\Delta t \sum_{l=2}^{k} \int_{\mathbb{R}^3} \mu \chi_{\Omega_{\Delta t}^{l-1}} M_{\Delta t}^{l-1} \cdot \left[\frac{(H_{\text{ext}})_{\Delta t}^{l} - (H_{\text{ext}})_{\Delta t}^{l-1}}{\Delta t} \right] dx$$
$$\leqslant \mu^{\frac{4}{3}} \|\partial_t H_{\text{ext}}\|_{L^{\frac{4}{3}}((0,\infty)\times\mathbb{R}^3)}^{\frac{4}{3}} + \Delta t \sum_{l=1}^{k-1} \int_{\Omega_0} 2 \left(\left| \frac{1}{\det\left(\nabla_X \eta_{\Delta t}^{l}\right)} \tilde{M}_{\Delta t}^{l} \right|^2 - 1 \right)^2 \det\left(\nabla_X \eta_{\Delta t}^{l}\right) \, dX$$
$$+ \Delta t \sum_{l=1}^{k-1} \int_{\Omega_0} \frac{1}{2} W\left(\nabla_X \eta_{\Delta t}^{l}\right) + \frac{1}{2q} \left| \nabla_X^2 \eta_{\Delta t}^{l} \right|^q \, dX + c_3 k \Delta t. \tag{5.4.13}$$

Applying the identity (5.4.10) as well as the estimates (5.4.11) and (5.4.13) to the inequality (5.4.9) we infer that

$$\frac{1}{2}\tilde{E}\left(\eta_{\Delta t}^{k}, \tilde{M}_{\Delta t}^{k}\right) + \sum_{l=1}^{k} \Delta t \tilde{R}_{\Delta t}^{l}\left(\eta_{\Delta t}^{l}, \tilde{M}_{\Delta t}^{l}\right)$$
$$\leqslant \tilde{E}\left(\eta_0, \tilde{M}_0\right) + 1 + \frac{c_2^2}{4c_1} + \int_{\mathbb{R}^3} \frac{3\left(\beta^2\mu^4\right)^{\frac{1}{3}} \left|(H_{\text{ext}})_{\Delta t}^{k}\right|^{\frac{4}{3}}}{4} dx - \int_{\Omega_{\Delta t}^{0}} \mu M_{\Delta t}^{0} \cdot (H_{\text{ext}})_{\Delta t}^{1} \, dx$$
$$+ \mu^{\frac{4}{3}} \|\partial_t H_{\text{ext}}\|_{L^{\frac{4}{3}}((0,\infty)\times\mathbb{R}^3)}^{\frac{4}{3}} + c_3 k \Delta t + \Delta t \sum_{l=1}^{k-1} \int_{\Omega_0} 2 \left(\left| \frac{1}{\det\left(\nabla_X \eta_{\Delta t}^{l}\right)} \tilde{M}_{\Delta t}^{l} \right|^2 - 1 \right)^2 \det\left(\nabla_X \eta_{\Delta t}^{l}\right) \, dX$$
$$+ \Delta t \sum_{l=1}^{k-1} \int_{\Omega_0} \frac{1}{2} W\left(\nabla_X \eta_{\Delta t}^{l}\right) + \frac{1}{2q} \left| \nabla_X^2 \eta_{\Delta t}^{l} \right|^q \, dX + \Delta t \sum_{l=1}^{k} \int_{\Omega_0} \rho f_{\Delta t}^{l}\left(\eta_{\Delta t}^{l-1}\right) \cdot \left(\frac{\eta_{\Delta t}^{l} - \eta_{\Delta t}^{l-1}}{\Delta t} \right) dX$$
$$\leqslant c_4 + c_3 k \Delta t + \Delta t \sum_{l=1}^{k-1} \int_{\Omega_0} 2 \left(\left| \frac{1}{\det\left(\nabla_X \eta_{\Delta t}^{l}\right)} \tilde{M}_{\Delta t}^{l} \right|^2 - 1 \right)^2 \det\left(\nabla_X \eta_{\Delta t}^{l}\right) + \frac{1}{2} W\left(\nabla_X \eta_{\Delta t}^{l}\right)$$
$$+ \frac{1}{2q} \left| \nabla_X^2 \eta_{\Delta t}^{l} \right|^q \, dX + \Delta t \sum_{l=1}^{k} \int_{\Omega_0} \rho f_{\Delta t}^{l}\left(\eta_{\Delta t}^{l-1}\right) \cdot \left(\frac{\eta_{\Delta t}^{l} - \eta_{\Delta t}^{l-1}}{\Delta t} \right) dX, \tag{5.4.14}$$

where the constant

$$c_4 = c_4\left(c_1, c_2, \eta_0, \tilde{M}_0, \beta, \mu, \Omega_0, \|H_{\text{ext}}\|_{W^{1,\frac{4}{3}}(0,\infty; L^{\frac{4}{3}}(\mathbb{R}^3))}\right) > 0 \tag{5.4.15}$$

is independent of Δt and k. Moreover, the f-dependent terms on the right-hand side of this inequality can be controlled by recalling the first inequality in (5.3.2): Indeed, the constant $c(\eta_{\Delta t}^{k-1})$ in this estimate is bounded away from zero for bounded values of $\tilde{E}\left(\eta_{\Delta t}^{k-1}, \tilde{M}_{\Delta t}^{k-1}\right)$. In our current situation, due to the induction assumption (5.4.8), we can replace this constant by a constant

$$c_5 = c_5(E_0, \eta_0, \tilde{M}_0, \mu, \beta, H_{\text{ext}}) > 0 \tag{5.4.16}$$

independent of Δt and k. It follows that

$$\int_{\Omega_0} \Delta t \nu \left| \nabla_X \left(\frac{\eta_{\Delta t}^{l} - \eta_{\Delta t}^{l-1}}{\Delta t} \right) \left(\nabla_X \eta_{\Delta t}^{l-1} \right)^{-1} \right|^2 \det\left(\nabla_X \eta_{\Delta t}^{l-1}\right) - \Delta t \rho f_{\Delta t}^{l}\left(\eta_{\Delta t}^{l-1}\right) \cdot \left(\frac{\eta_{\Delta t}^{l} - \eta_{\Delta t}^{l-1}}{\Delta t} \right) dX$$
$$\geqslant \Delta t \left[\frac{\nu c_5}{2} \int_{\Omega_0} \left| \nabla_X \left(\frac{\eta_{\Delta t}^{l} - \eta_{\Delta t}^{l-1}}{\Delta t} \right) \right|^2 dX - \frac{|\Omega_0| \left(\tilde{c}\rho \|f\|_{L^\infty((0,\infty)\times\mathbb{R}^3)} \right)^2}{2\nu c_5} \right].$$

Applying this estimate to the inequality (5.4.14) we infer that

$$\frac{1}{2}\tilde{E}\left(\eta^k_{\Delta t},\tilde{M}^k_{\Delta t}\right)+\sum_{l=1}^{k}\Delta t\left[\frac{\nu c_5}{2}\int_{\Omega_0}\left|\nabla_X\left(\frac{\eta^l_{\Delta t}-\eta^{l-1}_{\Delta t}}{\Delta t}\right)\right|^2+\det\left(\nabla_X\eta^{l-1}_{\Delta t}\right)\frac{1}{2}\right.$$
$$\left.\left|\frac{\frac{1}{\det\left(\nabla_X\eta^l_{\Delta t}\right)}\tilde{M}^l_{\Delta t}-\frac{1}{\det\left(\nabla_X\eta^{l-1}_{\Delta t}\right)}\tilde{M}^{l-1}_{\Delta t}}{\Delta t}+\frac{\operatorname{tr}\left(\nabla_X\left(\frac{\eta^l_{\Delta t}-\eta^{l-1}_{\Delta t}}{\Delta t}\right)\left(\nabla_X\eta^{l-1}_{\Delta t}\right)^{-1}\right)}{\det\left(\nabla_X\eta^{l-1}_{\Delta t}\right)}\tilde{M}^l_{\Delta t}\right|^2 dX\right]$$
$$\leqslant c_4+k\Delta t\left(c_3+\frac{|\Omega_0|\left(\tilde{c}\rho\,\|f\|_{L^\infty((0,\infty)\times\mathbb{R}^3)}\right)^2}{2\nu c_5}\right)+\Delta t\sum_{l=1}^{k-1}\int_{\Omega_0}\frac{1}{2}W\left(\nabla_X\eta^l_{\Delta t}\right)+\frac{1}{2q}\left|\nabla^2_X\eta^l_{\Delta t}\right|^q$$
$$+2\left(\left|\frac{1}{\det\left(\nabla_X\eta^l_{\Delta t}\right)}\tilde{M}^l_{\Delta t}\right|^2-1\right)^2\det\left(\nabla_X\eta^l_{\Delta t}\right)\,dX$$
$$\leqslant c_4+k\Delta t\left(c_3+\frac{|\Omega_0|\left(\tilde{c}\rho\,\|f\|_{L^\infty((0,\infty)\times\mathbb{R}^3)}\right)^2}{2\nu c_5}\right)+\Delta t\sum_{l=1}^{k-1}\max\left\{1,\ 8\beta^2\right\}\frac{1}{2}\tilde{E}\left(\eta^l_{\Delta t},\tilde{M}^l_{\Delta t}\right).$$

is independent of Δt and k. Thus the discrete Gronwall inequality (cf. [99, (1.67)]) implies that

$$\frac{1}{2}\tilde{E}\left(\eta^k_{\Delta t},\tilde{M}^k_{\Delta t}\right)+\sum_{l=1}^{k}\Delta t\left[\frac{\nu c_5}{2}\int_{\Omega_0}\left|\nabla_X\left(\frac{\eta^l_{\Delta t}-\eta^{l-1}_{\Delta t}}{\Delta t}\right)\right|^2+\det\left(\nabla_X\eta^{l-1}_{\Delta t}\right)\frac{1}{2}\right.$$
$$\left.\left|\frac{\frac{1}{\det\left(\nabla_X\eta^l_{\Delta t}\right)}\tilde{M}^l_{\Delta t}-\frac{1}{\det\left(\nabla_X\eta^{l-1}_{\Delta t}\right)}\tilde{M}^{l-1}_{\Delta t}}{\Delta t}+\frac{\operatorname{tr}\left(\nabla_X\left(\frac{\eta^l_{\Delta t}-\eta^{l-1}_{\Delta t}}{\Delta t}\right)\left(\nabla_X\eta^{l-1}_{\Delta t}\right)^{-1}\right)}{\det\left(\nabla_X\eta^{l-1}_{\Delta t}\right)}\tilde{M}^l_{\Delta t}\right|^2 dX\right]$$
$$\leqslant\left[c_4+k\Delta t\left(c_3+\frac{|\Omega_0|\left(\tilde{c}\rho\,\|f\|_{L^\infty((0,\infty)\times\mathbb{R}^3)}\right)^2}{2\nu c_5}\right)\right]e^{\max\{1,\ 8\beta^2\}k\Delta t}\leqslant E_0,$$

where the last inequality is due to our choice (5.4.7) of T and the fact that $k\Delta t\leqslant T$. This concludes the proof.

□

For the time $T>0$ given by Lemma 5.4.1 the energy estimate (5.4.6) implies the uniform bounds

$$\|\eta_{\Delta t}\|_{L^\infty(0,T;W^{2,q}(\Omega_0))}+\|\overline{\eta}_{\Delta t}\|_{L^\infty(0,T;W^{2,q}(\Omega_0))}+\|\overline{\eta}'_{\Delta t}\|_{L^\infty(0,T;W^{2,q}(\Omega_0))}\leqslant c, \quad (5.4.17)$$
$$\left\|\frac{1}{\det\left(\nabla_X\eta_{\Delta t}\right)}\right\|_{L^\infty((0,T)\times\Omega_0)}+\left\|\frac{1}{\det\left(\nabla_X\overline{\eta}_{\Delta t}\right)}\right\|_{L^\infty((0,T)\times\Omega_0)}+\left\|\frac{1}{\det\left(\nabla_X\overline{\eta}'_{\Delta t}\right)}\right\|_{L^\infty((0,T)\times\Omega_0)}\leqslant c, \quad (5.4.18)$$
$$\left\|\left(\nabla_X\eta_{\Delta t}\right)^{-1}\right\|_{L^\infty((0,T)\times\Omega_0)}+\left\|\left(\nabla_X\overline{\eta}_{\Delta t}\right)^{-1}\right\|_{L^\infty((0,T)\times\Omega_0)}+\left\|\left(\nabla_X\overline{\eta}'_{\Delta t}\right)^{-1}\right\|_{L^\infty((0,T)\times\Omega_0)}\leqslant c, \quad (5.4.19)$$
$$\|\partial_t\eta_{\Delta t}\|_{L^2(0,T;H^1(\Omega_0))}\leqslant c, \quad (5.4.20)$$
$$\left\|\tilde{M}_{\Delta t}\right\|_{L^\infty(0,T;H^1(\Omega_0))}+\left\|\overline{\tilde{M}}_{\Delta t}\right\|_{L^\infty(0,T;H^1(\Omega_0))}+\left\|\overline{\tilde{M}}'_{\Delta t}\right\|_{L^\infty(0,T;H^1(\Omega_0))}\leqslant c \quad (5.4.21)$$

and

$$\left\|\frac{\tilde{M}_{\Delta t}}{\det\left(\nabla_X\eta_{\Delta t}\right)}\right\|_{L^\infty(0,T;H^1(\Omega_0))}+\left\|\frac{\overline{\tilde{M}}_{\Delta t}}{\det\left(\nabla_X\overline{\eta}_{\Delta t}\right)}\right\|_{L^\infty(0,T;H^1(\Omega_0))}+\left\|\frac{\overline{\tilde{M}}'_{\Delta t}}{\det\left(\nabla_X\overline{\eta}'_{\Delta t}\right)}\right\|_{L^\infty(0,T;H^1(\Omega_0))}\leqslant c \quad (5.4.22)$$

with a constant $c > 0$ independent of Δt. Here, the bound (5.4.18) of the determinant of the deformation gradient away from zero follows from the estimate (5.4.6) via Lemma A.7.1 in the appendix. Moreover, the bound (5.4.21) follows by using the product rule and Jacobi's formula for the derivative of determinants to write

$$\partial_{X_{ij}} \tilde{M}_{\Delta t} = \det\left(\nabla_X \eta_{\Delta t}\right) \partial_{X_{ij}} \left(\frac{1}{\det\left(\nabla_X \eta_{\Delta t}\right)} \tilde{M}_{\Delta t} \right) + \operatorname{tr}\left(\partial_{X_{ij}} \nabla_X \eta_{\Delta t} \left(\nabla_X \eta_{\Delta t}\right)^{-1} \right) \tilde{M}_{\Delta t}$$

for all $i, j = 1, .., 3$ and estimating the right-hand side via the bounds (5.4.17)–(5.4.19) and (5.4.22). The bounds (5.4.17)–(5.4.22) allow us to find functions $\eta \in L^\infty(0, T; \mathcal{E})$, $\tilde{M} \in L^\infty(0, T; H^1(\Omega_0))$ such that, possibly after the extraction of a subsequence,

$$\eta_{\Delta t}, \overline{\eta}_{\Delta t}, \overline{\eta}'_{\Delta t} \overset{*}{\rightharpoonup} \eta \text{ in } L^\infty\left(0, T; W^{2,q}\left(\Omega_0\right)\right), \qquad \partial_t \eta_{\Delta t} \rightharpoonup \partial_t \eta \text{ in } L^2\left(0, T; H^1\left(\Omega_0\right)\right), \tag{5.4.23}$$

$$\tilde{M}_{\Delta t}, \overline{\tilde{M}}_{\Delta t}, \overline{\tilde{M}}'_{\Delta t} \overset{*}{\rightharpoonup} \tilde{M} \text{ in } L^\infty\left(0, T; H^1\left(\Omega_0\right)\right) \tag{5.4.24}$$

and

$$\frac{1}{\det\left(\nabla_X \eta_{\Delta t}\right)} \tilde{M}_{\Delta t}, \ \frac{1}{\det\left(\nabla_X \overline{\eta}_{\Delta t}\right)} \overline{\tilde{M}}_{\Delta t}, \ \frac{1}{\det\left(\nabla_X \overline{\eta}'_{\Delta t}\right)} \overline{\tilde{M}}'_{\Delta t} \overset{*}{\rightharpoonup} \frac{1}{\det\left(\nabla_X \eta\right)} \tilde{M} \qquad \text{in } L^\infty\left(0, T; H^1\left(\Omega_0\right)\right). \tag{5.4.25}$$

Here the equality of the weak limits of different interpolants of the same discrete functions follows from Lemma A.3.1 in the appendix. Due to the bound (5.4.17) of $\eta_{\Delta t}$ in $L^\infty(0, T; W^{2,q}(\Omega_0))$ and the bound (5.4.20) of $\partial_t \eta_{\Delta t}$ in $L^2(0, T; H^1(\Omega_0))$, the Aubin-Lions Lemma shows that the deformation also converges uniformly,

$$\eta_{\Delta t} \to \eta \quad \text{in } C\left([0, T]; C^1\left(\overline{\Omega_0}\right)\right). \tag{5.4.26}$$

In combination with the bound of $\det(\nabla_X \eta_{\Delta t})$ away from zero in (5.4.18) this further shows that

$$\left(\nabla_X \eta_{\Delta t}\right)^{-1} \to \left(\nabla_X \eta\right) \quad \text{in } C\left([0, T] \times \overline{\Omega_0}\right). \tag{5.4.27}$$

Moreover, as a consequence of the convergences (5.4.26) and (5.4.27) we see that

$$\overline{\eta}_{\Delta t}, \overline{\eta}'_{\Delta t} \to \eta \ \text{ in } L^\infty\left(0, T; C^1\left(\overline{\Omega_0}\right)\right), \quad \left(\nabla_X \overline{\eta}_{\Delta t}\right)^{-1}, \left(\nabla_X \overline{\eta}'_{\Delta t}\right)^{-1} \to \left(\nabla_X \eta\right)^{-1} \ \text{ in } L^\infty\left(0, T; C\left(\overline{\Omega_0}\right)\right). \tag{5.4.28}$$

Finally, the injectivity of $\overline{\eta}_{\Delta t}(t) \in \mathcal{E}$ in Ω_0 implies the existence of inverse functions $\overline{\eta}^{-1}_{\Delta t}(t, \cdot) : \overline{\overline{\Omega}}_{\Delta t}(t) \to \Omega_0$ of $\overline{\eta}_{\Delta t}(t, \cdot)$ for all $t \in [0, T]$. For these inverse functions we deduce pointwise convergence from the uniform bound (5.4.19) of the inverse deformation gradient and the convergence (5.4.28): Indeed, the bound (5.4.19) implies bi-Lipschitz continuity of $\overline{\eta}_{\Delta t}(t, \cdot)$ for all $t \in [0, T]$ with a bi-Lipschitz constant $c > 0$ independent of Δt and t. We choose some arbitrary point

$$x \in \Omega(t) := \eta\left(t, \Omega_0\right). \tag{5.4.29}$$

For any sufficiently small $\Delta t > 0$ it holds that $x \in \overline{\overline{\Omega}}_{\Delta t}(t)$ due to the convergence (5.4.28). For such Δt it follows that

$$\begin{aligned} \left|\overline{\eta}^{-1}_{\Delta t}(t, x) - \eta^{-1}(t, x)\right| \leqslant & c \left|\overline{\eta}_{\Delta t}\left(t, \overline{\eta}^{-1}_{\Delta t}(t, x)\right) - \overline{\eta}_{\Delta t}\left(t, \eta^{-1}(t, x)\right)\right| \\ = & c \left|x - \overline{\eta}_{\Delta t}\left(t, \eta^{-1}(t, x)\right)\right|, \end{aligned}$$

where $\eta^{-1}(t, \cdot) : \Omega(t) \to \Omega_0$ denotes the inverse of $\eta(t, \cdot) \in \mathcal{E}$. According to the convergence (5.4.28) the right-hand side of this inequality vanishes for $\Delta t \to 0$ and hence we have shown the desired pointwise convergence

$$\overline{\eta}^{-1}_{\Delta t}(t, \cdot) \to \eta^{-1}(t, \cdot) \qquad \text{pointwise in } \Omega\left(t\right) \text{ for all } t \in [0, T]. \tag{5.4.30}$$

We point out that with the current configuration $\Omega(t)$ of the limit system given by the formula (5.4.29) we may also express the magnetization of the limit system in the current configuration,

$$M := M_\eta \left[\tilde{M}\right] = \frac{1}{\det\left(\left[\nabla_X \eta\right]\left(\eta^{-1}\right)\right)} \tilde{M}\left(\eta^{-1}\right). \tag{5.4.31}$$

While the above uniform bounds and convergences hold true on the interval $[0,T]$ for $T>0$ given by Lemma 5.4.1, the discrete equation of motion (5.4.4) is satisfied on this interval only if $T>0$ is in addition chosen such that $\eta_{\Delta t}^k \in \operatorname{int}(\mathcal{E})$ for all $k=1,...,\frac{T}{\Delta t}$. The existence of a time $T>0$ independent of $\Delta t>0$ for which this is indeed true can be seen as another consequence of the uniform bounds (5.4.17)–(5.4.20) and is proved in the following Lemma:

Lemma 5.4.2. *There exists a time $T>0$, independent of $\Delta t>0$, such that $\eta_{\Delta t}^k \in \operatorname{int}(\mathcal{E})$ for all $k=1,...,\frac{T}{\Delta t}$. In particular, the equation of motion* (5.4.4) *holds true on the interval $[0,T]$ for all test functions $\chi \in \mathcal{D}((0,T)\times\Omega_0)$.*

Proof

From the uniform bounds (5.4.17), (5.4.20) of $\eta_{\Delta t}$ and the Morrey embedding it immediately follows that

$$\|\eta_{\Delta t}\|_{C^{0,\frac{1}{2}}([0,T];H^1(\Omega_0))} \leqslant c$$

for a constant $c>0$ independent of Δt. Consequently it holds that

$$\|\eta_{\Delta t}(t_1) - \eta_{\Delta t}(t_2)\|_{H^1(\Omega_0)} \leqslant c\sqrt{t_1 - t_2} \qquad \forall t_1, t_2 \in [0,T],\ t_1 > t_2. \tag{5.4.32}$$

Now let $t_1, t_2 \geqslant 0$ be such that $t_1 > t_2 + \Delta t$. Then there exist $k, l \in \mathbb{N}$, $k \geqslant l+1$, such that $t_1 \in ((k-1)\Delta t, k\Delta t]$ and $t_2 \in ((l-1)\Delta t, l\Delta t]$. Consequently,

$$(k-l)\Delta t \leqslant 2\max\{\Delta t, (k-1-l)\Delta t\} \leqslant 2|t_1 - t_2|. \tag{5.4.33}$$

Moreover, by definition of the piecewise affine interpolant $\eta_{\Delta t}$ in (5.4.1) and the piecewise constant interpolant $\overline{\eta}_{\Delta t}$ in (5.4.2), we know that

$$\overline{\eta}_{\Delta t}(t_1) = \eta_{\Delta t}(k\Delta t), \qquad \overline{\eta}_{\Delta t}(t_2) = \eta_{\Delta t}(l\Delta t).$$

Therefore, the Hölder continuity (5.4.32) of $\eta_{\Delta t}$ and the estimate (5.4.33) imply that

$$\|\overline{\eta}_{\Delta t}(t_1) - \overline{\eta}_{\Delta t}(t_2)\|_{H^1(\Omega_0)} = \|\eta_{\Delta t}(k\Delta t) - \eta_{\Delta t}(l\Delta t)\|_{H^1(\Omega_0)} \leqslant c\sqrt{(k-l)\Delta t} \leqslant \sqrt{2}c\sqrt{t_1 - t_2}. \tag{5.4.34}$$

for all $t_1, t_2 \geqslant 0$ satisfying $t_1 > t_2 + \Delta t$. Now let $\Gamma > 0$ be the constant given by Lemma A.7.2 in the appendix. From the estimate (5.4.34) we infer the existence of some sufficiently small time $T>0$, independent of Δt, such that

$$\left\|\eta_{\Delta t}^k - \eta_0\right\|_{H^1(\Omega_0)} < \Gamma \qquad \text{for all sufficiently small } \Delta t > 0 \text{ and all } k = 1,...,\frac{T}{\Delta t}.$$

Hence, Lemma A.7.2 implies that for all such k the deformation $\eta_{\Delta t}^k$ is injective on $\partial\Omega_0$ and in particular on N. From Remark 5.1.1 it follows that $\eta_{\Delta t}^k \in \operatorname{int}(\mathcal{E})$.

□

In the following we will first finish our existence proof on the interval $[0,T]$ where $T>0$ with $\frac{T}{\Delta t} \in \mathbb{N}$ is chosen according to Lemma 5.4.1 and Lemma 5.4.2. Subsequently, in Section 5.4.5 below, we will extend the resulting weak solution to the interval $[0,T')$ where $T'>0$ is chosen as in Theorem 5.1.1.

5.4.2 Convergence of the stray field

For the limit passage in both the equation of motion and the magnetic force balance we require (strong) convergence of the stray field. In order to deduce this we first show strong convergence of the magnetization. From the uniform bound (5.4.6) for the discrete dissipation potential and the bound (5.4.18) of $\det(\nabla_X\overline{\eta}_{\Delta t})$ away from zero we know that

$$\left\| \frac{\frac{1}{\det(\nabla_X\overline{\eta}_{\Delta t})}\overline{\tilde{M}}_{\Delta t} - \frac{1}{\det\left(\nabla_X\overline{\eta}'_{\Delta t}\right)}\overline{\tilde{M}}'_{\Delta t}}{\Delta t} + \frac{\operatorname{tr}\left(\nabla_X\left(\frac{\overline{\eta}_{\Delta t}-\overline{\eta}'_{\Delta t}}{\Delta t}\right)(\nabla_X\overline{\eta}'_{\Delta t})^{-1}\right)}{\det\left(\nabla_X\overline{\eta}'_{\Delta t}\right)}\overline{\tilde{M}}_{\Delta t} \right\|_{L^2((0,T)\times\Omega_0)} \leqslant c \quad (5.4.35)$$

for a constant $c > 0$ independent of Δt. Since $\partial_t\eta_{\Delta t}$ is bounded in $L^2(0,T;H^1(\Omega_0))$ (cf. (5.4.20)) and $\overline{\tilde{M}}_{\Delta t}$ is bounded in $L^\infty(0,T;L^6(\Omega_0))$ (cf. (5.4.21)) we further see that, under exploitation of the Hölder inequality,

$$\begin{aligned}
&\left\| \frac{\operatorname{tr}\left(\nabla_X\left(\frac{\overline{\eta}_{\Delta t}-\overline{\eta}'_{\Delta t}}{\Delta t}\right)(\nabla_X\overline{\eta}'_{\Delta t})^{-1}\right)}{\det\left(\nabla_X\overline{\eta}'_{\Delta t}\right)}\overline{\tilde{M}}_{\Delta t} \right\|_{L^2(0,T;L^{\frac{3}{2}}(\Omega_0))} \\
\leqslant & c\left\| \frac{(\nabla_X\overline{\eta}'_{\Delta t})^{-1}}{\det\left(\nabla_X\overline{\eta}'_{\Delta t}\right)} \right\|_{L^\infty((0,T)\times\Omega_0)} \left\| \nabla_X\left(\frac{\overline{\eta}_{\Delta t}-\overline{\eta}'_{\Delta t}}{\Delta t}\right) \right\|_{L^2((0,T)\times\Omega_0)} \left\|\overline{\tilde{M}}_{\Delta t}\right\|_{L^\infty(0,T;L^6(\Omega_0))} \leqslant c. \quad (5.4.36)
\end{aligned}$$

Combining the estimates (5.4.35) and (5.4.36) we infer the uniform bound

$$\left\| \frac{\frac{1}{\det(\nabla_X\overline{\eta}_{\Delta t}(\cdot))}\overline{\tilde{M}}_{\Delta t} - \frac{1}{\det\left(\nabla_X\overline{\eta}'_{\Delta t}(\cdot)\right)}\overline{\tilde{M}}'_{\Delta t}}{\Delta t} \right\|_{L^2(0,T;L^{\frac{3}{2}}(\Omega_0))} \leqslant c. \quad (5.4.37)$$

This, in combination with the $L^\infty(0,T;H^1(\Omega))$-bound (5.4.22), yields the conditions for both the classical and the discrete (cf. Lemma A.3.3 in the appendix) Aubin-Lions Lemma, which yield that

$$\begin{aligned}
\frac{1}{\det(\nabla_X\eta_{\Delta t})}\tilde{M}_{\Delta t} &\to \frac{1}{\det(\nabla_X\eta_{\Delta t})}\tilde{M}_{\Delta t} \quad \text{in } C\left(0,T;L^p\left(\Omega_0\right)\right) \\
\frac{1}{\det(\nabla_X\overline{\eta}_{\Delta t})}\overline{\tilde{M}}_{\Delta t} &\to \frac{1}{\det(\nabla_X\eta_{\Delta t})}\tilde{M}_{\Delta t} \quad \text{in } L^r\left(0,T;L^p\left(\Omega_0\right)\right) \quad (5.4.38)
\end{aligned}$$

for all $1 \leqslant p < 6$ and all $1 \leqslant r < \infty$. In combination with the uniform convergences (5.4.26) and (5.4.28) of the deformation gradient this further implies that

$$\tilde{M}_{\Delta t} \to \tilde{M} \quad \text{in } C\left(0,T;L^p\left(\Omega_0\right)\right), \qquad \overline{\tilde{M}}_{\Delta t} \to \tilde{M} \quad \text{in } L^r\left(0,T;L^p\left(\Omega_0\right)\right) \quad (5.4.39)$$

for all $1 \leqslant p < 6$ and all $1 \leqslant r < \infty$. Next we show that the discrete stray field converges weakly to a corresponding limit function. From the uniform bound (5.4.22) of $\frac{\overline{\tilde{M}}_{\Delta t}}{\det(\nabla_X\overline{\eta}_{\Delta t})}$ we know that

$$\begin{aligned}
\left\|\overline{M}_{\Delta t}\right\|_{L^\infty\left(0,T;L^2\left(\overline{\Omega}_{\Delta t}(\cdot)\right)\right)} &= \operatorname{esssup}_{t\in[0,T]}\left(\int_{\Omega_0}\det\left(\nabla_X\overline{\eta}_{\Delta t}(t)\right)\left|\frac{1}{\det\left(\nabla_X\overline{\eta}_{\Delta t}(t)\right)}\overline{\tilde{M}}_{\Delta t}(t)\right|^2 dX\right)^{\frac{1}{2}} \\
&\leqslant \left\|\det\left(\nabla_X\overline{\eta}_{\Delta t}\right)\right\|^{\frac{1}{2}}_{L^\infty((0,T)\times\Omega_0)}\left\|\frac{1}{\det\left(\nabla_X\overline{\eta}_{\Delta t}\right)}\overline{\tilde{M}}_{\Delta t}\right\|_{L^\infty(0,T;L^2(\Omega_0))} \leqslant c
\end{aligned}$$

for a constant $c > 0$ independent of Δt. This, together with the H^1-bound (A.2.9) for solutions to the Poisson equation (5.1.6) given by Lemma A.2.3 in the appendix implies that further

$$\left\|\phi\left[\overline{\tilde{M}}_{\Delta t},\overline{\eta}_{\Delta t}\right]\right\|_{L^\infty(0,T;\dot{H}^1(\mathbb{R}^3))} = \left\|\nabla\phi\left[\overline{\tilde{M}}_{\Delta t},\overline{\eta}_{\Delta t}\right]\right\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \leqslant c. \quad (5.4.40)$$

We conclude the existence of some function $\phi \in L^\infty(0,T;\dot{H}^1(\mathbb{R}^3))$ for which we may assume that

$$\phi\left[\overline{\tilde{M}}_{\Delta t},\overline{\eta}_{\Delta t}\right] \overset{*}{\rightharpoonup} \phi \quad \text{in } L^\infty\left(0,T;\dot{H}^1\left(\mathbb{R}^3\right)\right). \tag{5.4.41}$$

In order to identify ϕ we note that by the definition of $\phi[\overline{\tilde{M}}_{\Delta t},\overline{\eta}_{\Delta t}]$ as a solution to the Poisson equation (5.1.6) it holds that

$$\int_0^T \zeta \int_{\mathbb{R}^3} \nabla\phi\left[\overline{\tilde{M}}_{\Delta t},\overline{\eta}_{\Delta t}\right]\cdot\nabla\psi\ dxdt = \int_0^T \zeta\int_{\Omega_0} \overline{\tilde{M}}_{\Delta t}\cdot\nabla\psi\left(\overline{\eta}_{\Delta t}\right)\ dXdt$$

for any $\psi \in \mathcal{D}(\mathbb{R}^3)$, $\zeta \in \mathcal{D}(0,T)$. Due to the convergences (5.4.28) of the deformation, (5.4.39) of the magnetization and (5.4.41) of $\phi[\overline{\tilde{M}}_{\Delta t},\overline{\eta}_{\Delta t}]$ we may pass to the limit in both sides of this identity. After a transformation back to the current configuration $\Omega(t) = \eta(t,\Omega_0)$ we thus infer that

$$\int_{\mathbb{R}^3} \nabla\phi\cdot\nabla\psi\ dx = \int_{\Omega(t)} M\cdot\nabla\psi\ dX$$

for almost all $t \in [0,T]$ and all $\psi \in \mathcal{D}(\mathbb{R}^3)$ and, due to the density of $\mathcal{D}(\mathbb{R}^3)$ in $\dot{H}^1(\mathbb{R}^3)$, for all $\psi \in \dot{H}^1(\mathbb{R}^3)$. Therefore, we may identify $\phi = \phi[\tilde{M},\eta]$ and in particular, by (5.4.41),

$$H\left[\overline{\tilde{M}}_{\Delta t},\overline{\eta}_{\Delta t}\right] = -\nabla\phi\left[\overline{\tilde{M}}_{\Delta t},\overline{\eta}_{\Delta t}\right] \rightharpoonup -\nabla\phi\left[\tilde{M},\eta\right] = H\left[\tilde{M},\eta\right] \qquad \text{in } L^2\left((0,T)\times\mathbb{R}^3\right). \tag{5.4.42}$$

Our next goal is to improve the weak convergence (5.4.42) to strong convergence. To this end we use the definition of the stray field in (5.1.6) and the set $\overline{\overline{\Omega}}_{\Delta t}(t)$ in (5.4.3) to estimate

$$\begin{aligned}
&\left|\int_0^T\int_{\mathbb{R}^3}\left|H\left[\overline{\tilde{M}}_{\Delta t},\overline{\eta}_{\Delta t}\right]\right|^2\ dxdt - \int_0^T\int_{\mathbb{R}^3}\left|H\left[\tilde{M},\eta\right]\right|^2\ dxdt\right| \\
=&\left|\int_0^T\int_{\overline{\overline{\Omega}}_{\Delta t}(t)} \overline{M}_{\Delta t}\cdot H\left[\overline{\tilde{M}}_{\Delta t},\overline{\eta}_{\Delta t}\right]\ dxdt - \int_0^T\int_{\Omega(t)} M\cdot H\left[\tilde{M},\eta\right]\ dxdt\right| \\
=&\left|\int_0^T\int_{\Omega_0} \overline{\tilde{M}}_{\Delta t}\cdot H\left[\overline{\tilde{M}}_{\Delta t},\overline{\eta}_{\Delta t}\right](\overline{\eta}_{\Delta t})\ dXdt - \int_0^T\int_{\Omega_0}\tilde{M}\cdot H\left[\tilde{M},\eta\right](\eta)\ dXdt\right| \\
\leqslant&\left|\int_0^T\int_{\Omega_0}\left(\overline{\tilde{M}}_{\Delta t}-\tilde{M}\right)\cdot H\left[\overline{\tilde{M}}_{\Delta t},\overline{\eta}_{\Delta t}\right](\overline{\eta}_{\Delta t})\ dXdt\right| \\
&+\left|\int_0^T\int_{\Omega_0}\tilde{M}\cdot\left(H\left[\overline{\tilde{M}}_{\Delta t},\overline{\eta}_{\Delta t}\right](\overline{\eta}_{\Delta t}) - H\left[\tilde{M},\eta\right](\overline{\eta}_{\Delta t})\right)\ dXdt\right| \\
&+\left|\int_0^T\int_{\Omega_0}\tilde{M}\cdot\left(H\left[\tilde{M},\eta\right](\overline{\eta}_{\Delta t}) - H\left[\tilde{M},\eta\right](\eta)\right)\ dXdt\right|.
\end{aligned} \tag{5.4.43}$$

Here, for the first integral on the right-hand side, we immediately see that

$$\begin{aligned}
&\left|\int_0^T\int_{\Omega_0}\left(\overline{\tilde{M}}_{\Delta t}-\tilde{M}\right)\cdot H\left[\overline{\tilde{M}}_{\Delta t},\overline{\eta}_{\Delta t}\right](\overline{\eta}_{\Delta t})\ dXdt\right| \\
\leqslant&\left(\int_0^T\int_{\overline{\overline{\Omega}}_{\Delta t}(t)}\frac{1}{\det\left(\left[\nabla_X\overline{\eta}_{\Delta t}\right]\left((\overline{\eta}_{\Delta t})^{-1}\right)\right)}\left|H\left[\overline{\tilde{M}}_{\Delta t},\overline{\eta}_{\Delta t}\right]\right|^2\ dxdt\right)^{\frac{1}{2}}\left\|\overline{\tilde{M}}_{\Delta t}-\tilde{M}\right\|_{L^2((0,T)\times\Omega_0)} \\
\leqslant& c\left\|\overline{\tilde{M}}_{\Delta t}-\tilde{M}\right\|_{L^2((0,T)\times\Omega_0)} \to 0.
\end{aligned} \tag{5.4.44}$$

due to the uniform bounds (5.4.18), (5.4.40) and the strong convergence (5.4.39) of the magnetization. In order to show that also the second integral on the right-hand side of the inequality (5.4.43) vanishes we choose a sequence of functions $(\tilde{M}_n)_{n\in\mathbb{N}} \subset \mathcal{D}((0,T)\times\Omega_0)$ such that $\tilde{M}_n \to \tilde{M}$ in $L^6((0,T)\times\Omega_0)$. Then we estimate

$$\begin{aligned}
&\left|\int_0^T\int_{\Omega_0} \tilde{M}\cdot\left(H\left[\overline{\tilde{M}}_{\Delta t},\overline{\eta}_{\Delta t}\right](\overline{\eta}_{\Delta t}) - H\left[\tilde{M},\eta\right](\overline{\eta}_{\Delta t})\right)\,dX\,dt\right| \\
\leqslant&\left|\int_0^T\int_{\Omega_0}\left(\tilde{M}-\tilde{M}_n\right)\cdot\left(H\left[\overline{\tilde{M}}_{\Delta t},\overline{\eta}_{\Delta t}\right](\overline{\eta}_{\Delta t}) - H\left[\tilde{M},\eta\right](\overline{\eta}_{\Delta t})\right)\,dX\,dt\right| \\
&+\left|\int_0^T\int_{\Omega_0}\tilde{M}_n\cdot\left(H\left[\overline{\tilde{M}}_{\Delta t},\overline{\eta}_{\Delta t}\right](\overline{\eta}_{\Delta t}) - H\left[\tilde{M},\eta\right](\overline{\eta}_{\Delta t})\right)\,dX\,dt\right| \\
=&c\left\|\tilde{M}-\tilde{M}_n\right\|_{L^2((0,T)\times\Omega_0)} \\
&+\left|\int_0^T\int_{\overline{\overline{\Omega}}_{\Delta t}(t)}\frac{1}{\det\left(\left[\nabla_X\overline{\eta}_{\Delta t}\right]\left(\overline{\eta}_{\Delta t}^{-1}\right)\right)}\tilde{M}_n\left((\overline{\eta}_{\Delta t})^{-1}\right)\cdot\left(H\left[\overline{\tilde{M}}_{\Delta t},\overline{\eta}_{\Delta t}\right]-H\left[\tilde{M},\eta\right]\right)\,dx\,dt\right|,
\end{aligned}\tag{5.4.45}$$

using the same bounds as in the derivation of the estimate (5.4.44). In order to show that the second term on the right-hand side vanishes as Δt tends to zero, we denote by $g^n_{\Delta t} : [0,T]\times\mathbb{R}^3 \to \mathbb{R}^3$ the function

$$g^n_{\Delta t} := \begin{cases} \frac{1}{\det\left(\left[\nabla_X\overline{\eta}_{\Delta t}\right]\left(\overline{\eta}_{\Delta t}^{-1}\right)\right)}\tilde{M}_n\left((\overline{\eta}_{\Delta t})^{-1}\right) & \text{in } Q\left(\overline{\overline{\Omega}}_{\Delta t},T\right), \\ 0 & \text{in } \left([0,T]\times\mathbb{R}^3\right)\backslash Q\left(\overline{\overline{\Omega}}_{\Delta t},T\right). \end{cases}$$

Due to the uniform convergence (5.4.28) of $\overline{\eta}_{\Delta t}$, the pointwise convergence (5.4.30) of $(\overline{\eta}_{\Delta t})^{-1}$ and the compact support of $\tilde{M}_n$ we observe that

$$g^n_{\Delta t} \to g^n \qquad \text{pointwise in } (0,T)\times\mathbb{R}^3 \quad \text{for } \Delta t \to 0, \tag{5.4.46}$$

where

$$g^n := \begin{cases} \frac{1}{\det\left(\left[\nabla_X\eta\right]\left(\eta^{-1}\right)\right)}\tilde{M}_n\left((\eta)^{-1}\right) & \text{in } Q\left(\Omega,T\right), \\ 0 & \text{in } \left([0,T]\times\mathbb{R}^3\right)\backslash Q\left(\Omega,T\right). \end{cases}$$

We further estimate

$$\begin{aligned}
\|g^n_{\Delta t}\|_{L^6((0,T)\times\mathbb{R}^3)} &= \left(\int_0^T\int_{\overline{\overline{\Omega}}_{\Delta t}(t)}\left|\frac{1}{\det\left(\left[\nabla_X\overline{\eta}_{\Delta t}\right]\left(\overline{\eta}_{\Delta t}^{-1}\right)\right)}\tilde{M}_n\left(\overline{\eta}_{\Delta t}^{-1}\right)\right|^6\,dx\,dt\right)^{\frac{1}{6}} \\
&= \left(\int_0^T\int_{\Omega_0}\det\left(\nabla_X\overline{\eta}_{\Delta t}\right)\left|\frac{1}{\det\left(\nabla_X\overline{\eta}_{\Delta t}\right)}\tilde{M}_n\right|^6\,dX\,dt\right)^{\frac{1}{6}} \\
&\leqslant \left\|\det\left(\nabla_X\overline{\eta}_{\Delta t}\right)\right\|_{L^\infty((0,T)\times\Omega_0)}\left\|\frac{1}{\det\left(\nabla_X\overline{\eta}_{\Delta t}\right)}\tilde{M}_n\right\|_{L^6((0,T)\times\Omega_0)} \leqslant c
\end{aligned}$$

by exploiting the uniform bounds (5.4.17), (5.4.18) and the convergence of $\tilde{M}_n$ in $L^6((0,T)\times\Omega_0)$. This bound, together with the pointwise convergence (5.4.46) of $g^n_{\Delta t}$ allows us to apply the Vitali convergence theorem to infer strong convergence of $g^n_{\Delta t}$ in, for example, $L^2((0,T)\times\mathbb{R}^3)$,

$$g^n_{\Delta t} \to g^n \quad \text{in } L^2\left((0,T)\times\mathbb{R}^3\right) \qquad \text{for } \Delta t \to 0.$$

Combining this with the weak convergence (5.4.42) of the stray field, we conclude that

$$\begin{aligned}
&\left|\int_0^T\int_{\overline{\overline{\Omega}}_{\Delta t}(t)}\frac{1}{\det\left(\left[\nabla_X\overline{\eta}_{\Delta t}\right]\left(\overline{\eta}_{\Delta t}^{-1}\right)\right)}\tilde{M}_n\left(\overline{\eta}_{\Delta t}^{-1}\right)\cdot\left(H\left[\overline{\tilde{M}}_{\Delta t},\overline{\eta}_{\Delta t}\right]-H\left[\tilde{M},\eta\right]\right)\,dx\,dt\right| \\
=&\left|\int_0^T\int_{\mathbb{R}^3}g^n_{\Delta t}\cdot\left(H\left[\overline{\tilde{M}}_{\Delta t},\overline{\eta}_{\Delta t}\right]-H\left[\tilde{M},\eta\right]\right)\,dx\,dt\right| \to 0
\end{aligned}$$

for $n \in \mathbb{N}$ fixed and $\Delta t \to 0$. Consequently, letting first Δt tend to zero and subsequently n tend to infinity on the right-hand side of the inequality (5.4.45), we have shown that

$$\left| \int_0^T \int_{\Omega_0} \tilde{M} \cdot \left(H \left[\overline{\tilde{M}}_{\Delta t}, \overline{\eta}_{\Delta t} \right] (\overline{\eta}_{\Delta t}) - H \left[\tilde{M}, \eta \right] (\overline{\eta}_{\Delta t}) \right) \, dX dt \right| \to 0 \tag{5.4.47}$$

for $\Delta t \to 0$. It remains to prove that also the third integral on the right-hand side of the inequality (5.4.43) tends to zero. To this end we notice that, by Lemma A.7.4 for the convergence of compositions with the deformation,

$$H \left[\tilde{M}, \eta \right] (\overline{\eta}_{\Delta t}) \to H \left[\tilde{M}, \eta \right] (\eta) \qquad \text{in } L^2 \left((0,T) \times \Omega_0 \right). \tag{5.4.48}$$

In particular it follows that

$$\left| \int_0^T \int_{\Omega_0} \tilde{M} \cdot \left(H \left[\tilde{M}, \eta \right] (\overline{\eta}_{\Delta t}) - H \left[\tilde{M}, \eta \right] (\eta) \right) \, dX dt \right| \to 0.$$

Now, combining this convergence with the convergences (5.4.44) and (5.4.47), we see that the right-hand side of the estimate (5.4.43) tends to zero for $\Delta t \to 0$. This shows convergence of the $L^2((0,T) \times \mathbb{R}^3)$-norm of $H[\overline{\tilde{M}}_{\Delta t}, \overline{\eta}_{\Delta t}]$ to the one of $H[\tilde{M}, \eta]$, which, in combination with the weak convergence (5.4.42) of $H[\overline{\tilde{M}}_{\Delta t}, \overline{\eta}_{\Delta t}]$, implies the desired strong convergence

$$H \left[\overline{\tilde{M}}_{\Delta t}, \overline{\eta}_{\Delta t} \right] \to H \left[\tilde{M}, \eta \right] \qquad \text{in } L^2 \left((0,T) \times \mathbb{R}^3 \right). \tag{5.4.49}$$

Finally, since in the equations formulated in the reference configuration the stray field only appears in composition with the deformation, we point out that the strong convergence (5.4.49) also implies the convergence of this composition,

$$H \left[\overline{\tilde{M}}_{\Delta t}, \overline{\eta}_{\Delta t} \right] (\overline{\eta}_{\Delta t}) \to H \left[\tilde{M}, \eta \right] (\eta) \qquad \text{in } L^2 \left((0,T) \times \Omega_0 \right), \tag{5.4.50}$$

cf. Lemma A.7.4.

5.4.3 Magnetic force balance

Our next goal is the limit passage in the discrete magnetic force balance (5.4.5). In order to obtain the desired form (5.1.19) (cf. also the identity (5.1.23)) of the magnetic force balance after this limit passage, we test the discrete magnetic force balance (5.4.5) by functions of the form

$$\det \left(\nabla_X \overline{\eta}_{\Delta t} \right) \tilde{\tilde{M}}, \qquad \tilde{\tilde{M}} \in L^\infty \left(0,T; H^1 \left(\Omega_0 \right) \right).$$

Indeed, these functions constitute admissible test functions for the equation (5.4.5) since, by Jacobi's formula for the derivative of determinants and the $L^\infty(0,T; W^{2,q}(\Omega_0))$-regularity of $\overline{\eta}_{\Delta t}$,

$$\begin{aligned} &\partial_{X_i} \left(\det \left(\nabla_X \overline{\eta}_{\Delta t} \right) \tilde{\tilde{M}} \right) \\ &= \det \left(\nabla_X \overline{\eta}_{\Delta t} \right) \operatorname{tr} \left(\partial_{X_i} \nabla_X \overline{\eta}_{\Delta t} \left(\nabla_X \overline{\eta}_{\Delta t} \right)^{-1} \right) \tilde{\tilde{M}} + \det \left(\nabla_X \overline{\eta}_{\Delta t} \right) \partial_{X_i} \tilde{\tilde{M}} \quad \in L^\infty \left(0,T; L^2 \left(\Omega_0 \right) \right) \end{aligned}$$

for all $i = 1, ..., 3$, i.e.

$$\det \left(\nabla_X \overline{\eta}_{\Delta t} \right) \tilde{\tilde{M}} \in L^\infty \left(0,T; H^1 \left(\Omega_0 \right) \right).$$

Using them as such we infer the identity

$$
\begin{aligned}
&\int_0^T \int_{\Omega_0} \det(\nabla_X \overline{\eta}_{\Delta t}) \tilde{\Psi}_M \left(\nabla_X \overline{\eta}_{\Delta t}, \overline{\tilde{M}}_{\Delta t}\right) \cdot \tilde{\hat{M}} - \mu \det(\nabla_X \overline{\eta}_{\Delta t}) H \left[\overline{\tilde{M}}_{\Delta t}, \overline{\eta}_{\Delta t}\right] (\overline{\eta}_{\Delta t}) \cdot \tilde{\hat{M}} \\
&+ 2A \det\left(\nabla_X \overline{\eta}_{\Delta t}\right) \left[\nabla_X \left(\frac{1}{\det\left(\nabla_X \overline{\eta}_{\Delta t}\right)} \overline{\tilde{M}}_{\Delta t}\right) \left(\nabla_X \overline{\eta}_{\Delta t}\right)^{-1}\right] : \left(\nabla_X \tilde{\hat{M}} \left(\nabla_X \overline{\eta}_{\Delta t}\right)^{-1}\right) \\
&+ \frac{1}{\beta^2} \det\left(\nabla_X \overline{\eta}_{\Delta t}\right) \left(\left|\frac{1}{\det\left(\nabla_X \overline{\eta}_{\Delta t}\right)} \overline{\tilde{M}}_{\Delta t}\right|^2 - 1\right) \left(\frac{1}{\det\left(\nabla_X \overline{\eta}_{\Delta t}\right)} \overline{\tilde{M}}_{\Delta t}\right) \cdot \tilde{\hat{M}} \\
&+ \det\left(\nabla_X \overline{\eta}'_{\Delta t}\right) \left[\frac{\frac{1}{\det(\nabla_X \overline{\eta}_{\Delta t})} \overline{\tilde{M}}_{\Delta t} - \frac{1}{\det\left(\nabla_X \overline{\eta}'_{\Delta t}\right)} \overline{\tilde{M}}'_{\Delta t}}{\Delta t} + \frac{\operatorname{tr}\left(\nabla_X \left(\frac{\overline{\eta}_{\Delta t} - \overline{\eta}'_{\Delta t}}{\Delta t}\right) \left(\nabla_X \overline{\eta}'_{\Delta t}\right)^{-1}\right)}{\det\left(\nabla_X \overline{\eta}'_{\Delta t}\right)} \overline{\tilde{M}}_{\Delta t}\right] \\
&\cdot \left[\tilde{\hat{M}} + \frac{\det(\nabla_X \overline{\eta}_{\Delta t}) \operatorname{tr}\left(\left(\nabla_X \overline{\eta}_{\Delta t} - \nabla_X \overline{\eta}'_{\Delta t}\right) \left(\nabla_X \overline{\eta}'_{\Delta t}\right)^{-1}\right)}{\det\left(\nabla_X \overline{\eta}'_{\Delta t}\right)} \tilde{\hat{M}}\right] \\
&- \mu \det(\nabla_X \overline{\eta}_{\Delta t}) \overline{(H_{\text{ext}})}_{\Delta t} \left(\overline{\eta}_{\Delta t}\right) \cdot \tilde{\hat{M}} \; dX dt = 0
\end{aligned} \tag{5.4.51}
$$

for all $\tilde{\hat{M}} \in L^\infty(0, T; H^1(\Omega_0))$. In order to pass to the limit in this relation we study several of the terms individually: We first notice that by the boundedness assumption (5.1.12) on $\tilde{\Psi}_M$ it holds that

$$
\left|\tilde{\Psi}_M \left(\nabla_X \overline{\eta}_{\Delta t}, \overline{\tilde{M}}_{\Delta t}\right)\right| \leqslant c \left(1 + \left|\nabla_X \overline{\eta}_{\Delta t}\right|^{p_2} + \left|\overline{\tilde{M}}_{\Delta t}\right|^{p_4}\right),
$$

where $1 \leqslant p_2 < \infty$ and $1 \leqslant p_4 < 5$. Due to the uniform bounds (5.4.17) and (5.4.21) this shows that

$$
\left\|\tilde{\Psi}_M \left(\nabla_X \overline{\eta}_{\Delta t}, \overline{\tilde{M}}_{\Delta t}\right)\right\|_{L^p((0,T) \times \Omega_0)} \leqslant c
$$

for some $p > \frac{6}{5}$. In combination with the continuity (5.1.8) of $\tilde{\Psi}_M$, the convergences (5.4.28), (5.4.39) and the Vitali convergence theorem this shows that

$$
\tilde{\Psi}_M \left(\nabla_X \overline{\eta}_{\Delta t}, \overline{\tilde{M}}_{\Delta t}\right) \to \tilde{\Psi}_M \left(\nabla_X \eta, \tilde{M}\right) \qquad \text{in } L^p\left((0,T) \times \Omega_0\right) \quad \text{for some } p > \frac{6}{5}. \tag{5.4.52}
$$

Next, due to the uniform bounds (5.4.35) and (5.4.37), the uniform convergence (5.4.28) of $\overline{\eta}_{\Delta t}$ and $\overline{\eta}'_{\Delta t}$, the weak convergence (5.4.23) of $\partial_t \eta_{\Delta t}$ and the strong convergence (5.4.39) of $\overline{\tilde{M}}_{\Delta t}$ we see that

$$
\begin{aligned}
&\frac{\frac{1}{\det(\nabla_X \overline{\eta}_{\Delta t})} \overline{\tilde{M}}_{\Delta t} - \frac{1}{\det\left(\nabla_X \overline{\eta}'_{\Delta t}\right)} \overline{\tilde{M}}'_{\Delta t}}{\Delta t} + \frac{\operatorname{tr}\left(\nabla_X \left(\frac{\overline{\eta}_{\Delta t} - \overline{\eta}'_{\Delta t}}{\Delta t}\right) \left(\nabla_X \overline{\eta}'_{\Delta t}\right)^{-1}\right)}{\det\left(\nabla_X \overline{\eta}'_{\Delta t}\right)} \overline{\tilde{M}}_{\Delta t} \\
\to & \partial_t \left(\frac{1}{\det\left(\nabla_X \eta\right)} \tilde{M}\right) + \frac{\operatorname{tr}\left(\left(\nabla_X \partial_t \eta\right) \left(\nabla_X \eta\right)^{-1}\right)}{\det\left(\nabla_X \eta\right)} \tilde{M} \\
= & - \frac{\operatorname{tr}\left(\left(\nabla_X \partial_t \eta\right) \left(\nabla_X \eta\right)^{-1}\right)}{\det\left(\nabla_X \eta\right)} \tilde{M} + \frac{1}{\det\left(\nabla_X \eta\right)} \partial_t \tilde{M} + \frac{\operatorname{tr}\left(\left(\nabla_X \partial_t \eta\right) \left(\nabla_X \eta\right)^{-1}\right)}{\det\left(\nabla_X \eta\right)} \tilde{M} \\
= & \frac{1}{\det\left(\nabla_X \eta\right)} \partial_t \tilde{M} \qquad \text{in } L^2\left((0,T) \times \Omega_0\right).
\end{aligned} \tag{5.4.53}
$$

In combination with the uniform convergence (5.4.28) of $\overline{\eta}_{\Delta t}$ and $\overline{\eta}'_{\Delta t}$ this shows that

$$
\begin{aligned}
&\int_0^T \int_{\Omega_0} \det\left(\nabla_X \overline{\eta}'_{\Delta t}\right) \left[\frac{\frac{1}{\det(\nabla_X \overline{\eta}_{\Delta t})} \overline{\tilde{M}}_{\Delta t} - \frac{1}{\det\left(\nabla_X \overline{\eta}'_{\Delta t}\right)} \overline{\tilde{M}}'_{\Delta t}}{\Delta t} + \frac{\operatorname{tr}\left(\nabla_X \left(\frac{\overline{\eta}_{\Delta t} - \overline{\eta}'_{\Delta t}}{\Delta t}\right) \left(\nabla_X \overline{\eta}'_{\Delta t}\right)^{-1}\right)}{\det\left(\nabla_X \overline{\eta}'_{\Delta t}\right)} \overline{\tilde{M}}_{\Delta t}\right] \\
&\cdot \left[\tilde{\hat{M}} + \frac{\det(\nabla_X \overline{\eta}_{\Delta t}) \operatorname{tr}\left(\left(\nabla_X \overline{\eta} - \nabla_X \overline{\eta}'_{\Delta t}\right) \left(\nabla_X \overline{\eta}'_{\Delta t}\right)^{-1}\right)}{\det\left(\nabla_X \overline{\eta}'_{\Delta t}\right)} \tilde{\hat{M}}\right] \; dX dt \\
\to & \int_0^T \int_{\Omega_0} \partial_t \tilde{M} \cdot \tilde{\hat{M}} \; dX dt.
\end{aligned} \tag{5.4.54}
$$

Finally, for the discretized external function $\overline{(H_{\mathrm{ext}})}_{\Delta t}$, we note that, by Lemma A.3.2 (ii) in the appendix,

$$\overline{(H_{\mathrm{ext}})}_{\Delta t} \to H_{\mathrm{ext}} \qquad \text{in } C\left([0,T]; W^{1,\frac{4}{3}}(\mathbb{R}^3)\right). \tag{5.4.55}$$

This, together with Lemma A.7.4 for the convergence of compositions with the deformation, shows the convergence

$$\overline{(H_{\mathrm{ext}})}_{\Delta t}\left(\overline{\eta}_{\Delta t}\right) \to H_{\mathrm{ext}}(\eta) \qquad \text{in } L^{\frac{4}{3}}\left(0,T; W^{1,\frac{4}{3}}\left(\Omega_0\right)\right). \tag{5.4.56}$$

Now, combining the convergences (5.4.52), (5.4.54) and (5.4.56) with the uniform convergences (5.4.28) of $\overline{\eta}_{\Delta t}$, $\overline{\eta}'_{\Delta t}$ and $(\nabla_X \overline{\eta}_{\Delta t})^{-1}$, the weak and strong convergences (5.4.25) and (5.4.38) of $\frac{1}{\det(\nabla_X \overline{\eta}_{\Delta t})}\overline{\tilde{M}}_{\Delta t}$ and the convergence (5.4.50) of $H[\overline{\tilde{M}}_{\Delta t}, \overline{\eta}_{\Delta t}](\overline{\eta}_{\Delta t})$, we may pass to the limit in the magnetic force balance (5.4.51). This yields the limit identity

$$\begin{aligned}
&\int_0^T \int_{\Omega_0} \det\left(\nabla_X \eta\right) \tilde{\Psi}_M\left(\nabla_X \eta, \tilde{M}\right) \cdot \tilde{\hat{M}} - \mu \det\left(\nabla_X \eta\right) H\left[\tilde{M}, \eta\right](\eta) \cdot \tilde{\hat{M}} \\
&+ 2A \det\left(\nabla_X \eta\right)\left(\nabla_X\left(\frac{1}{\det\left(\nabla_X \eta\right)} \tilde{M}\right)\left(\nabla_X \eta\right)^{-1}\right) : \left(\nabla_X \tilde{\hat{M}}\left(\nabla_X \eta\right)^{-1}\right) \\
&+ \frac{1}{\beta^2} \det\left(\nabla_X \eta\right)\left(\left|\frac{1}{\det\left(\nabla_X \eta\right)} \tilde{M}\right|^2 - 1\right)\left(\frac{1}{\det\left(\nabla_X \eta\right)}\right) \tilde{M} \cdot \tilde{\hat{M}} \\
&+ \partial_t \tilde{M} \cdot \tilde{\hat{M}} - \mu \det\left(\nabla_X \eta\right) H_{\mathrm{ext}}(\eta) \cdot \tilde{\hat{M}} \, dX dt = 0
\end{aligned} \tag{5.4.57}$$

for all $\tilde{\hat{M}} \in L^\infty(0,T; H^1(\Omega_0))$. It remains to transform this equation to the current configuration. To this end we consider an arbitrary test function $\hat{M} \in L^\infty(0,T; H^1(\Omega(\cdot)))$. Due to the relations

$$\int_{\Omega_0} \left|\hat{M}\left(\eta(t)\right)\right|^2 \, dX = \int_{\Omega(t)} \frac{1}{\det\left(\left[\nabla_X \eta(t)\right]\left(\eta^{-1}(t)\right)\right)} \left|\hat{M}(t)\right|^2 \, dx \leqslant c \left\|\hat{M}\right\|^2_{L^\infty(0,T;L^2(\Omega(\cdot)))}$$

and

$$\begin{aligned}
\int_{\Omega_0} \left|\nabla_X \hat{M}\left(\eta(t)\right)\right|^2 \, dX &= \int_{\Omega(t)} \frac{1}{\det\left(\left[\nabla_X \eta(t)\right]\left(\eta^{-1}(t)\right)\right)} \left|\nabla \hat{M}(t)\left[\nabla_X \eta(t)\right]\left(\eta^{-1}(t)\right)\right|^2 \, dx \\
&\leqslant c \left\|\nabla \hat{M}\right\|^2_{L^\infty(0,T;L^2(\Omega(\cdot)))}
\end{aligned}$$

for almost all $t \in [0,T]$ it holds that $\hat{M}(\eta) \in L^\infty(0,T; H^1(\Omega_0))$. Thus we may use $\hat{M}(\eta)$ as a test function in the equation (5.4.57). Transforming the resulting equation to the current configuration we infer that the pair (η, M), where M denotes the magnetization in the current configuration defined in (5.4.31), satisfies the desired magnetic force balance (5.1.19) (cf. also the identity (5.1.23)),

$$\begin{aligned}
&\int_0^T \int_{\Omega(t)} \tilde{\Psi}_M\left(\left[\nabla_X \eta\right]\left(\eta^{-1}\right), \det\left(\left[\nabla_X \eta\right]\left(\eta^{-1}\right)\right) M\right) \cdot \hat{M} - \mu H\left[\tilde{M}, \eta\right] \cdot \hat{M} + 2A \nabla M : \nabla \hat{M} \\
&+ \frac{1}{\beta^2}\left(|M|^2 - 1\right) M \cdot \hat{M} + \left[\partial_t M + (v \cdot \nabla) M + (\nabla \cdot v) M\right] \cdot \hat{M} - \mu H_{\mathrm{ext}} \cdot \hat{M} \, dx dt = 0
\end{aligned} \tag{5.4.58}$$

for any test function $\hat{M} \in L^\infty(0,T; H^1(\Omega(\cdot)))$.

5.4.4 Equation of motion

It remains to pass to the limit in the discrete equation of motion (5.4.4). Again we study the individual terms appearing in the equation separately. In order to pass to the limit in the exchange energy terms we first show strong convergence of the magnetization gradient. To this end we make use of

Minty's trick: We return to the discrete magnetic force balance (5.4.5) and test it by the test function $\tilde{\tilde{M}} = \overline{\tilde{M}}_{\Delta t} \in L^\infty(0,T;H^1(\Omega_0))$, which results in the identity

$$\begin{aligned}
&\int_0^T \int_{\Omega_0} 2A \det\left(\nabla_X \overline{\eta}_{\Delta t}\right) \left| \nabla_X \left(\frac{1}{\det\left(\nabla_X \overline{\eta}_{\Delta t}\right)} \overline{\tilde{M}}_{\Delta t} \right) \left(\nabla_X \overline{\eta}_{\Delta t}\right)^{-1} \right|^2 \, dX dt \\
&= - \int_0^T \int_{\Omega_0} \tilde{\Psi}_M \left(\nabla_X \overline{\eta}_{\Delta t}, \overline{\tilde{M}}_{\Delta t} \right) \cdot \overline{\tilde{M}}_{\Delta t} - \mu H \left[\overline{\tilde{M}}_{\Delta t}, \overline{\eta}_{\Delta t} \right] \left(\overline{\eta}_{\Delta t}\right) \cdot \overline{\tilde{M}}_{\Delta t} \\
&+ \frac{1}{\beta^2} \det\left(\nabla_X \overline{\eta}_{\Delta t}\right) \left(\left| \frac{1}{\det\left(\nabla_X \overline{\eta}_{\Delta t}\right)} \overline{\tilde{M}}_{\Delta t} \right|^4 - \left| \frac{1}{\det\left(\nabla_X \overline{\eta}_{\Delta t}\right)} \overline{\tilde{M}}_{\Delta t} \right|^2 \right) \\
&+ \det\left(\nabla_X \overline{\eta}'_{\Delta t}\right) \left[\frac{\frac{1}{\det\left(\nabla_X \overline{\eta}_{\Delta t}\right)} \overline{\tilde{M}}_{\Delta t} - \frac{1}{\det\left(\nabla_X \overline{\eta}'_{\Delta t}\right)} \overline{\tilde{M}}'_{\Delta t}}{\Delta t} + \frac{\operatorname{tr}\left(\nabla_X \left(\frac{\overline{\eta}_{\Delta t} - \overline{\eta}'_{\Delta t}}{\Delta t} \right) \left(\nabla_X \overline{\eta}'_{\Delta t}\right)^{-1} \right)}{\det\left(\nabla_X \overline{\eta}'_{\Delta t}\right)} \overline{\tilde{M}}_{\Delta t} \right] \\
&\cdot \left[\frac{1}{\det\left(\nabla_X \overline{\eta}_{\Delta t}\right)} \overline{\tilde{M}}_{\Delta t} + \frac{\operatorname{tr}\left(\left(\nabla_X \overline{\eta}_{\Delta t} - \nabla_X \overline{\eta}'_{\Delta t} \right) \left(\nabla_X \overline{\eta}'_{\Delta t}\right)^{-1} \right)}{\det\left(\nabla_X \overline{\eta}'_{\Delta t}\right)} \overline{\tilde{M}}_{\Delta t} \right] - \mu \overline{(H_{\text{ext}})}_{\Delta t} \left(\overline{\eta}_{\Delta t}\right) \cdot \overline{\tilde{M}}_{\Delta t} \, dX dt.
\end{aligned}$$

Due to the strong convergence (5.4.39) of $\overline{\tilde{M}}_{\Delta t}$ we can pass to the limit in the right-hand side of this equation, under exploitation of the convergences (5.4.28), (5.4.38), (5.4.50), (5.4.52), (5.4.53), (5.4.56), in the same way as in the limit passage in the magnetic force balance in Section 5.4.3. This results in the relation

$$\begin{aligned}
&\lim_{\Delta t \to 0} \int_0^T \int_{\Omega_0} 2A \det\left(\nabla_X \overline{\eta}_{\Delta t}\right) \left| \nabla_X \left(\frac{1}{\det\left(\nabla_X \overline{\eta}_{\Delta t}\right)} \overline{\tilde{M}}_{\Delta t} \right) \left(\nabla_X \overline{\eta}_{\Delta t}\right)^{-1} \right|^2 \, dX dt \\
&= - \int_0^T \int_{\Omega_0} \tilde{\Psi}_M \left(\nabla_X \eta, \tilde{M} \right) \cdot \tilde{M} - \mu H \left[\tilde{M}, \eta \right] (\eta) \cdot \tilde{M} \\
&+ \frac{1}{\beta^2} \det\left(\nabla_X \eta\right) \left(\left| \frac{1}{\det\left(\nabla_X \eta\right)} \tilde{M} \right|^4 - \left| \frac{1}{\det\left(\nabla_X \eta\right)} \tilde{M} \right|^2 \right) + \partial_t \tilde{M} \cdot \left(\frac{1}{\det\left(\nabla_X \eta\right)} \tilde{M} \right) \\
&- \mu H_{\text{ext}} (\eta) \cdot \tilde{M} \, dX dt.
\end{aligned}$$

We compare this identity to the magnetic force balance (5.4.57) tested by $\frac{1}{\det(\nabla_X \eta)} \tilde{M} \in L^\infty(0,T;H^1(\Omega))$ and infer that

$$\begin{aligned}
&\lim_{\Delta t \to 0} \int_0^T \int_{\Omega_0} 2A \det\left(\nabla_X \overline{\eta}_{\Delta t}\right) \left| \nabla_X \left(\frac{1}{\det\left(\nabla_X \overline{\eta}_{\Delta t}\right)} \overline{\tilde{M}}_{\Delta t} \right) \left(\nabla_X \overline{\eta}_{\Delta t}\right)^{-1} \right|^2 \, dX dt \\
&= \int_0^T \int_{\Omega_0} 2A \det\left(\nabla_X \eta\right) \left| \nabla_X \left(\frac{1}{\det\left(\nabla_X \eta\right)} \tilde{M} \right) \left(\nabla_X \eta\right)^{-1} \right|^2 \, dX dt.
\end{aligned}$$

Since weak convergence in $L^2((0,T) \times \Omega_0)$ together with convergence of the $L^2((0,T) \times \Omega_0)$-norm implies strong convergence in $L^2((0,T) \times \Omega_0)$, we have thus shown that

$$\begin{aligned}
&\sqrt{\det\left(\nabla_X \overline{\eta}_{\Delta t}\right)} \nabla_X \left(\frac{1}{\det\left(\nabla_X \overline{\eta}_{\Delta t}\right)} \overline{\tilde{M}}_{\Delta t} \right) \left(\nabla_X \overline{\eta}_{\Delta t}\right)^{-1} \\
&\to \sqrt{\det\left(\nabla_X \eta\right)} \nabla_X \left(\frac{1}{\det\left(\nabla_X \eta\right)} \tilde{M} \right) \left(\nabla_X \eta\right)^{-1} \qquad \text{in } L^2\left((0,T) \times \Omega_0\right).
\end{aligned}$$

In combination with the bound (5.4.18) of $\det(\nabla_X \overline{\eta}_{\Delta t})$ away from zero and the uniform convergence (5.4.28) of both $\overline{\eta}_{\Delta t}$ and its gradient this implies the desired strong convergence

$$\nabla_X \left(\frac{1}{\det\left(\nabla_X \overline{\eta}_{\Delta t}\right)} \overline{\tilde{M}}_{\Delta t} \right) \to \nabla_X \left(\frac{1}{\det\left(\nabla_X \eta\right)} \tilde{M} \right) \qquad \text{in } L^2\left((0,T) \times \Omega_0\right). \tag{5.4.59}$$

Now, in order to pass to the limit in the exchange energy terms, we consider an arbitrary test function $\chi \in \mathcal{D}((0,T) \times \Omega_0)$ and use the identity

$$\partial_{X_i} (\nabla_X \overline{\eta}_{\Delta t})^{-1} = -(\nabla_X \overline{\eta}_{\Delta t})^{-1} \partial_{X_i} (\nabla_X \overline{\eta}_{\Delta t}) (\nabla_X \overline{\eta}_{\Delta t})^{-1} \quad \forall i = 1, ..., 3$$

to write

$$\begin{aligned}
&\partial_{X_i} \left(\frac{\operatorname{tr}\left(\nabla_X \chi (\nabla_X \overline{\eta}_{\Delta t})^{-1}\right)}{\det (\nabla_X \overline{\eta}_{\Delta t})} \overline{\tilde{M}}_{\Delta t} \right) \\
&= \operatorname{tr}\left(\partial_{X_i} \nabla_X \chi (\nabla_X \overline{\eta}_{\Delta t})^{-1} - \nabla_X \chi (\nabla_X \overline{\eta}_{\Delta t})^{-1} \partial_{X_i} (\nabla_X \overline{\eta}_{\Delta t}) (\nabla_X \overline{\eta}_{\Delta t})^{-1}\right) \frac{1}{\det (\nabla_X \overline{\eta}_{\Delta t})} \overline{\tilde{M}}_{\Delta t} \\
&\quad + \operatorname{tr}\left(\nabla_X \chi (\nabla_X \overline{\eta}_{\Delta t})^{-1}\right) \partial_{X_i} \left(\frac{1}{\det (\nabla_X \overline{\eta}_{\Delta t})} \overline{\tilde{M}}_{\Delta t} \right).
\end{aligned}$$

From the weak convergence (5.4.23) of $\nabla_X^2 \overline{\eta}_{\Delta t}$, the strong convergence (5.4.38) of $\frac{1}{\det(\nabla_X \overline{\eta}_{\Delta t})} \overline{\tilde{M}}_{\Delta t}$ and the (strong) convergence (5.4.59) of its gradient we thus infer that

$$\nabla_X \left(\frac{\operatorname{tr}\left(\nabla_X \chi (\nabla_X \overline{\eta}_{\Delta t})^{-1}\right)}{\det (\nabla_X \overline{\eta}_{\Delta t})} \overline{\tilde{M}}_{\Delta t} \right) \rightharpoonup \nabla_X \left(\frac{\operatorname{tr}\left(\nabla_X \chi (\nabla_X \eta)^{-1}\right)}{\det (\nabla_X \eta)} \tilde{M} \right) \qquad \text{in } L^2((0,T) \times \Omega_0).$$

This, in combination with the strong convergence (5.4.59) of the magnetization gradient, immediately implies that

$$\begin{aligned}
&\int_0^T \int_{\Omega_0} A \det (\nabla_X \overline{\eta}_{\Delta t}) \left| \nabla_X \left(\frac{1}{\det (\nabla_X \overline{\eta}_{\Delta t})} \overline{\tilde{M}}_{\Delta t} \right) (\nabla_X \overline{\eta}_{\Delta t})^{-1} \right|^2 \left((\nabla_X \overline{\eta}_{\Delta t})^{-1} \right)^T : \nabla_X \chi \\
&- 2A \det (\nabla_X \overline{\eta}_{\Delta t}) \left[\nabla_X \left(\frac{1}{\det (\nabla_X \overline{\eta}_{\Delta t})} \overline{\tilde{M}}_{\Delta t} \right) (\nabla_X \overline{\eta}_{\Delta t})^{-1} \right] : \left[\nabla_X \left(\frac{\operatorname{tr}\left(\nabla_X \chi (\nabla_X \overline{\eta}_{\Delta t})^{-1}\right)}{\det (\nabla_X \overline{\eta}_{\Delta t})} \overline{\tilde{M}}_{\Delta t} \right) \right. \\
&\left. (\nabla_X \overline{\eta}_{\Delta t})^{-1} + \nabla_X \left(\frac{1}{\det (\nabla_X \overline{\eta}_{\Delta t})} \overline{\tilde{M}}_{\Delta t} \right) (\nabla_X \overline{\eta}_{\Delta t})^{-1} \nabla_X \chi (\nabla_X \overline{\eta}_{\Delta t})^{-1} \right] dX dt \\
\to &\int_0^T \int_{\Omega_0} A \det (\nabla_X \eta) \left| \nabla_X \left(\frac{1}{\det (\nabla_X \eta)} \tilde{M} \right) (\nabla_X \eta)^{-1} \right|^2 \left((\nabla_X \eta)^{-1} \right)^T : \nabla_X \chi \\
&- 2A \det (\nabla_X \eta) \left[\nabla_X \left(\frac{1}{\det (\nabla_X \eta)} \tilde{M} \right) (\nabla_X \eta)^{-1} \right] : \left[\nabla_X \left(\frac{\operatorname{tr}\left(\nabla_X \chi (\nabla_X \eta)^{-1}\right)}{\det (\nabla_X \eta)} \tilde{M} \right) (\nabla_X \eta)^{-1} \right. \\
&\left. + \nabla_X \left(\frac{1}{\det (\nabla_X \eta)} \tilde{M} \right) (\nabla_X \eta)^{-1} \nabla_X \chi (\nabla_X \eta)^{-1} \right] dX dt \qquad (5.4.60)
\end{aligned}$$

for any $\chi \in \mathcal{D}((0,T) \times \Omega_0)$. For the limit passage in the stray field term we require (local) weak convergence of the gradient of the stray field. In order to show this we consider an arbitrary compact set $K \subset \Omega_0$. From the uniform bound (5.4.18) of $\det(\nabla_X \overline{\eta}_{\Delta t})$ away from zero we infer that

$$\begin{aligned}
&\int_0^T \int_K \left| \nabla_X H \left[\overline{\tilde{M}}_{\Delta t}, \overline{\eta}_{\Delta t} \right] (\overline{\eta}_{\Delta t}) \right|^2 dX dt \\
&= \int_0^T \int_{\overline{\eta}_{\Delta t}(t,K)} \frac{1}{\det \left([\nabla_X \overline{\eta}_{\Delta t}] (\overline{\eta}_{\Delta t}^{-1}) \right)} \left| \left(\nabla H \left[\overline{\tilde{M}}_{\Delta t}, \overline{\eta}_{\Delta t} \right] \right) \left([\nabla_X \overline{\eta}_{\Delta t}] (\overline{\eta}_{\Delta t}^{-1}) \right) \right|^2 dx dt \\
&\leqslant c \left\| \nabla H \left[\overline{\tilde{M}}_{\Delta t}, \overline{\eta}_{\Delta t} \right] \right\|_{L^2(0,T;L^2(\overline{\eta}_{\Delta t}(\cdot,K)))}^2 \qquad (5.4.61)
\end{aligned}$$

for a constant $c > 0$ independent of Δt. Further, from the compactness of K and the uniform convergence (5.4.28) of $\overline{\eta}_{\Delta t}$ we infer the existence of some constant $\delta > 0$ independent of Δt such that

$$\operatorname{dist}\left(\overline{\eta}_{\Delta t}(t,K), \partial \overline{\Omega}_{\Delta t}(t) \right) \geqslant \delta \qquad \forall t \in [0,T]$$

for all sufficiently small $\Delta t > 0$. This allows us to apply the bound (A.2.10), given by Lemma A.2.3 in the appendix for solutions to the Poisson equation, to the right-hand side of the inequality (5.4.61). Hence we infer that, for another constant $c > 0$ independent of Δt,

$$\begin{aligned}
&\left\|\nabla_X H\left[\widetilde{\overline{M}}_{\Delta t}, \overline{\eta}_{\Delta t}\right](\overline{\eta}_{\Delta t})\right\|^2_{L^2((0,T)\times K)} \\
\leqslant & c\int_0^T \int_{\overline{\overline{\Omega}}_{\Delta t}(t)} \left|\overline{M}_{\Delta t}\right|^2 + \left|\nabla \overline{M}_{\Delta t}\right|^2 \, dxdt \\
\leqslant & c\int_0^T \int_{\Omega_0} \det\left(\nabla_X \overline{\eta}_{\Delta t}\right)\left[\left|\frac{1}{\det\left(\nabla_X \overline{\eta}_{\Delta t}\right)}\widetilde{\overline{M}}_{\Delta t}\right|^2 + \left|\nabla_X\left(\frac{1}{\det\left(\nabla_X \overline{\eta}_{\Delta t}\right)}\widetilde{\overline{M}}_{\Delta t}\right)\left(\nabla_X \overline{\eta}_{\Delta t}\right)^{-1}\right|^2\right] dXdt \leqslant c
\end{aligned}$$

thanks to the uniform bounds (5.4.17), (5.4.19) and (5.4.22). Thus, using a diagonal argument, we may extract another subsequence for which it holds that

$$\nabla_X H\left[\widetilde{\overline{M}}_{\Delta t}, \overline{\eta}_{\Delta t}\right](\overline{\eta}_{\Delta t}) \rightharpoonup \nabla_X H\left[\tilde{M}, \eta\right](\eta) \qquad \text{in } L^2\left((0,T)\times K\right)$$

for any compact set $K \subset \Omega_0$, where the identification of the limit function results from the already known convergence (5.4.50). In combination with the uniform convergence (5.4.28) of the inverse deformation gradient and the strong convergence (5.4.39) of the magnetization this implies that, for any compactly supported test function $\chi \in \mathcal{D}((0,T)\times\Omega_0)$,

$$\begin{aligned}
&\int_0^T \int_{\Omega_0} \mu\left[\left(\left(\nabla_X H\left[\widetilde{\overline{M}}_{\Delta t}, \overline{\eta}_{\Delta t}\right](\overline{\eta}_{\Delta t})\right)\left(\nabla_X \overline{\eta}_{\Delta t}\right)^{-1}\right)^T \widetilde{\overline{M}}_{\Delta t}\right]\cdot\chi \, dXdt \\
\to &\int_0^T \int_{\Omega_0} \mu\left[\left(\left(\nabla_X H\left[\tilde{M}, \eta\right](\eta)\right)\left(\nabla_X \eta\right)^{-1}\right)^T \tilde{M}\right]\cdot\chi \, dXdt.
\end{aligned} \tag{5.4.62}$$

Next we remark that due to the uniform bound (5.4.17) we can extract another subsequence and find some limit function $z \in L^\infty(0,T; L^{\frac{q}{q-1}}(\Omega_0))$, which will be identified in (5.4.70) below, such that

$$\left|\nabla_X^2 \overline{\eta}_{\Delta t}\right|^{q-2} \nabla_X^2 \overline{\eta}_{\Delta t} \overset{*}{\rightharpoonup} z \qquad \text{in } L^\infty\left(0,T; L^{\frac{q}{q-1}}\left(\Omega_0\right)\right). \tag{5.4.63}$$

For the limit passage in the terms involving the anisotropy energy density $\tilde{\Psi}$ and the elastic energy density W we recall the boundedness assumptions (5.1.10), (5.1.11) on W' and $\tilde{\Psi}_F$, which imply that

$$\left|W'\left(\nabla_X \overline{\eta}_{\Delta t}\right)\right| + \left|\tilde{\Psi}_F\left(\nabla_X \overline{\eta}_{\Delta t}, \widetilde{\overline{M}}_{\Delta t}\right)\right| \leqslant c\left(1 + \left|\nabla_X \overline{\eta}_{\Delta t}\right|^{p_2} + \left|\widetilde{\overline{M}}_{\Delta t}\right|^{p_3}\right),$$

where $1 \leqslant p_2 < \infty$ and $1 \leqslant p_3 < 6$. Together with the uniform bounds (5.4.17) and (5.4.21) this implies that

$$\left\|W'\left(\nabla_X \overline{\eta}_{\Delta t}\right)\right\|_{L^p((0,T)\times\Omega_0)} + \left\|\tilde{\Psi}_F\left(\nabla_X \overline{\eta}_{\Delta t}, \widetilde{\overline{M}}_{\Delta t}\right)\right\|_{L^p((0,T)\times\Omega_0)} \leqslant c$$

for some $p > 1$. Therefore, the strong convergences (5.4.28) and (5.4.39), together with the continuity (5.1.8) of W' and $\tilde{\Psi}_F$ and the Vitali convergence theorem, yield

$$W'\left(\nabla_X \overline{\eta}_{\Delta t}\right) \to W'\left(\nabla_X \eta\right) \qquad \text{in } L^p\left((0,T)\times\Omega_0\right), \tag{5.4.64}$$

$$\tilde{\Psi}_F\left(\nabla_X \overline{\eta}_{\Delta t}, \widetilde{\overline{M}}_{\Delta t}\right) \to \tilde{\Psi}_F\left(\nabla_X \eta, \tilde{M}\right) \qquad \text{in } L^p\left((0,T)\times\Omega_0\right) \tag{5.4.65}$$

for some $p > 1$. For the forcing term f we know from Lemma A.3.2 (i) in the appendix that

$$\overline{f}_{\Delta t} \to f \qquad \text{in } C\left([0,T]; L^p\left(\mathbb{R}^3\right)\right) \quad \text{for all } 1 \leqslant p < \infty,$$

which, as in the derivation of the convergence (5.4.56) of the quantity $H_{\text{ext}}(\overline{\eta}_{\Delta t})$, implies that

$$\overline{f}_{\Delta t}\left(\overline{\eta}'_{\Delta t}\right) \to f(\eta) \qquad \text{in } L^p\left((0,T)\times\Omega_0\right) \quad \text{for all } 1 \leqslant p < \infty. \tag{5.4.66}$$

Finally we notice that the term involving the material derivative of the magnetization vanishes as Δt tends to zero: Indeed, from the uniform bound (5.4.35) as well as the strong convergences (5.4.28) of the deformation gradient and its inverse and (5.4.39) of the magnetization we see that

$$\int_0^T \int_{\Omega_0} \det\left(\nabla_X \overline{\eta}'_{\Delta t}\right) \left[\frac{\frac{1}{\det(\nabla_X \overline{\eta}_{\Delta t})} \overline{\widetilde{M}}_{\Delta t} - \frac{1}{\det(\nabla_X \overline{\eta}'_{\Delta t})} \overline{\widetilde{M}}'_{\Delta t}}{\Delta t} + \frac{\operatorname{tr}\left(\nabla_X \left(\frac{\overline{\eta}_{\Delta t} - \overline{\eta}'_{\Delta t}}{\Delta t}\right) \left(\nabla_X \overline{\eta}'_{\Delta t}\right)^{-1}\right)}{\det\left(\nabla_X \overline{\eta}'_{\Delta t}\right)} \overline{\widetilde{M}}_{\Delta t} \right]$$
$$\cdot \left[- \frac{\operatorname{tr}\left(\nabla_X \chi \left(\nabla_X \overline{\eta}_{\Delta t}\right)^{-1}\right)}{\det\left(\nabla_X \overline{\eta}_{\Delta t}\right)} \overline{\widetilde{M}}_{\Delta t} + \frac{\operatorname{tr}\left(\nabla_X \chi \left(\nabla_X \overline{\eta}'_{\Delta t}\right)^{-1}\right)}{\det\left(\nabla_X \overline{\eta}'_{\Delta t}\right)} \overline{\widetilde{M}}_{\Delta t} \right] dX dt \to 0. \tag{5.4.67}$$

Now we combine the convergences (5.4.60) and (5.4.62)–(5.4.67) with the weak convergence (5.4.23) of $\partial_t \eta_{\Delta t}$ as well as the strong convergences (5.4.28) of $\nabla_X \overline{\eta}_{\Delta t}$, $\nabla_X \overline{\eta}'_{\Delta t}$ and their inverses, (5.4.39) of $\overline{\widetilde{M}}_{\Delta t}$ and (5.4.56) of $H_{\text{ext}}(\overline{\eta}_{\Delta t})$ and pass to the limit in the discrete equation of motion (5.4.4). As a result we obtain the identity

$$\begin{aligned}
&\int_0^T \int_{\Omega_0} \left[W'\left(\nabla_X \eta\right) - a \frac{\operatorname{cof}\left(\nabla_X \eta\right)}{\left(\det\left(\nabla_X \eta\right)\right)^a} \right] : \nabla_X \chi + z \,\vdots\, \nabla_X^2 \chi \\
&+ \tilde{\Psi}_F\left(\nabla_X \eta, M\right) : \nabla_X \chi - \mu \left[\left(\left(\nabla_X H\left[\tilde{M}, \eta\right](\eta) \right) \left(\nabla_X \eta\right)^{-1} \right)^T \tilde{M} \right] \cdot \chi \\
&+ A \det\left(\nabla_X \eta\right) \left| \nabla_X \left(\frac{1}{\det\left(\nabla_X \eta\right)} \tilde{M} \right) \left(\nabla_X \eta\right)^{-1} \right|^2 \left(\left(\nabla_X \eta\right)^{-1} \right)^T : \nabla_X \chi \\
&- 2A \det\left(\nabla_X \eta\right) \left[\nabla_X \left(\frac{1}{\det\left(\nabla_X \eta\right)} \tilde{M} \right) \left(\nabla_X \eta\right)^{-1} \right] : \left[\nabla_X \left(\frac{\operatorname{tr}\left(\nabla_X \chi \left(\nabla_X \eta\right)^{-1}\right)}{\det\left(\nabla_X \eta\right)} \tilde{M} \right) \left(\nabla_X \eta\right)^{-1} \right. \\
&\left. + \nabla_X \left(\frac{1}{\det\left(\nabla_X \eta\right)} \tilde{M} \right) \left(\nabla_X \eta\right)^{-1} \nabla_X \chi \left(\nabla_X \eta\right)^{-1} \right] \\
&+ \frac{1}{4\beta^2} \det\left(\nabla_X \eta\right) \left(\left| \frac{1}{\det\left(\nabla_X \eta\right)} \tilde{M} \right|^2 - 1 \right)^2 \left(\left(\nabla_X \eta\right)^{-1} \right)^T : \nabla_X \chi \\
&- \frac{1}{\beta^2 \det\left(\nabla_X \eta\right)} \left(\left| \frac{1}{\det\left(\nabla_X \eta\right)} \tilde{M} \right|^2 - 1 \right) \left| \tilde{M} \right|^2 \left(\left(\nabla_X \eta\right)^{-1} \right)^T : \nabla_X \chi \\
&+ 2\nu \det\left(\nabla_X \eta\right) \left[\nabla_X \partial_t \eta \left(\nabla_X \eta\right)^{-1} \left(\left(\nabla_X \eta\right)^{-1} \right)^T \right] : \nabla_X \chi - \rho f(\eta) \cdot \chi \\
&- \mu \left[\left(\nabla_X \left(H_{\text{ext}}(\eta)\right) \left(\nabla_X \eta\right)^{-1} \right)^T \tilde{M} \right] \cdot \chi \, dX dt = 0
\end{aligned} \tag{5.4.68}$$

for all $\chi \in \mathcal{D}((0,T) \times \Omega_0)$. It remains to identify the weak-$*$ limit z of $|\nabla_X^2 \overline{\eta}_{\Delta t}|^{q-2} \nabla_X^2 \overline{\eta}_{\Delta t}$. To this end we choose an arbitrary non-negative smooth cutoff function $\phi \in \mathcal{D}((0,T) \times \Omega_0)$. After a density argument we may use the function $(\overline{\eta}_{\Delta t} - \eta)\phi$ as a test function for the equation of motion (5.4.4) on the Δt-level. Since

$$\left(\overline{\eta}_{\Delta t} - \eta\right) \phi \overset{*}{\rightharpoonup} 0 \quad \text{in } L^\infty\left(0, T; W^{2,q}\left(\Omega_0\right)\right), \qquad \left(\overline{\eta}_{\Delta t} - \eta\right) \phi \to 0 \quad \text{in } C\left([0,T]; C^1\left(\overline{\Omega_0}\right)\right),$$

we may repeat the above limit passage leading to the identity (5.4.68) and deduce that

$$\int_0^T \int_{\Omega_0} \left(\left| \nabla_X^2 \overline{\eta}_{\Delta t} \right|^{q-2} \nabla_X^2 \overline{\eta}_{\Delta t} \right) \,\vdots\, \nabla_X^2 \left(\left(\overline{\eta}_{\Delta t} - \eta\right) \phi \right) \, dX dt \to 0.$$

This in combination with the weak convergence (5.4.23) of $\overline{\eta}_{\Delta t}$ shows that

$$\begin{aligned}
&\left\| \left(\nabla_X^2 \overline{\eta}_{\Delta t} - \nabla_X^2 \eta \right) \phi^{\frac{1}{q}} \right\|_{L^q((0,T) \times \Omega_0)}^q \\
\leqslant c &\int_0^T \int_{\Omega_0} \left(\left| \nabla_X^2 \overline{\eta}_{\Delta t} \right|^{q-2} \nabla_X^2 \overline{\eta}_{\Delta t} - \left| \nabla_X^2 \eta \right|^{q-2} \nabla_X^2 \eta \right) \,\vdots\, \nabla_X^2 \left(\overline{\eta}_{\Delta t} - \eta\right) \phi \, dX dt \to 0.
\end{aligned}$$

Since ϕ can be chosen as an approximation of the constant function 1 on $(0,T)\times\Omega_0$ this implies that

$$\nabla_X^2\overline{\eta}_{\Delta t}\to\nabla_X^2\eta \qquad \text{in } L^q\left((0,T)\times\Omega_0\right). \tag{5.4.69}$$

This is sufficient to identify the limit function z in the convergence (5.4.63) as

$$z=\left|\nabla_X^2\eta\right|^{q-2}\nabla_X^2\eta \qquad \text{a.e. in } (0,T)\times\Omega_0. \tag{5.4.70}$$

Applying this identity to the limit (5.4.68) of the discrete equation of motion, we conclude that our limit functions satisfy the desired equation of motion (5.1.15) (cf. also the identity (5.1.21)).

5.4.5 Proof of the main result

We are now in the position to prove Theorem 5.1.1. Summarizing the results from Sections 5.4.1–5.4.4, we have so far shown the existence of a weak solution in the sense of Definition 5.1.1 - except for the regularity (5.1.18) of M - on the interval $[0,T)$ with $T>0$ chosen according to Lemma 5.4.1 and Lemma 5.4.2. Indeed, the regularity (5.1.13), (5.1.14) of the deformation and the magnetization follows from the convergences (5.4.23), (5.4.24), (5.4.53) and the fact that the set $\mathcal{E}$ is closed with respect to weak convergence in $W^{2,q}(\Omega_0)$. The equation of motion (5.1.15) (cf. also the identity (5.1.21)) follows from the identity (5.4.68) together with the identification (5.4.70). From the uniform in time convergences (5.4.26) and (5.4.39) of the deformation and the magnetization we further conclude the initial conditions (5.1.16), (5.1.17). The magnetic force balance (5.1.19) (cf. also the identity (5.1.23)) is shown in (5.4.58). Next we check the $L^\infty(0,T;H^1(\Omega(\cdot)))$-regularity (5.1.18) of M: Due to the $L^\infty(0,T;H^1(\Omega_0))$-regularity of $\frac{1}{\det(\nabla_X\eta)}\tilde{M}$ (cf. (5.4.25)) and the essential bounds of $\det(\nabla_X\eta)$ and $(\nabla_X\eta)^{-1}$ (cf. (5.4.28)) we know that

$$\begin{aligned}\operatorname{esssup}_{t\in[0,T]}\|M(t)\|^2_{L^2(\Omega(t))}&=\operatorname{esssup}_{t\in[0,T]}\int_{\Omega(t)}|M(t)|^2\,dx\\&=\operatorname{esssup}_{t\in[0,T]}\int_{\Omega_0}\det\left(\nabla_X\eta(t)\right)\left|\frac{1}{\det\left(\nabla_X\eta(t)\right)}\tilde{M}\right|^2\,dX<\infty\end{aligned}$$

and

$$\begin{aligned}\operatorname{esssup}_{t\in[0,T]}\|\nabla M(t)\|^2_{L^2(\Omega(t))}&=\operatorname{esssup}_{t\in[0,T]}\int_{\Omega(t)}|\nabla M(t)|^2\,dx\\&=\operatorname{esssup}_{t\in[0,T]}\int_{\Omega_0}\det\left(\nabla_X\eta(t)\right)\left|\nabla_X\left(\frac{1}{\det\left(\nabla_X\eta(t)\right)}\tilde{M}\right)\left(\nabla_X\eta\right)^{-1}\right|^2\,dX\\&<\infty.\end{aligned}$$

This proves the desired $L^\infty(0,T;H^1(\Omega(\cdot)))$-regularity (5.1.18) of M.

Finally, it remains to show that the existence time T can be chosen as $T=T'$ for $T'>0$ as in Theorem 5.1.1. To this end we denote by $T_{\max}>0$ the maximal time such that $(\eta,\tilde{M})$ is a weak solution on each interval $[0,T)$ with $0<T<T_{\max}$. We assume that $T_{\max}<\infty$ and

$$\liminf_{t\to T_{\max}}\tilde{E}\left(\eta(t),\tilde{M}(t)\right)<\infty. \tag{5.4.71}$$

We first show that the pair $(\eta,\tilde{M})$ satisfies an energy inequality on the interval $[0,T_{\max}]$. In order to do so we return to the discrete energy inequality in the form (5.4.14), which holds true for arbitrarily large discrete time indices $k\in\mathbb{N}$. In terms of the interpolants of the discrete solution this inequality can be expressed as

$$\begin{aligned}&\frac{1}{2}\tilde{E}\left(\overline{\eta}_{\Delta t}(\tau),\overline{\tilde{M}}_{\Delta t}(\tau)\right)+\int_0^\tau\overline{\tilde{R}}_{\Delta t}\left(\overline{\eta}_{\Delta t}(t),\overline{\tilde{M}}_{\Delta t}(t)\right)\,dt\\&\leqslant c_4+c_3\left(\tau+\Delta t\right)+\int_0^\tau\int_{\Omega_0}2\left(\left|\frac{1}{\det\left(\nabla_X\overline{\eta}_{\Delta t}\right)}\overline{\tilde{M}}_{\Delta t}\right|^2-1\right)^2\det\left(\nabla_X\overline{\eta}_{\Delta t}\right)+\frac{1}{2}W\left(\overline{\eta}_{\Delta t}\right)\\&\quad+\frac{1}{2q}\left|\nabla_X^2\overline{\eta}_{\Delta t}\right|^q\,dXdt+\int_0^{\tau+\Delta t}\int_{\Omega_0}\rho\overline{f}_{\Delta t}\left(\overline{\eta}'_{\Delta t}\right)\cdot\partial_t\eta_{\Delta t}\,dXdt\end{aligned}$$

for any $\tau \in [0,\infty)$, where $c_3, c_4 > 0$ denote the constants introduced in (5.4.12) and (5.4.15), respectively, independent of Δt and τ. For almost all $\tau \in [0, T_{\max})$ we can use the convergences (5.4.23), (5.4.28), (5.4.38), (5.4.39), (5.4.50), (5.4.53), (5.4.59), (5.4.66) and (5.4.69) in combination with the weak lower semicontinuity of norms to pass to the limit in this inequality. This yields the estimate

$$\begin{aligned}&\frac{1}{2}\tilde{E}\left(\eta(\tau),\tilde{M}(\tau)\right)+\int_0^\tau \tilde{R}\left(\eta,\tilde{M}\right)\,dt\\ \leqslant & c_4+c_3T_{\max}+\int_0^\tau\int_{\Omega_0} 2\left(\left|\frac{1}{\det\left(\nabla_X\eta\right)}\tilde{M}\right|^2-1\right)^2\det\left(\nabla_X\eta\right)+\frac{1}{2}W\left(\eta\right)+\frac{1}{2q}\left|\nabla_X^2\eta\right|^q\,dXdt\\ &+\int_0^\tau\int_{\Omega_0}\rho f\left(\eta\right)\cdot\partial_t\eta\,dXdt \end{aligned} \tag{5.4.72}$$

for almost all $\tau \in [0, T_{\max}]$. Next, due to the bound (5.4.71), we find a sequence $(t_i)_{i\in\mathbb{N}} \subset [0, T_{\max})$ and a constant $c > 0$ independent of i such that $t_i \to T_{\max}$ for $i \to \infty$ and $\tilde{E}(\eta(t_i), \tilde{M}(t_i)) \leqslant c$ for all $i \in \mathbb{N}$. In particular, we may extract a subsequence and find functions $\eta(T_{\max}) \in \mathcal{E}$ and $\tilde{M}(T_{\max}) \in H^1(\Omega_0)$ such that

$$\eta\left(t_i\right)\rightharpoonup\eta\left(T_{\max}\right)\ \text{ in } W^{2,q}\left(\Omega_0\right),\quad \eta\left(t_i\right)\to\eta\left(T_{\max}\right)\ \text{ in } C^1\left(\overline{\Omega_0}\right),\quad \tilde{M}\left(t_i\right)\to\tilde{M}\left(T_{\max}\right)\ \text{ in } L^4\left(\Omega_0\right),$$
$$\frac{1}{\det\left(\nabla_X\eta\left(t_i\right)\right)}\tilde{M}\left(t_i\right)\rightharpoonup\frac{1}{\det\left(\nabla_X\eta\left(T_{\max}\right)\right)}\tilde{M}\left(T_{\max}\right)\ \text{ in } H^1\left(\Omega_0\right)$$

for $t_i \to T_{\max}$ and

$$E\left(\eta\left(T_{\max}\right),\tilde{M}\left(T_{\max}\right)\right)\leqslant\liminf_{t_i\to T_{\max}}E\left(\eta\left(t_i\right),\tilde{M}\left(t_i\right)\right)<\infty. \tag{5.4.73}$$

The regularity

$$\eta\in L^\infty\left(0,T;W^{2,q}\left(\Omega_0\right)\right)\bigcap C\left([0,T];C^1\left(\overline{\Omega_0}\right)\right),\qquad \tilde{M}\in L^\infty\left(0,T;H^1\left(\Omega_0\right)\right)\bigcap C\left([0,T];L^2\left(\Omega_0\right)\right)$$

for all $0 \leqslant T < T_{\max}$, cf. (5.4.23), (5.4.24), (5.4.26) and (5.4.39), implies that, cf. Remark A.4.1,

$$\eta\in C_{\text{weak}}\left([0,T_{\max});W^{2,q}\left(\Omega_0\right)\right),\qquad \tilde{M}\in C_{\text{weak}}\left([0,T_{\max});H^1\left(\Omega_0\right)\right).$$

Consequently the functions $\eta(T_{\max})$ and $\tilde{M}(T_{\max})$ can be understood not only as the limits of $\eta(t_i)$ and $\tilde{M}(t_i)$ for $t_i \to T_{\max}$ but as the limits of $\eta(t)$ and $\tilde{M}(t)$ for $t \to T_{\max}$ respectively. In particular, since weakly convergent sequences are bounded, it follows that

$$\tilde{E}_{\text{el}}\left(\eta(t)\right)\leqslant c\qquad\text{for all } t\in[0,T_{\max}]$$

for a constant $c > 0$ independent of t. Hence, by Lemma A.7.1 in the appendix, $\det(\nabla_X\eta(t))$ is bounded away from zero in Ω_0, uniformly with respect to $t \in [0, T_{\max}]$. In combination with the Poincaré inequality and the Young inequality this yields, exactly as in the derivation of the first inequality in (5.3.2),

$$\begin{aligned}&\int_{\Omega_0}\nu\left|\nabla_X\partial_t\eta(t)\left(\nabla_X\eta(t)\right)^{-1}\right|^2\det\left(\nabla_X\eta(t)\right)-\rho f\left(t,\eta(t)\right)\cdot\partial_t\eta(t)\,dX\\ \geqslant&\frac{\nu c}{2}\int_{\Omega_0}\left|\nabla_X\partial_t\eta(t)\right|^2\,dX-\frac{|\Omega_0|\left(\tilde{c}\rho\left\|f\right\|_{L^\infty((0,\infty)\times\mathbb{R}^3)}\right)^2}{2\nu c}\end{aligned}$$

for almost all $t \in [0, T_{\max}]$, the constant $\tilde{c} > 0$ from the Poincaré inequality on Ω_0 and another constant $c > 0$ independent of t. Applying this to the energy estimate (5.4.72) in the limit system we infer that

$$\begin{aligned}&\frac{1}{2}\tilde{E}\left(\eta(\tau),\tilde{M}(\tau)\right)+\int_0^\tau\int_{\Omega_0}\frac{\nu c}{2}\left|\nabla_X\partial_t\eta\right|^2+\frac{1}{2}\left|\frac{1}{\det\left(\nabla_X\eta\right)}\partial_t\tilde{M}\right|^2\det\left(\nabla_X\eta\right)\,dXdt\\ \leqslant&c+\int_0^\tau\max\left\{1,\ 8\beta^2\right\}\frac{1}{2}\tilde{E}\left(\eta(t),\tilde{M}(t)\right)\,dt\end{aligned}$$

for almost all $\tau \in [0, T_{\max}]$ and yet another constant $c > 0$ independent of τ. From the Gronwall inequality we thus infer that

$$\frac{1}{2}\tilde{E}\left(\eta(\tau), \tilde{M}(\tau)\right) + \int_0^\tau \int_{\Omega_0} \frac{\nu c}{2} \left|\nabla_X \partial_t \eta\right|^2 + \frac{1}{2}\left|\frac{1}{\det\left(\nabla_X \eta\right)} \partial_t \tilde{M}\right|^2 \det\left(\nabla_X \eta\right)\ dXdt \leqslant c e^{\max\{1,\ 8\beta^2\} T_{\max}}$$

for almost all $\tau \in [0, T_{\max}]$ and in particular

$$\eta \in L^\infty\left(0, T_{\max}; \mathcal{E}\right) \bigcap C\left([0, T_{\max}]; C^1\left(\overline{\Omega_0}\right)\right), \tag{5.4.74}$$

$$\tilde{M} \in L^\infty\left(0, T_{\max}; H^1\left(\Omega_0\right)\right) \bigcap C\left([0, T_{\max}]; L^2\left(\Omega_0\right)\right), \tag{5.4.75}$$

$$\partial_t \eta \in L^2\left(0, T_{\max}; H^1\left(\Omega_0\right)\right), \qquad \partial_t \tilde{M} \in L^2\left((0, T_{\max}) \times \Omega_0\right). \tag{5.4.76}$$

With the regularity (5.4.74)–(5.4.76) at hand we may conclude the proof via a contradiction argument: We assume in addition that $\eta(T_{\max}) \in \mathrm{int}(\mathcal{E})$. Then we may apply the local existence result we have proved so far to construct a solution on the interval $[T_{\max}, T_{\max} + \epsilon)$ for some small $\epsilon > 0$. Due to the regularity (5.4.74)–(5.4.76) the two solutions can be assembled to a solution on the interval $[0, T_{\max} + \epsilon)$. This, however, results in a contradiction to the maximality of $T_{\max}$. Therefore, the assumption $\eta(T_{\max}) \in \mathrm{int}(\mathcal{E})$ is wrong. It follows that $\eta(T_{\max}) \in \partial\mathcal{E}$ and hence $T_{\max} = T'$, which finishes the proof of Theorem 5.1.1.

Chapter 6

Conclusion

In the first part of this thesis we dealt with the interaction problem between an electrically conducting fluid, insulating rigid bodies as well as the electromagnetic fields inside both of these materials. After a brief look at the derivation of the models for this problem - which consist of a coupling between the Navier-Stokes equations, the Maxwell equations and the balances of linear and angular momentum of the rigid bodies - in Chapter 2, we turned to the associated mathematical analysis.

In Chapter 3 we proved the local-in-time existence of weak solutions to the problem in the case of an incompressible electrically conducting fluid and one insulating rigid body. Our proof relied on a hybrid discrete-continuous approximation scheme: The main part of the system was discretized with respect to time via the Rothe method, which allowed us to deal with the high coupling of the problem caused by the solution-dependent test functions in the induction equation. The transport equation for the characteristic function instead was solved directly as a continuous equation on the small intervals between all consecutive discrete time points. This was necessary to prevent the function from taking values apart from 0 and 1, so that the position of the solid body could be determined precisely at all discrete times. The solution-dependent test functions in the momentum equation could be handled via a use of the Brinkman penalization. In this penalization method the rigid body is approximated by a permeable rigid body, the velocity field of which is determined as a rigid projection of the fluid velocity.

In Chapter 4 we were able to extend our previous findings to the setting of a compressible fluid and multiple rigid bodies. Additionally, instead of local-in-time we obtained global-in-time existence. Besides the difficulties already known from the incompressible case, the proof of this result beared some further problems. Most strikingly, it seemed impossible to discretize the continuity equation in such a way that non-negativity of the density could be guaranteed. This lead us to shift the weighting in the hybrid approximation scheme: While we still approximated the induction equation discretely, the whole mechanical part of the system was treated as a continuous problem right from the start. Consequently, we were able to construct a non-negative density by following the classical existence theory for the time-dependent compressible Navier-Stokes equations. By an adept combination of the discrete and the continuous parts of the system, we could then derive a meaningful energy inequality and thereby the uniform bounds required for the limit passage in the time discretization. Moreover, the fact that the density is not bounded away from zero in a compressible fluid caused us to switch the penalization method. Instead of the Brinkman penalization we used a method in which the original problem is replaced by an all-fluid problem and the rigid bodies are approximated by letting the viscosity of the fluid rise to infinity in certain parts of the domain.

Potential future research goals, building up on our results, arise in view of the possible applications of the models studied in this thesis. Indeed, as discussed in Section 1.1, such applications include the medical procedures of capsule endoscopy and remote drug delivery. In these procedures, robots of a microscopic scale are navigated through the electrically conducting blood in the human body. The robots used for such purposes can typically be expected to be electrically conducting, cf. [59]. The solid bodies we took into consideration in our examination of the fluid-rigid body interaction problem, however, were assumed to be insulating. Consequently, a logical next step could be an extension of our results to the proof of the existence of weak solutions to the fluid-rigid body interaction problem with not only an electrically conducting fluid but also electrically conducting rigid bodies. Similarly, an extension of our results to a setting which includes deformable solid objects would be desirable. This is of interest for another application mentioned in Section 1.1, namely the study of the interaction between the membranes of cells and extracellular or intracellular fluids in organisms: While our work in the rigid body case can be understood as an intermediate step on the way to the comprehension of this occurrence, it needs to be kept in mind that the membranes of cells in real organisms are rather deformable.

Additionally, we mention the possibility of investigating the fluid-rigid body interaction problem with an electrically conducting fluid under different boundary and interface conditions than the classical no-slip condition. A specific example we have in mind is the slip boundary and interface condition of friction type, which is considered as particularly interesting in the modeling of fluid-structure interactions. This also establishes a connection to the work discussed in Section 1.4, wherein the existence of weak solutions to a fluid-only problem subject to such a boundary condition is proved.

Moreover, we point out that there remains an open problem connected to our simplifying assumptions on the magnetic permeability μ in the models in this thesis. Indeed, despite the material differences in the solid and the fluid region, we assumed μ to take the same value in both of these domains, cf. Section 1.3.1 and Section 1.3.2. Thanks to this we could assume the magnetic induction B to be continuous across the interface between the fluid and the solid bodies. The latter condition, in turn, was crucial for the Sobolev regularity of B assumed in our weak formulation of the problem. A further potential aim for future works could thus be the set-up of a suitable definition as well as the proof of the existence of weak solutions to the problem in the - physically more accurate - case of a magnetic permeability taking different values in the fluid and the solid region. However an advancement in this direction would probably need a new methodology.

In the second part of the thesis we analyzed the evolution of a magnetoelastic material. More specifically, in Chapter 5, we proved the local-in-time existence of weak solutions to a model of the interaction between the deformation and the magnetization of such a material. In this proof, in order to handle the non-convexity of the energy functional in the model, we exchanged the approach via the Rothe method for an implementation of De Giorgi's minimizing movement scheme. In the latter approach the problem is also discretized with respect to the time, however, instead of solving the discrete equations directly we obtain them as the Euler-Lagrange equations of a suitable discrete minimization problem. The solution to the discrete problem - as the solution to a minimization problem - was then easily seen to satisfy a suitable energy inequality. A derivation of an energy inequality from the discrete equations instead would have been prevented by the non-convexity of the energy. The greatest difficulty in our application of De Giorgi's method was to find the correct choice of the discrete minimization problem. The minimization problem needed to be constructed in such a way that its Euler-Lagrange equations constitute a suitable approximation of the original system. We succeeded in this construction by realizing that already on the continuous level both equations of the system can be expressed in terms of the same energy and dissipation potentials, despite the transport terms in the magnetic force balance, which do not appear in the equation of motion. We could subsequently use these potentials as a template for the functional to be minimized on the discrete level.

Similar to the first part of this thesis, our analysis of the evolution of magnetoelastic materials offers some opportunities for future research as well. First of all, we recall that in our investigations we had to limit ourselves to compactly supported test functions in the equation of motion, cf. Remark 5.1.5. An extension of our results to the proof of the existence of weak solutions allowing for non-compactly supported test functions would be desirable. Next, we recall that in our considerations we restricted ourselves to the quasi-static case, in which inertial effects are ignored. Our proof in this case followed, on the whole, the implementation of De Giorgi's minimizing movements method in [7, Section 2]. Following further the extension of the minimizing movements method from [7, Section 3], a generalization of our results to the fully dynamical setting - with the inertial effects included - then appears to be a practicable task for future works. Finally, we point out that the solid studied in its own right in the present thesis might be supplemented by a surrounding electrically conducting fluid. This would turn the problem into a fluid-structure interaction problem and in particular establish a connection to the first part of the thesis. While the interaction between the fluid and the deformable solid in this setting could probably be handled as in [7], more difficulties are to be expected from the electromagnetic part of the problem: The theory of magnetohydrodynamics in the fluid would have to be combined with the theory of micromagnetics in the solid, which would require the development of another methodology. Overcoming these issues, however, one could prove the existence of weak solutions to a fluid-structure interaction problem with deformable solids and with electromagnetic effects taken into account in both the fluid and the solid domain.

A Appendix

A.1 Carathéodory solutions

In Chapter 3 and Chapter 4 we make use of the concept of Carathéodory solution to ordinary differential equations (see [95, Section 3.2], [99, Section 1.6]) in order to characterize the motion of the solid bodies via the characteristics of the associated velocity fields. For the convenience of the reader we give a brief summary of this concept in the present section. Let $T > 0$ and $s \in [0,T]$. A function $u : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ is called a Carathéodory function if $u(\cdot,x)$ is measurable for all fixed $x \in \mathbb{R}^3$ and if $u(t,\cdot)$ is continuous for almost all fixed $t \in [0,T]$. For such a function u we study the initial value problem

$$\frac{d\eta(t;x)}{dt} = u\left(t, \eta(t;x)\right), \qquad X(s) = x \tag{A.1.1}$$

for $t \in [0,T]$, $x \in \mathbb{R}^3$. A function $\eta(\cdot;x) : [0,T] \to \mathbb{R}^3$ is said to be a Carathéodory solution to (A.1.1) if it is absolutely continuous and if it satisfies the initial value problem (A.1.1) for almost all $t \in [0,T]$. Under certain regularity assumptions on u the existence and uniqueness of Carathéodory solutions is well-known. For our purposes we use the following version of [95, Theorem 3.4].

Theorem A.1.1. *Let $T > 0$ and $s \in [0,T]$. Let either*

$$u \in L^2(0,T; W^{1,\infty}(\mathbb{R}^3)) \tag{A.1.2}$$

or let

$$u : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3, \quad u(t,x) := v(t) + w(t) \times x, \qquad v, w \in L^2(0,T). \tag{A.1.3}$$

Then, for all $x \in \mathbb{R}^3$, the initial value problem (A.1.1) *has a unique Carathéodory solution $\eta(\cdot,x) : [0,T] \to \mathbb{R}^3$.*

Proof
In both the cases (A.1.2) and (A.1.3) the function u constitutes a Carathéodory function such that, for all $c > 0$, there exists a function $h_c \in L^2(0,T)$ with

$$|u(t,x)| \leqslant h_c(t)$$

for all $t \in [0,T]$ and all $x \in \mathbb{R}^3$ satisfying $|x| \leqslant c$. Moreover, in both the cases (A.1.2) and (A.1.3) there exists a function $\alpha \in L^2(0,T)$ such that

$$|u\left(t,x_1\right) - u\left(t,x_2\right)| \leqslant \alpha(t)\left|x_1 - x_2\right|$$

for all $t \in [0,T]$ and all $x_1, x_2 \in \mathbb{R}^3$. Under these circumstances the existence of a unique solution $\eta \in H^1(0,T)$ to the initial vale problem (A.1.1) is implied by [95, Theorem 3.4]. Since $\eta \in H^1(0,T)$ is moreover absolutely continuous, this concludes the proof.

□

A.2 Results related to the Poisson equation

In Chapter 3 we estimate the H^2-norm of the velocity field in the regularized system by the L^2-norm of its Laplacian. This is possible, in C^2-domains, due to the following well-known inequality.

Lemma A.2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class C^2. Then there exists a constant $c > 0$ such that, for any function $\phi \in H^2(\Omega) \bigcap H_0^1(\Omega)$,*

$$\|\phi\|_{H^2(\Omega)} \leqslant c \left(\|\phi\|_{L^2(\Omega)} + \|\Delta\phi\|_{L^2(\Omega)} \right). \tag{A.2.1}$$

Proof
Any function $\phi \in H^2(\Omega) \bigcap H_0^1(\Omega)$ solves the Poisson problem

$$\Delta\phi = f \quad \text{in } \Omega, \qquad \phi = 0 \quad \text{on } \partial\Omega$$

with $f = \Delta\phi \in L^2(\Omega)$. Therefore, the estimate (A.2.1) is given by the classical (boundary) regularity theory for the Poisson equation, cf. [40, Section 6.3.2, Theorem 4].

□

In order to include non-solenoidal functions into the set of admissible test functions in the variational form of the induction equation in Chapter 3 and Chapter 4, we make use of the following variant of the Helmholtz decomposition [107, Theorem 4.2].

Lemma A.2.2. *Let $\Omega \subset \mathbb{R}^3$ be a simply connected bounded domain of class $C^{1,1}$ with outer unit normal vector* n. *Then any function $b \in H^1(\Omega;\mathbb{R}^3)$ satisfying $\operatorname{curl} b \in H^1(\Omega;\mathbb{R}^3)$ admits a decomposition of the form*

$$b = \nabla q + \operatorname{curl} w,$$

where the functions on the right-hand side satisfy

$$q \in H^1(\Omega;\mathbb{R}), \quad w \in L^2\left(\Omega;\mathbb{R}^3\right), \quad \operatorname{curl} w \in H^1\left(\Omega;\mathbb{R}^3\right), \quad \operatorname{div} w = 0 \quad \textit{in } \Omega, \quad (\operatorname{curl} w)\cdot \mathrm{n}|_{\partial\Omega} = 0. \tag{A.2.2}$$

Proof
According to the Helmholtz decomposition [107, Theorem 4.2] for L^2-functions we may write

$$b = \nabla q + \operatorname{curl} w,$$

where

$$q \in H^1(\Omega;\mathbb{R}), \qquad w \in L^2\left(\Omega;\mathbb{R}^3\right), \qquad \operatorname{curl} w \in L^2\left(\Omega;\mathbb{R}^3\right), \qquad \operatorname{div} w = 0 \quad \text{in } \Omega \tag{A.2.3}$$

and in addition, cf. [107, (4.11)],

$$\int_\Omega \operatorname{curl} w \cdot \nabla\phi \; dx = 0 \qquad \forall \phi \in H^1(\Omega;\mathbb{R}). \tag{A.2.4}$$

We denote by γ the trace operator on the classical Sobolev spaces and by γ_{n} the trace operator

$$\gamma_{\mathrm{n}}: \; \left\{\psi \in L^2\left(\Omega;\mathbb{R}^3\right): \; \operatorname{div}\psi \in L^2(\Omega;\mathbb{R})\right\} \to \left(H^{\frac{1}{2}}(\partial\Omega;\mathbb{R})\right)^*, \qquad \gamma_{\mathrm{n}}(\psi) = \gamma(\psi)\cdot \mathrm{n} \quad \forall \psi \in \mathcal{D}\left(\mathbb{R}^3\right),$$

which satisfies the Stokes formula

$$\int_\Omega \phi \operatorname{div}\psi \; dx = \langle \gamma_{\mathrm{n}}(\psi), \gamma(\phi)\rangle_{(H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}))^* \times H^{\frac{1}{2}}(\partial\Omega;\mathbb{R})} - \int_\Omega \nabla\phi\cdot\psi \; dx \tag{A.2.5}$$

for all $\psi \in L^2(\Omega;\mathbb{R}^3)$ with $\operatorname{div}\psi \in L^2(\Omega;\mathbb{R})$ and all $\phi \in H^1(\Omega)$, cf. [94, Lemma 3.10]. Since $\operatorname{div}(\operatorname{curl} w) = 0 \in L^2(\Omega;\mathbb{R})$, $\gamma_{\mathrm{n}}(\operatorname{curl} w)$ is well-defined in $(H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}))^*$ and from the identity (A.2.4) and the Stokes formula (A.2.5) it follows that in fact

$$\gamma_{\mathrm{n}}(\operatorname{curl} w) = 0. \tag{A.2.6}$$

This allows us to estimate

$$\|\operatorname{curl} w\|_{H^1(\Omega)}^2 \leqslant c\left(\|\operatorname{curl} w\|_{L^2(\Omega)}^2 + \|\operatorname{curl}(\operatorname{curl} w)\|_{L^2(\Omega)}^2\right) < \infty, \tag{A.2.7}$$

cf. [107, Theorem 3.1], where the boundedness of $\|\operatorname{curl}(\operatorname{curl} w)\|_{L^2(\Omega)}^2$ follows from the inclusion

$$\operatorname{curl}(\operatorname{curl} w) = \operatorname{curl}(b - \nabla q) = \operatorname{curl} b \in H^1\left(\Omega; \mathbb{R}^3\right).$$

The relations (A.2.3), the boundary condition (A.2.6) - which can now be understood in the sense of traces of H^1-functions - and the estimate (A.2.7) combined yield the relations (A.2.2), which concludes the proof.

□

For the existence and the uniform bounds of the stray field in Chapter 5 we make use of the following classical results for the Poisson equation, cf. [98, Theorem 5.1], [57, Theorem 8.8].

Lemma A.2.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. Let $M \in H^1(\Omega)$ be extended by 0 outside of Ω. Then there exists a unique solution $\phi \in \dot{H}^1(\mathbb{R}^3)$ to the Poisson equation*

$$\int_{\mathbb{R}^3} \nabla\phi \cdot \nabla\psi \, dx = \int_\Omega M \cdot \nabla\psi \, dx \qquad \forall \psi \in \dot{H}^1\left(\mathbb{R}^3\right), \tag{A.2.8}$$

where $\dot{H}^1(\mathbb{R}^3)$ denotes the Hilbert space defined in (5.1.5). Further there exists a constant $c > 0$, independent of Ω, such that

$$\|\phi\|_{\dot{H}^1(\mathbb{R}^3)} = \|\nabla\phi\|_{L^2(\mathbb{R}^3)} \leqslant c\|M\|_{L^2(\Omega)} \tag{A.2.9}$$

Moreover, for any $\delta > 0$ there exists a constant $c(\delta) > 0$ independent of M and Ω such that

$$\left\|\nabla^2\phi\right\|_{L^2(K)} \leqslant c(\delta)\|M\|_{H^1(\Omega)} \tag{A.2.10}$$

for all compact subsets $K \subset \Omega$ satisfying $\operatorname{dist}(K, \partial\Omega) \geqslant \delta$.

Proof
The existence of a unique solution ϕ to the Poisson equation (A.2.8) as well as its $\dot{H}^1(\mathbb{R}^3)$-bound (A.2.9) is proved in [98, Theorem 5.1]. A proof for a local H^2-bound in terms of the $H^1(\Omega)$-norm of M and the $L^2(\Omega)$-norm of $\nabla\phi$ is given in [57, Theorem 8.8], so the local L^2-bound (A.2.10) of $\nabla^2\phi$ follows immediately from the $\dot{H}^1(\mathbb{R}^3)$-bound (A.2.9). We point out that, strictly speaking, the bound of the second gradient in [57, Theorem 8.8] depends on the $H^1(\Omega)$-norm of ϕ instead of only the $L^2(\Omega)$-norm of $\nabla\phi$. However, [57, Theorem 8.8] is formulated for the case of more general elliptic equations and a look into its proof shows that boundedness of the $L^2(\Omega)$-norm of $\nabla\phi$ is indeed sufficient for the setting of the Poisson equation.

□

A.3 Auxiliary results for the Rothe method

The proofs of all our main results in Chapters 3–5 are based on time discretizations of the respective systems of partial differential equations via the Rothe method. In this section we present miscellaneous auxiliary results for the application of this technique. We begin with the following variant of [99, Theorem 8.9], which is used in the limit passage from the discretized systems back to continuous in time settings, in order to guarantee that the weak limits of different interpolants of the same discrete functions coincide.

Lemma A.3.1. *Let $T > 0$, $\Delta t > 0$, with $\frac{T}{\Delta t} \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^3$ be a domain. Let further $h_{\Delta t}^k : \Omega \to \mathbb{R}^l$, $k = 0, ..., \frac{T}{\Delta t}$, $l \in \mathbb{N}$, be time-independent functions with piecewise affine and piecewise constant* *interpolants*

$$h_{\Delta t}(t) := \left(\frac{t}{\Delta t} - (k-1)\right) h^k_{\Delta t} + \left(k - \frac{t}{\Delta t}\right) h^{k-1}_{\Delta t} \qquad \textit{for } (k-1)\Delta t < t \leqslant k\Delta t, \quad k = 1, ..., \frac{T}{\Delta t},$$

$$\overline{h}_{\Delta t}(t) := h^k_{\Delta t} \qquad \textit{for } (k-1)\Delta t < t \leqslant k\Delta t, \quad k = 0, ..., \frac{T}{\Delta t},$$

$$\overline{h}'_{\Delta t}(t) := h^{k-1}_{\Delta t} \qquad \textit{for } (k-1)\Delta t < t \leqslant k\Delta t, \quad k = 1, ..., \frac{T}{\Delta t}.$$

Assume moreover that

$$h_{\Delta t} \overset{*}{\rightharpoonup} h \quad \textit{in } L^\infty(0,T;L^2(\Omega)), \quad \overline{h}_{\Delta t} \overset{*}{\rightharpoonup} \overline{h} \quad \textit{in } L^\infty(0,T;L^2(\Omega)), \quad \overline{h}'_{\Delta t} \overset{*}{\rightharpoonup} \overline{h}' \quad \textit{in } L^\infty(0,T;L^2(\Omega)).$$

Then it holds that

$$h = \overline{h} = \overline{h}'. \tag{A.3.1}$$

Proof
The proof, which is performed by comparing the limit of the functions $\overline{h}_{\Delta t}$, $\overline{h}'_{\Delta t}$ to the one of $h_{\Delta t}$ in the pairing with piecewise constant in time functions respectively, can be found in the proof of [99, Theorem 8.9]. For the convenience of the reader, we restate the argument here: Without loss of generality, we only consider the subsequences with indices $\Delta t = 2^{-l}T$, $l \in \mathbb{N}$. We pick $L \in \mathbb{N}$, $k_1 < k_2 < 2^L$ and $\psi \in L^2(\Omega)$ and consider functions of the form $\chi_{[\tau k_1, \tau k_2]}\psi$, where $\tau := 2^{-L}T > 0$ and $\chi_{[\tau k_1,\tau k_2]}$ denotes the characteristic function of the interval $[\tau k_1, \tau k_2]$. By [99, Proposition 1.36], linear combinations of such functions are dense in $L^2(0,T;L^2(\Omega))$. For $\Delta t \leqslant \tau$, i.e. $l \geqslant L$, we calculate

$$\left|\int_0^T \int_\Omega \left(h_{\Delta t} - \overline{h}_{\Delta t}\right) \cdot \chi_{[\tau k_1,\tau k_2]}\psi \, dxdt\right| = \left|\sum_{k=\frac{\tau k_1}{\Delta t}+1}^{\frac{\tau k_2}{\Delta t}} \int_{(k-1)\Delta t}^{k\Delta t} \int_\Omega \left[\left(h^k_{\Delta t} - h^{k-1}_{\Delta t}\right) \frac{t - k\Delta t}{\Delta t}\right] \cdot \psi \, dxdt\right|$$
$$= \left|-\frac{\Delta t}{2} \sum_{k=\frac{\tau k_1}{\Delta t}+1}^{\frac{\tau k_2}{\Delta t}} \int_\Omega \left(h^k_{\Delta t} - h^{k-1}_{\Delta t}\right) \cdot \psi \, dx\right| = \left|-\frac{\Delta t}{2} \int_\Omega \left(h_{\Delta t}(\tau k_2) - h_{\Delta t}(\tau k_1)\right) \cdot \psi \, dx\right| \leqslant c\Delta t, \tag{A.3.2}$$

with c independent of Δt, since $h_{\Delta t}$ is bounded uniformly in $L^\infty(0,T;L^2(\Omega))$. We conclude

$$h_{\Delta t} - \overline{h}_{\Delta t} \rightharpoonup 0 \quad \text{in } L^2(0,T;L^2(\Omega)),$$

which implies the first identity from (A.3.1). Using the same kind of test function again, we also see

$$\left|\int_0^T \int_\Omega \left(h_{\Delta t} - \overline{h}'_{\Delta t}\right) \cdot \chi_{[\tau k_1,\tau k_2]}\psi \, dxdt\right|$$
$$= \left|\int_0^T \int_\Omega \left(h_{\Delta t} - \overline{h}_{\Delta t}\right) \cdot \chi_{[\tau k_1,\tau k_2]}\psi \, dxdt + \int_{\tau k_2 - \Delta t}^{\tau k_2} \int_\Omega h^{\frac{\tau k_2}{\Delta t}}_{\Delta t} \cdot \psi \, dxdt - \int_{\tau k_1}^{\tau k_1 + \Delta t} \int_\Omega h^{\frac{\tau k_1}{\Delta t}}_{\Delta t} \cdot \psi \, dxdt\right|$$
$$\leqslant c\Delta t + 2\Delta t \|\overline{h}_{\Delta t}\|_{L^\infty(0,T;L^2(\Omega))} \|\psi\|_{L^2(\Omega)} \leqslant c\Delta t,$$

exploiting in the first inequality the estimate we already know from (A.3.2). This implies $h = \overline{h}'$ and hence the second identity in (A.3.1).

□

When passing to the limit in the time discretizations we further face the situation of having to pass to the limit in discretized versions of given time-dependent functions (see e.g. (3.4.42)) as well as in discretized versions of time-dependent functions which are already known to converge uniformly with respect to the time (see e.g. (3.4.59)). In order to deal with this we use the following version of [99, Lemma 8.7].

Lemma A.3.2.
Let $T > 0$, $\Delta t > 0$ with $\frac{T}{\Delta t} \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^3$ be a domain.

(i) Let $\gamma = \gamma(\Delta t) > 0$, $\gamma(\Delta t) \to 0$ *for* $\Delta t \to 0$. *Let* $h \in L^\infty((0,T) \times \Omega)$ *and define*

$$h_\gamma(t) := \int_0^T \theta_\gamma\left(t + \xi_\gamma(t) - s\right) h(s)\ ds, \qquad \xi_\gamma(t) := \gamma \frac{T - 2t}{T},$$

where $\theta_\gamma : \mathbb{R} \to \mathbb{R}$ *denotes a mollifier with support in* $[-\gamma, \gamma]$. *Set further*

$$\overline{h}_{\Delta t}(t) := h_{\gamma(\Delta t)}(k\Delta t) \qquad \text{for } (k-1)\Delta t < t \leqslant k\Delta t, \quad k = 1, ..., \frac{T}{\Delta t}.$$

Then

$$\overline{h}_{\Delta t} \to h \quad \text{in } L^p((0,T) \times \Omega) \qquad \forall 1 \leqslant p < \infty.$$

(ii) Let $h \in W^{l,q}([0,T]; W^{j,p}(\mathbb{R}^3))$, $l, j \in \mathbb{N}_0$, $1 \leqslant p, q < \infty$ *and define*

$$\overline{h}_{\Delta t}(t) := \frac{1}{\Delta t} \int_{(k-1)\Delta t}^{k\Delta t} h(s)\ ds \qquad \text{for } (k-1)\Delta t < t \leqslant k\Delta t, \quad k = 1, ..., \frac{T}{\Delta t}.$$

Then

$$\overline{h}_{\Delta t} \to h \quad \text{in } C\left([0,T]; W^{j,p}(\mathbb{R}^3)\right).$$

(iii) Let $(h_{\Delta t})_{\Delta t > 0} \subset C([0,T]; W^{j,p}(\mathbb{R}^3))$, $j \in \mathbb{N}_0$, $1 \leqslant p < \infty$, *be a sequence such that*

$$h_{\Delta t} \to h \quad \text{in } C\left([0,T]; W^{j,p}(\mathbb{R}^3)\right).$$

Let further

$$\begin{aligned} \overline{h}_{\Delta t}(t) :=& h_{\Delta t}(k\Delta t) && \text{for } (k-1)\Delta t < t \leqslant k\Delta t, \quad k = 1, ..., \frac{T}{\Delta t}, \\ \overline{h}'_{\Delta t}(t) :=& h_{\Delta t}((k-1)\Delta t) && \text{for } (k-1)\Delta t < t \leqslant k\Delta t, \quad k = 1, ..., \frac{T}{\Delta t}. \end{aligned}$$

Then

$$\overline{h}_{\Delta t}(t),\ \overline{h}'_{\Delta t}(t) \to h \quad \text{in } C\left([0,T]; W^{j,p}(\mathbb{R}^3)\right).$$

Proof
The statement (i) is proved in [99, Lemma 8.7]. The statements (ii) and (iii) follow by similar arguments: For (ii) we fix some arbitrary value $\epsilon > 0$ and write

$$\begin{aligned} \left\|\overline{h}_{\Delta t} - h\right\|_{C([0,T];W^{j,p}(\mathbb{R}^3))} &= \sup_{t\in[0,T]} \left\| \frac{1}{\Delta t} \int_{(k_{\Delta t,t}-1)\Delta t}^{k_{\Delta t,t}\Delta t} h(s) - h(t)\ ds \right\|_{W^{j,p}(\mathbb{R}^3)} \\ &\leqslant \sup_{t\in[0,T]} \frac{1}{\Delta t} \int_{(k_{\Delta t,t}-1)\Delta t}^{k_{\Delta t,t}\Delta t} \|h(s) - h(t)\|_{W^{j,p}(\mathbb{R}^3)}\ ds, \end{aligned} \tag{A.3.3}$$

where $k_{\Delta t,t} \in \{1, ..., \frac{T}{\Delta t}\}$ is chosen such that $t \in [(k_{\Delta t,t}-1)\Delta t, (k_{\Delta t,t})\Delta t]$. Since $h \in W^{l,q}([0,T]; W^{j,p}(\mathbb{R}^3)) \subset C([0,T]; W^{j,p}(\mathbb{R}^3))$ we find some $\tau > 0$ such that $\|h(s) - h(t)\|_{W^{j,p}(\mathbb{R}^3)} \leqslant \epsilon$ for $|s - t| < \tau$. Choosing $\Delta t < \tau$ it follows from the estimate (A.3.3) that

$$\left\|\overline{h}_{\Delta t} - h\right\|_{C([0,T];W^{j,p}(\mathbb{R}^3))} \leqslant \epsilon$$

for all such Δt, which concludes the proof of (ii). For the third statement we again fix an arbitrary value $\epsilon > 0$ and pick $\tau > 0$ sufficiently small to have $\|h_{\Delta t} - h\|_{C([0,T];W^{j,p}(\mathbb{R}^3))} \leqslant \frac{\epsilon}{2}$ for all $\Delta t < \tau$.

Further, we choose τ sufficiently small such that $\|h_{\Delta t}(t_1) - h_{\Delta t}(t_2)\|_{W^{j,p}(\mathbb{R}^3)} \leqslant \frac{\epsilon}{2}$ for $|t_1 - t_2|$, $\Delta t \leqslant \tau$. It follows that $\|\overline{h}_{\Delta t}(t) - h_{\Delta t}(t)\|_{W^{j,p}(\mathbb{R}^3)} \leqslant \frac{\epsilon}{2}$ for any $\Delta t \leqslant \tau$, $t \in [0,T]$ and for such Δt we conclude

$$\left\|\overline{h}_{\Delta t} - h\right\|_{C([0,T];W^{j,p}(\mathbb{R}^3))} \leqslant \left\|\overline{h}_{\Delta t} - h_{\Delta t}\right\|_{C([0,T];W^{j,p}(\mathbb{R}^3))} + \left\|h_{\Delta t} - h\right\|_{C([0,T];W^{j,p}(\mathbb{R}^3))} \leqslant \epsilon.$$

By the same arguments a corresponding estimate can also be derived for $\overline{h}'_{\Delta t}$ instead of $\overline{h}_{\Delta t}$, which concludes the proof.

□

Moreover, we make use of the following discrete version of the Aubin-Lions Lemma, cf. [36, Theorem 1].

Lemma A.3.3. *Let $T > 0$, $\Delta t > 0$ with $\frac{T}{\Delta t} \in \mathbb{N}$. Let $0 \leqslant t_1 \leqslant t_2 \leqslant T$ be such that $\frac{t_1}{\Delta t}, \frac{t_2}{\Delta t} \in \mathbb{N}$ and set $I := (t_1, t_2)$. Let $1 \leqslant p, q < \infty$ and let $Z_1 \subset Z_2 \subset Z_3$ be Banach spaces such that Z_1 is embedded compactly into Z_2 and Z_2 is embedded continuously into Z_3. Moreover, let $h^k_{\Delta t} \in Z_1$, $k = 0, ..., \frac{T}{\Delta t}$, be time-independent functions with piecewise constant interpolants*

$$\overline{h}_{\Delta t}(t) := h^k_{\Delta t} \qquad \text{for } (k-1)\Delta t < t \leqslant k\Delta t, \quad k = 0, ..., \frac{T}{\Delta t},$$
$$\overline{h}'_{\Delta t}(t) := h^{k-1}_{\Delta t} \qquad \text{for } (k-1)\Delta t < t \leqslant k\Delta t, \quad k = 1, ..., \frac{T}{\Delta t}.$$

(i) If there exists a constant $c > 0$, independent of Δt, such that the estimate

$$\left\|\frac{\overline{h}_{\Delta t}(\cdot) - \overline{h}_{\Delta t}(\cdot - \Delta t)}{\Delta t}\right\|_{L^p(t_1+\Delta t, t_2; Z_3)} + \left\|\overline{h}_{\Delta t}\right\|_{L^q(I;Z_1)} \leqslant c$$

holds true for all sufficiently small values of Δt, then there exists a function $h \in L^q(I; Z_2)$ such that, possibly after the extraction of a subsequence,

$$\overline{h}_{\Delta t} \to h \quad \text{in } L^q(I; Z_2). \tag{A.3.4}$$

(ii) If there exists a constant $c > 0$, independent of Δt, such that the estimate

$$\left\|\frac{\overline{h}'_{\Delta t}(\cdot) - \overline{h}'_{\Delta t}(\cdot - \Delta t)}{\Delta t}\right\|_{L^p(t_1+\Delta t, t_2; Z_3)} + \left\|\overline{h}'_{\Delta t}\right\|_{L^q(I;Z_1)} \leqslant c$$

holds true, respectively, for all sufficiently small values of Δt, then there exists a function $h \in L^q(I; Z_2)$ such that, possibly after the extraction of a subsequence,

$$\overline{h}'_{\Delta t} \to h \quad \text{in } L^q(I; Z_2). \tag{A.3.5}$$

Proof
A proof of Lemma A.3.3 is given, for example, in [36, Theorem 1].

□

Remark A.3.1. *A version of Lemma A.3.3 remains true even if the condition $\frac{t_1}{\Delta t}, \frac{t_2}{\Delta t} \in \mathbb{N}$ is not satisfied: In this case, instead of the convergence* (A.3.4) *or* (A.3.5)*, the extracted subsequence satisfies the convergence*

$$\overline{h}_{\Delta t} \to h \quad \text{in } L^q(I_C; Z_2) \qquad \text{or} \qquad \overline{h}'_{\Delta t} \to h \quad \text{in } L^q(I_C; Z_2),$$

respectively, for all compact intervals $I_C \subset I$.

In the derivation of the discrete energy estimate in Section 5.4.1 we control the discrete difference quotient of the external magnetic field H_{ext} through its classical time derivative $\partial_t H_{\text{ext}}$ via the following result, which constitutes a version of the estimate (8.72) in the proof of [99, Theorem 8.18].

Lemma A.3.4. *Let $\Delta t > 0$, let $H_{\text{ext}} \in W^{1,\frac{4}{3}}(0,\infty; W^{1,\frac{4}{3}}(\mathbb{R}^3))$ and set*

$$(H_{\text{ext}})^k_{\Delta t} := \frac{1}{\Delta t}\int_{(k-1)\Delta t}^{k\Delta t} H_{\text{ext}}(t)\ dt \qquad \forall k \in \mathbb{N}.$$

Then

$$\Delta t \sum_{l=2}^{k} \left\| \frac{(H_{ext})^l_{\Delta t} - (H_{ext})^{l-1}_{\Delta t}}{\Delta t} \right\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(\mathbb{R}^3)} \leqslant \|\partial_t H_{ext}\|^{\frac{4}{3}}_{L^{\frac{4}{3}}((0,\infty)\times\mathbb{R}^3)} \qquad \forall k \in \mathbb{N}.$$

Proof
The statement can be proved in the same way as the corresponding estimate (8.72) in the proof of [99, Theorem 8.18]. Indeed, for arbitrary $k \in \mathbb{N}$, we estimate, under exploitation of Jensen's inequality,

$$\begin{aligned}
\Delta t \sum_{l=2}^{k} \left\| \frac{(H_{\text{ext}})^l_{\Delta t} - (H_{\text{ext}})^{l-1}_{\Delta t}}{\Delta t} \right\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(\mathbb{R}^3)} &= \Delta t \sum_{l=2}^{k} \frac{1}{(\Delta t)^{\frac{8}{3}}} \left\| \int_{(l-1)\Delta t}^{l\Delta t} H_{\text{ext}}(t) - H_{\text{ext}}(t-\Delta t)\ dt \right\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(\mathbb{R}^3)} \\
&= \Delta t \sum_{l=2}^{k} \frac{1}{(\Delta t)^{\frac{8}{3}}} \left\| \int_{(l-1)\Delta t}^{l\Delta t} \int_0^{\Delta t} -\frac{d}{d\tau} H_{\text{ext}}(t-\tau)\ d\tau dt \right\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(\mathbb{R}^3)} \\
&= \Delta t \sum_{l=2}^{k} \frac{1}{(\Delta t)^{\frac{8}{3}}} \left\| \int_{(l-1)\Delta t}^{l\Delta t} \int_0^{\Delta t} \frac{d}{dt} H_{\text{ext}}(t-\tau)\ d\tau dt \right\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(\mathbb{R}^3)} \\
&\leqslant \Delta t \sum_{l=2}^{k} \frac{1}{(\Delta t)^{\frac{8}{3}}} \left(\int_{(l-1)\Delta t}^{l\Delta t} \int_0^{\Delta t} \left\| \frac{d}{dt} H_{\text{ext}}(t-\tau) \right\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} d\tau dt \right)^{\frac{4}{3}} \\
&\leqslant \sum_{l=2}^{k} \frac{1}{\Delta t} \int_{(l-1)\Delta t}^{l\Delta t} \int_0^{\Delta t} \left\| \frac{d}{dt} H_{\text{ext}}(t-\tau) \right\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(\mathbb{R}^3)} d\tau dt \\
&= \sum_{l=2}^{k} \frac{1}{\Delta t} \int_0^{\Delta t} \int_{(l-1)\Delta t-\tau}^{l\Delta t-\tau} \|\partial_t H_{\text{ext}}(s)\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(\mathbb{R}^3)} ds d\tau \\
&\leqslant \sum_{l=2}^{k} \frac{1}{\Delta t} \int_0^{\Delta t} \int_{(l-2)\Delta t}^{l\Delta t} \|\partial_t H_{\text{ext}}(s)\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(\mathbb{R}^3)} ds d\tau \\
&\leqslant \|\partial_t H_{\text{ext}}\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(0,\infty;L^{\frac{4}{3}}(\mathbb{R}^3))}.
\end{aligned}$$

□

A.4 Compactness results

In this section we summarize several compactness results used all throughout Chapters 3 and 4. They are required for the extraction of strongly convergent subsequences in the limit passages in the various approximation levels in the proofs of the respective main results Theorem 3.1.1 and Theorem 4.1.1. We start with the following lemma, which constitutes a variant of the well-known tools for the derivation of C_{weak}-convergence, cf. for example [94, Lemma 6.2].

Lemma A.4.1. *Let $I \subset \mathbb{R}$ be an interval, let X be a reflexive Banach space and let Y^* be a dense subset of the dual space X^* of X. Let further $(f_n)_{n\in\mathbb{N}} \subset L^\infty(I;X)$ be a sequence of functions satisfying*

$$f_n \overset{*}{\rightharpoonup} f \qquad \text{in } L^\infty(I;X)$$

and

$$\left\| \partial_t \langle f_n(t), \phi \rangle_{X\times X^*} \right\|_{L^p(I)} \leqslant c \qquad \forall \phi \in Y^* \tag{A.4.1}$$

for some $p > 1$ and a constant $c > 0$ dependent on ϕ but independent of n. Then, possibly after the extraction of a non-relabeled subsequence,

$$f_n \to f \qquad \text{in } C_{\text{weak}}\left(\overline{I}; X\right). \tag{A.4.2}$$

Moreover, if in addition Z denotes a Banach space such that X is embedded compactly into Z, then

$$f_n \to f \qquad \text{in } L^q\left(I; Z\right) \qquad \forall 1 \leqslant q < \infty. \tag{A.4.3}$$

Proof
We fix $\phi \in Y^*$. The estimate (A.4.1) and the Morrey embedding imply that

$$\left\|\langle f_n(\cdot), \phi\rangle_{X\times X^*}\right\|_{C(\overline{I})}, \ \left\|\langle f(\cdot), \phi\rangle_{X\times X^*}\right\|_{C(\overline{I})} \leqslant c \tag{A.4.4}$$

and

$$\left|\langle f_n(t), \phi\rangle_{X\times X^*} - \langle f_n(s), \phi\rangle_{X\times X^*}\right| \leqslant c\left|t-s\right|^{\frac{1}{p'}} \qquad \forall t, s \in \overline{I}$$

for a constant $c > 0$ depending on ϕ but not on n, s and t as well as $1 \leqslant p' < \infty$ such that $\frac{1}{p} + \frac{1}{p'} = 1$. The latter two estimates allow us to apply the Arzelà-Ascoli theorem and infer that, after the extraction of a non-relabeled subsequence,

$$\langle f_n(\cdot), \phi\rangle_{X\times X^*} \to \langle f(\cdot), \phi\rangle_{X\times X^*} \quad \text{in } C\left(\overline{I}\right) \qquad \forall \phi \in Y^*. \tag{A.4.5}$$

From this relation we may deduce the desired convergence (A.4.2) by following the arguments of the proof of [97, Lemma 2.2.5]: For any $n \in \mathbb{N}$ we define a bounded linear functional $L_n(t) : X^* \mapsto \mathbb{R}$ by

$$L_n(t)\phi := \liminf_{h\to 0} \frac{1}{h}\int_t^{t+h} \langle f_n(s), \phi\rangle_{X\times X^*}\, ds, \quad |L_n(t)\phi| \leqslant \|f_n\|_{L^\infty(I;X)}\|\phi\|_{X^*} \qquad \forall \phi \in X^*. \tag{A.4.6}$$

In particular it holds that $L_n(t) \in X^{**}$, so there exists a unique $l_n(t) \in X$ such that

$$\langle l_n(t), \phi\rangle_{X\times X^*} = L_n(t)\phi, \quad \|l_n(t)\|_X = \|L_n(t)\|_{X^{**}} \qquad \forall \phi \in X^*.$$

From the estimate (A.4.6) it follows that

$$\|l_n(t)\|_X = \sup_{\|\phi\|_{X^*}\leqslant 1} |L_n(t)\phi| \leqslant \|f_n\|_{L^\infty(I;X)}. \tag{A.4.7}$$

Further, from the definition of $L_n(t)$ in (A.4.6) and the continuity given by (A.4.4) we see that

$$\langle l_n(t), \phi\rangle_{X\times X^*} = L_n(t)\phi = \langle f_n(t), \phi\rangle_{X\times X^*} \qquad \forall \phi \in Y^*. \tag{A.4.8}$$

For almost all $t \in \overline{I}$ it holds that $f_n(t) \in X$. For such t the identity (A.4.8) shows that $f_n(t)$ coincides with $l_n(t)$ as an operator on X^* due to the density of Y^* in X^*. For the remaining $t \in \overline{I}$ we set, without loss of generality, $f_n(t)$ equal to its closure on X^*, which exists and is continuous because of the density of Y^* in X^*, cf. [94, 1.4.7.6]. Then the equation (A.4.8) again implies that $f_n(t) = l_n(t)$. Now the estimate (A.4.7) shows that

$$\|f_n(t)\|_{C(\overline{I};X)} \leqslant \|f_n(t)\|_{L^\infty(I;X)} \leqslant c \tag{A.4.9}$$

for a constant $c > 0$ independent of n. Arguing in the same way for f instead of f_n, we also see that

$$\|f(t)\|_{C(\overline{I};X)} \leqslant \|f(t)\|_{L^\infty(I;X)} \leqslant c. \tag{A.4.10}$$

Next we fix an arbitrary element $\phi \in X^*$, choose $\eta \in Y^*$ and write

$$\begin{aligned}
&\sup_{t\in\overline{I}} \left|\langle f_n(t) - f(t), \phi\rangle_{X\times X^*}\right| \\
&\leqslant \sup_{t\in\overline{I}} \left|\langle f_n(t) - f(t), \eta\rangle_{X\times X^*}\right| + \sup_{t\in\overline{I}} \left|\langle f_n(t) - f(t), \phi - \eta\rangle_{X\times X^*}\right| \\
&\leqslant \sup_{t\in\overline{I}} \left|\langle f_n(t) - f(t), \eta\rangle_{X\times X^*}\right| + \sup_{t\in\overline{I}} \|f_n(t) - f(t)\|_X \|\phi - \eta\|_{X^*}.
\end{aligned}$$

Due to the density of Y^* in X^* the quantity $\|\phi - \eta\|_{X^*}$ can be made arbitrarily small by a suitable choice of $\eta \in Y^*$. Consequently the desired convergence (A.4.2) follows from the uniform bounds (A.4.9), (A.4.10) and the convergence (A.4.5). Finally, the convergence (A.4.3) follows directly from the compactness of Z in X and the abstract Arzelá-Ascoli theorem, cf. [94, Theorem 1.70].

□

Remark A.4.1. *The proof of Lemma A.4.1 shows that the statement remains true if the condition* (A.4.1) *is replaced by the condition*

$$\langle f_n(\cdot), \phi\rangle_{X\times X^*} \to \langle f(\cdot), \phi\rangle_{X\times X^*} \quad \text{in } C\left(\overline{I}\right) \qquad \forall \phi \in Y^*.$$

Consequently, Lemma A.4.1 in particular implies the following statement: If $I \subset \mathbb{R}$ is an interval, X is a reflexive Banach space and Y is a Banach space which is dense in X, then

$$f \in C_{\text{weak}}\left(\overline{I}; X\right) \qquad \text{for all} \qquad f \in L^\infty(I; X) \bigcap C_{\text{weak}}\left(\overline{I}; Y\right),$$

which is the statement of [97, Lemma 2.2.5].

For the limit passage in the transport equation and hence for the convergence of the position of the rigid body in the incompressible fluid-structure interaction problem in Chapter 3 we use the following result, which is a variant of [103, Lemma 5.2, Corollary 5.2, Corollary 5.3].

Lemma A.4.2. *Let $T > 0$ and let $\chi_0 \in L^\infty(\mathbb{R}^3)$. Assume that for any $n \in \mathbb{N}$, the function*

$$\Pi_n : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3, \quad \Pi_n(t,x) := v_n(t) + w_n(t) \times x, \qquad v_n, w_n \in L^\infty(0,T),$$

satisfies

$$\|v_n\|_{L^\infty(0,T)}, \ \|w_n\|_{L^\infty(0,T)} \leqslant c \tag{A.4.11}$$

with $c > 0$ independent of n. Denote further by η_n the Carathéodory solution of

$$\frac{d\eta_n(s;t,x)}{dt} = \Pi_n\left(t, \eta_n(s;t,x)\right), \qquad \eta_n(s;s,x) = x \tag{A.4.12}$$

for $x \in \mathbb{R}^3$ and $s,t \in [0,T]$ and by $\chi_n(t,x) = \chi_0(\eta_n(t;0,x))$ the corresponding solution to

$$-\int_0^T \int_{\mathbb{R}^3} \chi_n \partial_t \Theta dx dt - \int_{\mathbb{R}^3} \chi_0 \Theta(0,x)\, dx = \int_0^T \int_{\mathbb{R}^3} (\chi_n \Pi_n) \cdot \nabla\Theta \, dx dt \quad \forall \Theta \in \mathcal{D}([0,T) \times \mathbb{R}^3). \tag{A.4.13}$$

Then, passing to subsequences if necessary, it holds that

$$\eta_n \to \eta \quad \text{in } C\left([0,T] \times [0,T]; C_{\text{loc}}\left(\mathbb{R}^3\right)\right), \tag{A.4.14}$$

$$\chi_n \to \chi \quad \text{in } C\left([0,T]; L^p_{loc}\left(\mathbb{R}^3\right)\right) \quad \forall 1 \leqslant p < \infty \tag{A.4.15}$$

with η denoting the unique solution of

$$\frac{d\eta(s;t,x)}{dt} = \Pi\left(t, \eta(s;t,x)\right), \qquad \eta(s;s,x) = x \tag{A.4.16}$$

for $x \in \mathbb{R}^3$ and $s,t \in [0,T]$, χ the one of

$$-\int_0^T \int_{\mathbb{R}^3} \chi \partial_t \Theta dx dt - \int_{\mathbb{R}^3} \chi_0 \Theta(0,x)\, dx = \int_0^T \int_{\mathbb{R}^3} (\chi \Pi) \cdot \nabla\Theta \, dx dt \quad \forall \Theta \in \mathcal{D}([0,T) \times \mathbb{R}^3) \tag{A.4.17}$$

and with Π given by

$$\Pi_n \overset{*}{\rightharpoonup} \Pi \quad \text{in } L^\infty\left(0,T; W^{1,\infty}_{\text{loc}}\left(\mathbb{R}^3\right)\right). \tag{A.4.18}$$

Moreover,

$$\chi(t,x) = \chi_0\left(\eta(t;0,x)\right). \tag{A.4.19}$$

Proof
First we note that the existence of the solution η_n to (A.4.12) and the fact that χ_n is the solution to (A.4.13) are guaranteed by [35, Theorem 3.2]. The relation (A.4.18) is clear by (A.4.11). The convergence (A.4.15) and the equation (A.4.17) then immediately follow from [86, Theorem 2.5]. From the Gronwall inequality, the bounds (A.4.11) and the relation (A.4.12) it is possible to check that for each compact set $K \subset \mathbb{R}^3$

$$\{\eta_n(s;t,\cdot)\} \text{ is relatively compact in } C(K) \text{ for all fixed } (s,t) \in [0,T] \times [0,T]$$

and further to show equicontinuity of the mapping

$$(s,t) \mapsto \eta_n(s;t,\cdot)$$

from $[0,T] \times [0,T]$ to $C(K)$. This gives us the conditions for a generalized version of the Arzelà-Ascoli theorem, [115, A_1(24i)], which allows us to infer (A.4.14). The fact that $\eta(s;\cdot,x)$ is the Carathéodory solution to the initial value problem (A.4.16) then follows by writing (A.4.12) in a variational form and passing to the limit with the help of (A.4.14) and (A.4.18). Since this solution is unique, it follows that the solution of (A.4.17) is given by the right-hand side of (A.4.19). But since we already determined the unique solution of (A.4.17) as the function χ given by (A.4.15), the equation (A.4.19) holds true, which concludes the proof.

□

Remark A.4.2. *If χ_0 has compact support in $\mathbb{R}^3$, the relation $\chi_n(t,x) = \chi_0(\eta_n(t;0,x))$ allows us to improve the local convergence (A.4.15) to*

$$\chi_n \to \chi \quad \text{in } C([0,T]; L^p(\mathbb{R}^3)) \quad \forall 1 \leqslant p < \infty.$$

In the compressible fluid-structure interaction problem in Chapter 4 we need a modified version of the previous result in order to show convergence of the positions of the solid bodies: Since the velocity field $R_\delta[u]$, determining the motion of the approximate solid region in the penalization method used in our proof of Theorem 3.1.1, is not a rigid velocity field, we need to generalize Lemma A.4.2 to a broader class of velocity fields. In fact, we need to take into account velocity fields with non-vanishing divergence, which in turn prevents us from using the transport theory and the description of the solid region via a characteristic function given by a transport equation. The result we use instead of Lemma A.4.2 is the following lemma, which can be found in [43, Proposition 5.1].

Lemma A.4.3. *Let $T > 0$, let $O \subset \mathbb{R}^3$ be a bounded domain of class $C^2 \bigcap C^{0,1}$ and let $\kappa > 0$. Let further $(u_n)_{n\in\mathbb{N}}$ be a sequence of vector fields bounded in $L^2(0,T;W^{1,\infty}(\mathbb{R}^3))$ uniformly with respect to n. Moreover, denote by η_n the Carathéodory solution to the initial value problem*

$$\frac{d\eta_n(t,x)}{dt} = u_n\left(t, \eta_n(t,x)\right), \qquad \eta_n(0,x) = x \tag{A.4.20}$$

for $x \in \mathbb{R}^3$ and $t \in [0,T]$ and set $O_n(t) := \eta_n(t,O)$ and $S_n(t) := (O_n(t))^\kappa = \{x \in \mathbb{R}^3 : \operatorname{dist}(x, O_n(t)) < \kappa\}$. Then, passing to subsequences if necessary, it holds that

$$\eta_n \to \eta \qquad \text{in } C\left([0,T]; C_{\text{loc}}\left(\mathbb{R}^3\right)\right), \tag{A.4.21}$$

$$\boldsymbol{db}_{O_n(t)} \to \boldsymbol{db}_{O(t)} \qquad \text{in } C_{\text{loc}}\left(\mathbb{R}^3\right), \tag{A.4.22}$$

$$\boldsymbol{db}_{S_n(t)} \to \boldsymbol{db}_{S(t)} \qquad \text{in } C_{\text{loc}}\left(\mathbb{R}^3\right) \tag{A.4.23}$$

uniformly with respect to $t \in [0,T]$, with $O(t) := \eta(t,O)$ and $S(t) := (O(t))^\kappa$, with η denoting the unique solution to

$$\frac{d\eta(t,x)}{dt} = u\left(t, \eta(t,x)\right), \qquad \eta(0,x) = x \tag{A.4.24}$$

for $x \in \mathbb{R}^3$, $t \in [0,T]$, with u given by

$$u_n \overset{*}{\rightharpoonup} u \quad \text{in } L^2\left(0,T; W^{1,\infty}\left(\mathbb{R}^3\right)\right)$$

and with $\boldsymbol{db}_U$ denoting the signed distance function of a set $U \subset \mathbb{R}^3$ defined in (4.2.17).

Proof
The convergences (A.4.21) and (A.4.22) can be concluded via an application of the Arzelà-Ascoli theorem, the convergence (A.4.23) follows directly from the convergence (A.4.22). The relation (A.4.24) then follows by passing to the limit in the corresponding relation (A.4.20). For the details of the proof we refer to [43, Proposition 5.1].

□

Since, according to Lemma 4.1.1, the rigid bodies can never leave the domain Ω, we moreover have the following corollary for the case that the velocity fields u_n in Lemma A.4.3 are compatible with suitable systems of isometries.

Corollary A.4.1. *Let $T > 0$ and let $\Omega, S_0^1, ..., S_0^N \subset \mathbb{R}^3$, $N \in \mathbb{N}$, be bounded domains of class $C^2 \bigcap C^{0,1}$. Let further $(u_n)_{n\in\mathbb{N}}$ be a sequence of vector fields bounded in $L^2(0,T;H^{1,2}(\Omega))$ uniformly with respect to n and let each u_n be compatible with the system $\{S_0^i, \eta_n^i\}_{i=1}^N$ where $\eta_n^i(t) : \mathbb{R}^3 \to \mathbb{R}^3$, $t \in [0,T]$, $i = 1, ..., N$ denotes an isometry. Then there exist isometries $\eta^i(t) : \mathbb{R}^3 \to \mathbb{R}^3$ such that, for a suitable subsequence, it holds that*

$$\eta_n^i \to \eta^i \quad \text{in } C\left([0,T]; C_{\text{loc}}\left(\mathbb{R}^3\right)\right)$$

and if the extracted subsequence is chosen such that

$$u_n \overset{*}{\rightharpoonup} u \quad \text{in } L^2\left(0,T;H^{1,2}(\Omega)\right)$$

for some $u \in L^2(0,T;H^{1,2}(\Omega))$, then u is compatible with the system $\{S_0^i, \eta^i\}_{i=1}^N$.

A.5 Auxiliary results for the Brinkman penalization

In the proof of the main result Theorem 3.1.1 of the incompressible fluid-structure interaction problem in Chapter 3 we make use of the Brinkman penalization, penalizing the deviation of the velocity field from its projection onto velocity fields which are rigid in the solid part of the domain. This projection is defined as follows, cf. the formula (3.1.6): For $t \in [0,T]$ let $\chi(t) \in L^\infty(\mathbb{R}^3;\{0,1\})$ denote the characteristic function of a bounded domain $S(t) \subset \mathbb{R}^3$, let $\rho(t) \in L^\infty(\mathbb{R}^3;\mathbb{R})$ satisfy $\rho(t) \geqslant \underline{\rho}$ almost everywhere in $S(t)$ for some constant $\underline{\rho} > 0$ and let $u(t) \in L^2(\mathbb{R}^3;\mathbb{R}^3)$. Then we define the rigid velocity field

$$\Pi_{[\chi,\rho,u]}(t,x) := (u_G)_{[\chi,\rho,u]}(t) + \omega_{[\chi,\rho,u]}(t) \times \left(x - a_{[\chi,\rho]}(t)\right) \qquad \forall x \in \mathbb{R}^3, \tag{A.5.1}$$

where

$$\begin{aligned}
(u_G)_{[\chi,\rho,u]}(t) &:= \frac{\int_{\mathbb{R}^3} \rho(t,x)\chi(t,x)u(t,x)\ dx}{\int_{\mathbb{R}^3} \rho(t,x)\chi(t,x)\ dx},\\
\omega_{[\chi,\rho,u]}(t) &:= \left(I_{[\chi,\rho]}(t)\right)^{-1} \int_{\mathbb{R}^3} \rho(t,x)\chi(t,x) \left(x - a_{[\chi,\rho]}(t)\right) \times u(t,x)\ dx,\\
I_{[\chi,\rho]}(t) &:= \int_{\mathbb{R}^3} \rho(t,x)\chi(t,x) \left(\left|x - a_{[\chi,\rho]}(t)\right|^2 \text{id} - \left(x - a_{[\chi,\rho]}(t)\right) \otimes \left(x - a_{[\chi,\rho]}(t)\right)\right)\ dx,\\
a_{[\chi,\rho]}(t) &:= \frac{\int_{\mathbb{R}^3} \rho(t,x)\chi(t,x)x\ dx}{\int_{\mathbb{R}^3} \rho(t,x)\chi(t,x)\ dx}.
\end{aligned}$$

The fact that $\Pi_{[\chi,\rho,u]}(t)$ indeed constitutes an orthogonal projection of u is proved in [15, Lemma 3.1]. For the convenience of the reader we restate this result in the following lemma.

Lemma A.5.1. *Let $t \in [0,T]$, let $\chi(t) \in L^\infty(\mathbb{R}^3;\{0,1\})$ denote the characteristic function of a bounded domain $S(t) \subset \mathbb{R}^3$, let $\rho(t) \in L^\infty(\mathbb{R}^3;\mathbb{R})$ satisfy $\rho(t) \geqslant \underline{\rho}$ almost everywhere in $S(t)$ for some constant $\underline{\rho} > 0$ and let $u(t) \in L^2(\mathbb{R}^3;\mathbb{R}^3)$. Let further the rigid velocity field $\Pi_{[\chi,\rho,u]}(t)$ be defined by the formula (A.5.1). Then*

$$\int_{\mathbb{R}^3} \rho(t,x)\chi(t,x) \left(u(t,x) - \Pi_{[\chi,\rho,u]}(t,x)\right) \cdot \Pi(t,x)\ dx = 0$$

for any rigid velocity field

$$\Pi(t,x) := v(t) + w(t) \times x, \qquad v(t), w(t) \in \mathbb{R}^3.$$

Proof
Setting $\tilde{v}(t) := v(t) + w(t) \times a_{[\chi,\rho]}(t)$ we write

$$\Pi(t,x) = \tilde{v}(t) + w(t) \times \left(x - a_{[\chi,\rho]}(t)\right).$$

Then, as shown in [15, Lemma 3.1], the desired identity can be proved by a straight-forward calculation:

$$\begin{aligned}
&\int_{\mathbb{R}^3} \rho(t,x)\chi(t,x)\left(u(t,x) - \Pi_{[\chi,\rho,u]}(t,x)\right) \cdot \Pi(t,x)\ dx \\
=&\int_{\mathbb{R}^3} \rho(t,x)\chi(t,x)\left[u(t,x) - \left((u_G)_{[\chi,\rho,u]}(t) + \omega_{[\chi,\rho,u]}(t) \times \left(x - a_{[\chi,\rho]}(t)\right)\right)\right] \\
&\cdot \left(\tilde{v}(t) + w(t) \times \left(x - a_{[\chi,\rho]}(t)\right)\right)\ dx \\
=&\tilde{v}(t) \cdot \int_{\mathbb{R}^3} \rho(t,x)\chi(t,x)u(t,x)\ dx + w(t) \cdot \int_{\mathbb{R}^3} \rho(t,x)\chi(t,x)\left(x - a_{[\chi,\rho]}(t)\right) \times u(t,x) dx \\
&- (u_G)_{[\chi,\rho,u]}(t) \cdot \tilde{v}(t) \int_{\mathbb{R}^3} \rho(t,x)\chi(t,x)\ dx \\
&- (u_G)_{[\chi,\rho,u]}(t) \cdot \left(w(t) \times \int_{\mathbb{R}^3} \rho(t,x)\chi(t,x)\left(x - a_{[\chi,\rho]}(t)\right)\ dx\right) \\
&- \tilde{v}(t) \cdot \left(\omega_{[\chi,\rho,u]}(t) \times \int_{\mathbb{R}^3} \rho(t,x)\chi(t,x)\left(x - a_{[\chi,\rho]}(t)\right) dx\right) \\
&- \int_{\mathbb{R}^3} \rho(t,x)\chi(t,x)\left[\omega_{[\chi,\rho,u]}(t) \times \left(x - a_{[\chi,\rho]}(t)\right)\right] \cdot \left[w(t) \times \left(x - a_{[\chi,\rho]}(t)\right)\right]\ dx \\
=&\tilde{v}(t) \cdot (u_G)_{[\chi,\rho,u]}(t) \int_{\mathbb{R}^3} \rho(t,x)\chi(t,x)\ dx + w(t) \cdot \left(I_{[\chi,\rho]}(t)\omega_{[\chi,\rho,u]}(t)\right) \\
&- (u_G)_{[\chi,\rho,u]}(t) \cdot \tilde{v}(t) \int_{\mathbb{R}^3} \rho(t,x)\chi(t,x)\ dx \\
&- (u_G)_{[\chi,\rho,u]}(t) \cdot \left(w(t) \times \int_{\mathbb{R}^3} \rho(t,x)\chi(t,x)\left(x - a_{[\chi,\rho]}(t)\right)\ dx\right) \\
&- \tilde{v}(t) \cdot \left(\omega_{[\chi,\rho,u]}(t) \times \int_{\mathbb{R}^3} \rho(t,x)\chi(t,x)\left(x - a_{[\chi,\rho]}(t)\right) dx\right) \\
&- \int_{\mathbb{R}^3} \rho(t,x)\chi(t,x)\left[\omega_{[\chi,\rho,u]}(t) \times \left(x - a_{[\chi,\rho]}(t)\right)\right] \cdot \left[w(t) \times \left(x - a_{[\chi,\rho]}(t)\right)\right]\ dx
\end{aligned} \tag{A.5.2}$$

Here on the right-hand side we see that

$$\int_{\mathbb{R}^3} \rho(t,x)\chi(t,x)\left(x - a_{[\chi,\rho]}(t)\right)\ dx = \int_{\Omega} \rho(t,x)\chi(t,x)x\ dx - a_{[\chi,\rho]}(t)\int_{\mathbb{R}^3} \rho(t,x)\chi(t,x)\ dx = 0. \tag{A.5.3}$$

Moreover, due to the identity

$$(a \times b) \cdot (c \times b) = c \cdot \left(\left[|b|^2 \mathrm{id} - b \otimes b\right] a\right) \qquad \forall a, b, c \in \mathbb{R}^3$$

it holds that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \rho(t,x)\chi(t,x)\left[\omega_{[\chi,\rho,u]}(t) \times \left(x - a_{[\chi,\rho]}(t)\right)\right] \cdot \left[w(t) \times \left(x - a_{[\chi,\rho]}(t)\right)\right]\ dx \\
=&w(t) \cdot \left(I_{[\chi,\rho]}(t)\omega_{[\chi,\rho,u]}(t)\right).
\end{aligned} \tag{A.5.4}$$

Combining the identities (A.5.3) and (A.5.4) with the fact that the second and the third term on the right-hand side of the equation (A.5.2) cancel each other we infer that

$$\int_{\mathbb{R}^3} \rho(t,x)\chi(t,x)\left(u(t,x) - \Pi_{[\chi,\rho,u]}(t,x)\right)\cdot \Pi(t,x)\, dx$$
$$= w(t)\cdot\left(I_{[\chi,\rho]}(t)\omega_{[\chi,\rho,u]}(t)\right) - w(t)\cdot\left(I_{[\chi,\rho]}(t)\omega_{[\chi,\rho,u]}(t)\right) = 0,$$

which concludes the proof.

□

Moreover, for the proof of Lemma 3.6.2 in the limit passage in the Brinkman penalization in Section 3.6 we use two auxiliary results, which we summarize here. The following Lemma provides a trace inequality as well as a Poincaré-type estimate on thin domains. The notation used therein corresponds to the notation used in Section 3.6.

Lemma A.5.2. *Let $0 < T_0 < T'$ be fixed, where T' is defined by (3.1.21). Let further $\kappa_{\sup} = \kappa_{\sup}(T_0) > 0$ be the supremum of all κ which satisfy (3.6.20) and let $m_{\min} \in \mathbb{N}$ be chosen as in the inequality (3.6.27). Then there exist constants $\frac{\kappa_{\sup}}{4} > \kappa_0 > 0$ and $c > 0$ such that for all $t \in [0, T_0]$, $\kappa \in [0, \kappa_0]$ and $m \geqslant m_{\min}$ the trace inequality*

$$\|f(t,\cdot)\|^2_{L^2(\partial(S_m(t))^\kappa)} \leqslant c\,\|f(t,\cdot)\|^{\frac{1}{2}}_{L^2((S_m(t))^\kappa)}\,\|f(t,\cdot)\|^{\frac{3}{2}}_{H^1((S_m(t))^\kappa)} \tag{A.5.5}$$

holds true for functions $f(t,\cdot) \in H^1((S_m(t))^\kappa)$ and the Poincaré-type estimate

$$\|f(t,\cdot)\|^2_{L^2((S_m(t))^\kappa\setminus S_m(t))} \leqslant c\left(\kappa\,\|f(t,\cdot)\|^2_{L^2(\partial S_m(t))} + \kappa^2\,\|\nabla f(t,\cdot)\|^2_{L^2((S_m(t))^\kappa\setminus S_m(t))}\right) \tag{A.5.6}$$

holds true for functions $f(t,\cdot) \in H^1((S_m(t))^\kappa\setminus S_m(t))$.

Proof

We first sketch the proof of (A.5.5). The idea is to consider, for $\kappa_0 > 0$ sufficiently small, a mapping $\Phi_{t,m}:\ \partial S_m(t)\times[-\kappa_0,\kappa_0] \to \mathbb{R}^3$ such that $\Phi_{t,m}(\cdot,0) = \mathrm{id}$ and

$$\Phi_{t,m}\left(\partial S_m(t),\kappa\right) = \begin{cases} \{x \in S_m(t):\ \operatorname{dist}(x,\partial S_m(t)) = -\kappa\} & \text{for } \kappa < 0,\\ \{x \in \Omega\setminus S_m(t):\ \operatorname{dist}(x,\partial S_m(t)) = \kappa\} & \text{for } \kappa > 0.\end{cases}$$

We further choose $\Phi_{t,m}$ to be bi-Lipschitz continuous uniformly with respect to t and m, i.e. both $\Phi_{t,m}$ and its inverse are Lipschitz-continuous with Lipschitz-constants independent of t and m. Such a mapping exists, since $S_m(t)$ is a Lipschitz domain by the assumptions of Theorem 3.1.1. For $a, b \in [-\kappa_0,\kappa_0]$ we denote by $S_{t,m,[a,b]}$ the set $\Phi_{t,m}(\partial S_m(t),[a,b])$. By means of some integral transformations, we can now transfer the problem to $S_m(t)$, where we can make use of the trace inequality

$$\|\cdot\|_{L^2(\partial S_m(t))} \leqslant \|\cdot\|_{L^2\left(\partial S_{t,m,[-\kappa_0,0]}\right)} \leqslant c\,\|\cdot\|_{H^{\frac{3}{4}}\left(S_{t,m,[-\kappa_0,0]}\right)}, \tag{A.5.7}$$

cf. [31, Theorem 2.3], with a constant $c > 0$ independent of t, κ and m. The estimate (A.5.7) leads to

$$\|f(t,\cdot)\|^2_{L^2(\partial(S_m(t))^\kappa)} \leqslant c\,\|f(t,\Phi_{t,m}(\cdot,\kappa))\|^2_{L^2(\partial S_m(t))} \leqslant c\,\|f(t,\cdot)\|^2_{H^{\frac{3}{4}}((S_m(t))^\kappa)} \quad \forall\kappa \in [0,\kappa_0],$$

where the constants $c > 0$ are independent of t, κ and m due to the uniform bi-Lipschitz continuity of $\Phi_{t,m}$. The inequality (A.5.5) then follows by an interpolation between L^2, $H^{\frac{3}{4}}$ and H^1. For the proof of (A.5.6), which follows the proof of [7, Lemma A.5], we also exploit the uniform bi-Lipschitz continuity of $\Phi_{t,m}$, which implies that

$$\int_{\partial S_m(t)}\int_0^\kappa |f(t,\Phi_{t,m}(\cdot,0))|^2|\mathrm{det}D\Phi_{t,m}(\cdot,s)|\,ds\,dA \leqslant c\kappa\,\|f(t,\cdot)\|^2_{L^2(\partial S_m(t))}$$

with a constant $c > 0$ uniform in t, κ and m. Using Young's inequality we can therefore estimate

$$
\begin{aligned}
&\|f(t,\cdot)\|^2_{L^2(S_{t,m,[0,\kappa]})} - 2c\kappa \|f(t,\cdot)\|^2_{L^2(\partial S_m(t))}\\
\leqslant &\int_{\partial S_m(t)} \int_0^\kappa |f(t,\Phi_{t,m}(\cdot,s))|^2 \left|\det D\Phi_{t,m}(\cdot,s)\right| dsdA - \int_{\partial S_m(t)} \int_0^\kappa 2|f(t,\Phi_{t,m}(\cdot,0))|^2 |\det D\Phi_{t,m}(\cdot,s)| dsdA\\
\leqslant 2 &\int_{\partial S_m(t)} \int_0^\kappa \left(|f(t,\Phi_{t,m}(\cdot,s))| - |f(t,\Phi_{t,m}(\cdot,0))|\right)^2 \left|\det D\Phi_{t,m}(\cdot,s)\right| dsdA. \qquad \text{(A.5.8)}
\end{aligned}
$$

Making use of the uniform bi-Lipschitz continuity of $\Phi_{t,m}$ once more and applying Jensen's inequality, we may further estimate

$$
\begin{aligned}
&\int_{\partial S_m(t)} \int_0^\kappa \left(|f(t,\Phi_{t,m}(\cdot,s))| - |f(t,\Phi_{t,m}(\cdot,0))|\right)^2 \left|\det D\Phi_{t,m}(\cdot,s)\right| dsdA\\
\leqslant c &\int_{\partial S_m(t)} \int_0^\kappa \left(\int_0^s |\nabla f(t,\Phi_{t,m}(\cdot,\tilde{s}))|\, d\tilde{s} \right)^2 \left|\det D\Phi_{t,m}(\cdot,s)\right| dsdA \leqslant c\kappa^2 \int_{S_{t,m,[0,\kappa]}} |\nabla f(t,y)|^2 dy.
\end{aligned}
$$

Applying this to the right-hand side of the inequality (A.5.8), we infer the inequality (A.5.6).

□

The second auxiliary result we require for the proof of Lemma 3.6.2 is the following variant of [15, Lemma 3.3], which yields an estimate for solutions to the Stokes problem with 0-right-hand side in terms of the boundary data. Again, the notation corresponds to the notation used in Section 3.6

Lemma A.5.3. *Let $0 < T_0 < T'$ be fixed, where T' is defined by* (3.1.21)*. Let $\kappa_{\sup} = \kappa_{\sup}(T_0) > 0$ be the supremum of all κ which satisfy* (3.6.20)*, let $m_{\min} \in \mathbb{N}$ be chosen as in the inequality* (3.6.27) *and let κ_0 denote the constant from Lemma A.5.2. Let further, for $t \in [0,T_0]$, $\kappa \in [0,\kappa_0]$ and $m \geqslant m_{\min}$, the functions $v(t) \in H^1(\Omega\backslash(S_m(t))^\kappa)$, $p(t) \in L^2(\Omega\backslash(S_m(t))^\kappa)$ denote the solution to the Stokes problem*

$$
\begin{aligned}
-\Delta v(t,\cdot) + \nabla p(t,\cdot) &= 0 \quad \text{in } \Omega\backslash\left(S_m(t)\right)^\kappa,\\
\operatorname{div} v(t,\cdot) &= 0 \quad \text{in } \Omega\backslash\left(S_m(t)\right)^\kappa,\\
v(t,\cdot) &= \begin{cases} w(t,\cdot) & \text{on } \partial\left(S_m(t)\right)^\kappa,\\ 0 & \text{on } \partial\Omega, \end{cases}
\end{aligned}
$$

for $w(t) \in H^{1,2}((S_m(t))^\kappa)$. Then there exists a constant $c > 0$, independent of t, κ and m, such that

$$
\|v(t)\|_{L^2(\Omega\backslash(S_m(t))^\kappa)} \leqslant c\,\|w(t)\|^{\frac{1}{4}}_{L^2((S_m(t))^\kappa)}\,\|w(t)\|^{\frac{3}{4}}_{H^1((S_m(t))^\kappa)}. \qquad \text{(A.5.9)}
$$

The same estimate also holds true for the corresponding solution to the Stokes problem in the limit $m \to \infty$, i.e. with $S_m(t)$ replaced by $S(t)$.

Proof

The proof essentially follows [15, Lemma 3.3]. The idea is to consider the Stokes problem on $\Omega\backslash(S_m(t))^\kappa$ with no-slip boundary condition and arbitrary right-hand side $\phi(t) \in L^2(\Omega\backslash(S_m(t))^\kappa)$. The unique solution $\tilde{v}(t) \in H^2(\Omega\backslash(S_m(t))^\kappa)$, $\tilde{p}(t) \in H^1(\Omega\backslash(S_m(t))^\kappa)$ to this problem can be seen to satisfy

$$
\int_{\Omega\backslash(S_m(t))^\kappa} v(t,x)\cdot\phi(t,x)\;dx = -\int_{\partial(\Omega\backslash(S_m(t))^\kappa)} (\nabla\tilde{v}(t)v(t))\cdot \mathrm{n}\;dx + \int_{\partial(\Omega\backslash(S_m(t))^\kappa)} \tilde{p}(t)v(t)\cdot\mathrm{n}\;dx,
$$

where n denotes the outer unit normal vector on $\partial(\Omega\backslash(S_m(t))^\kappa)$. The arbitrary choice of $\phi(t)$ then yields a dual estimate for $v(t,\cdot)$ from which, together with the trace inequality (A.5.5) and the standard estimates for the Stokes problem (cf. [112, Proposition 2.2, Proposition 2.3]), the assertion follows.

□

A.6 The parabolic Neumann problem

For the construction of a non-negative density in the compressible fluid-rigid body interaction problem in Chapter 4 we use the classical parabolic regularization of the continuity equation, cf. [94, Section 7.6.2]. The existence of the density as well as the associated uniform bounds are guaranteed by the following lemma.

Lemma A.6.1. *Let $T > 0$, $\epsilon > 0$, $n \in \mathbb{N}$, $\xi \in (0,1)$ and assume $\Omega \subset \mathbb{R}^3$ to be a bounded domain of class $C^{2,\xi}$. Let V_n, defined by (4.2.1), denote the Galerkin space from our approximate system in Section 4.2 and let $w \in C([0,T]; V_n)$. Finally, consider the initial data $\rho_0 \in C^{2,\xi}(\overline{\Omega})$ such that*

$$\nabla \rho_0 \cdot \mathrm{n}|_{\partial\Omega} = 0, \qquad \underline{\rho} \leqslant \rho_0 \leqslant \overline{\rho} \quad \text{in } \overline{\Omega}$$

for two constants $0 < \underline{\rho} \leqslant \overline{\rho} < \infty$ and the outer unit normal vector n on $\partial\Omega$. Then the Neumann problem

$$\partial_t \rho + \operatorname{div}(\rho w) = \epsilon \Delta \rho \quad \text{in } (0,T) \times \Omega, \qquad \nabla \rho_0 \cdot \mathrm{n}|_{\partial\Omega} = 0, \qquad \rho(0) = \rho_0 \quad \text{in } \Omega \tag{A.6.1}$$

admits a unique solution $\rho = \rho(w)$ in the class

$$\rho \in C\left([0,T]; C^{2,\xi}\left(\overline{\Omega}\right)\right) \bigcap C^1\left([0,T]; C^{0,\xi}\left(\overline{\Omega}\right)\right). \tag{A.6.2}$$

In addition, the estimates

$$0 < \underline{\rho} \exp\left(-\int_0^t \|w(\tau)\|_{W^{1,\infty}(\Omega)}\, d\tau\right) \leqslant \rho(w)(t) \leqslant \overline{\rho} \exp\left(\int_0^t \|w(\tau)\|_{W^{1,\infty}(\Omega)}\, d\tau\right) < \infty \quad \forall t \in [0,T], \tag{A.6.3}$$

$$\|\rho(w)\|_{C([0,T];C^{2,\xi}(\overline{\Omega}))} + \|\rho(w)\|_{C^1([0,T];C^{0,\xi}(\overline{\Omega}))} \leqslant c(w,\epsilon) \quad \forall w \in C([0,T]; V_n) \tag{A.6.4}$$

$$\|\rho(w_1) - \rho(w_2)\|_{C([0,T];L^2(\Omega))} \leqslant c(\epsilon) \|w_1 - w_2\|_{C([0,T];W^{1,\infty}(\Omega))} \quad \forall w_1, w_2 \in C([0,T]; V_n) \tag{A.6.5}$$

are satisfied for some constant $c(w,\epsilon) > 0$ bounded (for ϵ fixed) on bounded subsets of $C([0,T]; V_n)$ and some constant $c(\epsilon) > 0$ independent of w_1, w_2.

Proof
The existence of a unique solution ρ to the Neumann problem (A.6.1) in the class

$$\rho \in C\left([0,T]; W^{1,2}(\Omega)\right) \bigcap L^2\left(0,T; W^{2,2}(\Omega)\right), \quad \partial_t \rho \in L^2\left((0,T) \times \Omega\right),$$

which satisfies the estimates (A.6.3) and (A.6.5), is well known, cf. [94, Proposition 7.39]. The additional regularity (A.6.2) together with the corresponding estimate (A.6.4) then follows by classical results ([45, Theorem 10.22, Theorem 10.23]) on the maximal regularity for parabolic problems, cf. [45, Lemma 3.1].

□

A.7 Deformable/moving domains

In the setting of Chapter 5 we study the evolution of a magnetoelastic material. In the present section we summarize several auxiliary results we use throughout this investigation. The deformation of the material is described via a mapping $\eta : (0,\infty) \times \Omega_0 \to \mathbb{R}^3$, where the reference configuration $\Omega_0 \subset \mathbb{R}^3$ is assumed to be a bounded domain of class $C^{0,1}$. For almost all $t \in [0,T]$ the mapping $\eta(t)$ is further assumed to be an element of the set

$$\mathcal{E} := \left\{\eta \in W^{2,q}\left(\Omega_0; \mathbb{R}^3\right) : \; \tilde{E}_{\mathrm{el}}(\eta) < \infty, \; |\eta(\Omega_0)| = \int_{\Omega_0} \det(\nabla_X \eta)\, dX, \; \eta|_P = \gamma\right\},$$

where $q > 3$, the elastic energy $\tilde{E}_{\mathrm{el}}$ is defined as

$$\tilde{E}_{\mathrm{el}}(\eta) := \int_{\Omega_0} W(\nabla_X \eta) + \frac{1}{(\det(\nabla_X \eta))^a} + \frac{1}{q}\left|\nabla_X^2 \eta\right|^q \, dX$$

with

$$a > \frac{3q}{q-3} \tag{A.7.1}$$

and the elastic energy density $W \in C^1(\mathbb{R}^{3\times 3}; \mathbb{R}_0^+)$, $P \subset \partial\Omega_0$ has positive 2-dimensional Hausdorff measure $\mathcal{H}^2(P) > 0$ and $\gamma : P \to \mathbb{R}^3$ denotes a prescribed deformation.
For the proof of the main result Theorem 5.1.1 of Chapter 5 it is crucial to know that the determinants of the gradients of deformations with uniformly bounded energy are uniformly bounded away from zero. Such a bound is proved in [7, Section 2.3], the proof therein in turn follows ideas from [68]. For the convenience of the reader we restate the result in the following lemma.

Lemma A.7.1. *Let $E_0 > 0$ be given. Then there exists a constant $c = c(E_0) > 0$ such that for all deformations*

$$\eta \in \mathcal{E} \quad \textit{with} \quad \tilde{E}_{el}(\eta) \leqslant E_0 \tag{A.7.2}$$

it holds that

$$\det(\nabla_X \eta) \geqslant c \quad \textit{in } \Omega_0.$$

Proof
We follow the proof given in [7, Section 2.3]: From the energy bound (A.7.2) and the Morrey embedding we infer the existence of a constant $c = c(E_0)$, independent of η, such that

$$\|\det(\nabla_X \eta)\|_{C^{0,\alpha}(\Omega)} \leqslant c\,\|\det(\nabla_X \eta)\|_{W^{1,q}(\Omega_0)} \leqslant c \tag{A.7.3}$$

for $\alpha = 1 - \frac{3}{q}$. We choose some arbitrarily small value $\epsilon > 0$ and assume the existence of $X_0 \in \Omega_0$ such that $\det[\nabla_X \eta](X_0) = \epsilon$. We further choose $\delta > 0$ sufficiently small such that the ball $B_\delta(X_0)$ centered at X_0 with radius δ is contained in Ω_0 and satisfies $|B_\delta(X_0)| \leqslant 1$. Then, by the bound (A.7.2), Jensen's inequality and the (uniform) Hölder continuity given by the estimate (A.7.3), it holds that

$$\begin{aligned} E_0 \geqslant \int_{B_\delta(X_0)} \frac{1}{(\det(\nabla_X \eta))^a}\, dX &\geqslant \frac{\left(\frac{4}{3}\pi\right)^{a+1} \delta^{3a+3}}{\left(\int_{B_\delta(X_0)} \det(\nabla_X \eta)\; dX\right)^a} \\ &\geqslant \frac{\left(\frac{4}{3}\pi\right)^{a+1} \delta^{3a+3}}{\left(\det[\nabla_X \eta](X_0) + \int_{B_\delta(X_0)} \left|\det(\nabla_X \eta) - \det[\nabla_X \eta](X_0)\right|\; dX\right)^a} \\ &\geqslant \frac{\left(\frac{4}{3}\pi\right)^{a+1} \delta^{3a+3}}{\left(\epsilon + c\frac{4}{3}\pi\delta^{3+\alpha}\right)^a}. \end{aligned} \tag{A.7.4}$$

We recall that, by its definition in (A.7.1), $a > \frac{3q}{q-3}$ and therefore

$$\alpha a > \left(1 - \frac{3}{q}\right)\frac{3q}{q-3} = 3.$$

Consequently, by choosing ϵ and δ sufficiently small, the right-hand side of the inequality (A.7.4) can be made arbitrarily large, which leads to a contradiction and thus proves the statement of the lemma. □

Moreover, in order to find a small time interval $[0, T]$ on which a deformation $\eta(t)$, $t \in [0, T]$, remains injective on $\partial\Omega_0$ provided that $\eta(0)$ is injective on $\partial\Omega_0$, we use the following version of [7, Lemma 2.5, Proposition 2.7].

Lemma A.7.2. *Let $E_0 > 0$ be given.*

(i) There exists a constant $\delta = \delta(E_0) > 0$ such that for all deformations

$$\eta \in \mathcal{E} \quad \textit{with} \quad \tilde{E}_{\mathrm{el}}(\eta) \leqslant E_0$$

it holds that

$$X_1, X_2 \in \partial\Omega_0 \quad \textit{with} \quad |X_1 - X_2| < \delta \qquad \Rightarrow \qquad \eta(X_1) \neq \eta(X_2)$$

(ii) Let $\delta > 0$ be as in (i). Let in addition $\eta_0 \in \mathrm{int}(\mathcal{E})$ be given such that $\tilde{E}_{\mathrm{el}}(\eta_0) \leqslant E_0$ and η_0 is injective on $\partial\Omega_0$. Then there exists a constant $\Gamma = \Gamma(\eta_0, E_0)$ such that

$$X_1, X_2 \in \partial\Omega_0 \quad \textit{with} \quad |X_1 - X_2| \geqslant \delta \qquad \Rightarrow \qquad |\eta(X_1) - \eta(X_2)| \geqslant \frac{\epsilon}{2}$$

for all deformations

$$\eta \in \mathcal{E} \quad \textit{with} \quad \tilde{E}_{\mathrm{el}}(\eta) \leqslant E_0 \quad \textit{and} \quad \|\eta - \eta_0\|_{L^2(\Omega_0)} < \Gamma.$$

Proof
For the proof of (i) we refer to [7, Lemma 2.5]. For the statement (ii) we recall the proof from [7, Proposition 2.7] for the convenience of the reader. The injectivity of η_0 on $\partial\Omega_0$ implies the existence of $\epsilon > 0$ such that

$$X_1, X_2 \in \partial\Omega_0 \quad \text{with} \quad |X_1 - X_2| \geqslant \delta \qquad \Rightarrow \qquad |\eta_0(X_1) - \eta_0(X_2)| \geqslant \epsilon.$$

Now let $\eta \in \mathcal{E}$ be such that $\tilde{E}_{\mathrm{el}}(\eta) \leqslant E_0$ and assume there are $X_1, X_2 \in \partial\Omega_0$ with $|X_1 - X_2| \geqslant \delta$ and $|\eta(X_1) - \eta(X_2)| < \frac{\epsilon}{2}$. Then the triangle inequality implies that

$$|\eta(X_1) - \eta_0(X_1)| + |\eta(X_2) - \eta_0(X_2)| \geqslant \frac{\epsilon}{2}.$$

In particular it holds that $|\eta(X_i) - \eta_0(X_i)| \geqslant \frac{\epsilon}{4}$ for either $i = 1$ or $i = 2$. Without loss of generality, we assume that $i = 1$. Next, we point out that, since $\tilde{E}_{\mathrm{el}}(\eta) \leqslant E_0$, the function $\eta_0 - \eta$ is uniformly continuous independently of the specific choice of η. Thus we find $r > 0$ such that

$$|\eta(X) - \eta_0(X)| \geqslant \frac{\epsilon}{8} \qquad \forall X \in B_r(X_1) \bigcap \Omega_0,$$

where $B_r(X_1) \subset \mathbb{R}^3$ denotes the ball centered at X_1 with radius r. This allows us to define the desired constant $\Gamma > 0$ by

$$\Gamma := \frac{\epsilon}{8} \left| B_r(X_1) \bigcap \Omega_0 \right|^{\frac{1}{2}} \leqslant \|\eta - \eta_0\|_{L^2(\Omega_0)},$$

which concludes the proof.

□

Furthermore, the mathematical investigation of a (deformable) magnetoelastic material requires a generalization of the classical Bochner spaces to the case of moving domains, which is given via the formula (5.1.4). The following Lemma shows that this generalization is in fact a Banach space.

Lemma A.7.3. *Let $\eta : [0,T] \times \Omega_0 \to \mathbb{R}^3$ be a deformation satisfying*

$$\eta \in L^\infty(0,T;\mathcal{E}) \bigcap C\left([0,T]; C^1\left(\overline{\Omega_0}\right)\right), \quad \eta(t) \in \mathrm{int}\,(\mathcal{E}) \;\; \textit{and} \;\; \tilde{E}_{el}(\eta(t)) \leqslant c \;\textit{for a.a.}\; t \in [0,T], \tag{A.7.5}$$

where $c > 0$ denotes a constant independent of $t \in [0,T]$. Let $\Omega(t) := \eta(t, \Omega_0)$ and let

$$\eta^{-1}(t,\cdot) : \Omega(t) \to \Omega_0. \tag{A.7.6}$$

denote the inverse of the mapping $X \mapsto \eta(t,X)$ *for* $t \in [0,T]$. *Then, for all values* $1 \leqslant p < \infty$, $1 \leqslant r \leqslant \infty$ *and* $k = 0,1$, *the set*

$$L^r\left(0,T;W^{k,p}\left(\Omega\left(\cdot\right)\right)\right) = \left\{ m : [0,T] \to \bigcup_{t\in[0,T]} W^{k,p}\left(\Omega(t)\right) : \ m\left(\cdot,\eta^{-1}(\cdot,\cdot)\right) \in L^r\left(0,T;W^{k,p}\left(\Omega_0\right)\right) \right\}$$

constitutes a Banach space with the norm

$$\|m\|_{L^r(0,T;W^{k,p}(\Omega(\cdot)))} := \begin{cases} \left(\int_0^T \|m(t)\|^r_{W^{k,p}(\Omega(t))}\right)^{\frac{1}{r}} & \text{if } 1 \leqslant r < \infty \\ \operatorname{esssup}_{t\in[0,T]} \|m(t)\|_{W^{k,p}(\Omega(t))} & \text{if } r = \infty \end{cases}.$$

Proof

The existence of the inverse functions (A.7.6) follows directly from the injectivity of the mapping $X \mapsto \eta(t,X)$ implied by the assumptions (A.7.5). For $t \in [0,T]$ we define a linear mapping

$$\Phi_t : W^{k,p}\left(\Omega_0\right) \to W^{k,p}\left(\Omega(t)\right), \qquad \Phi_t\left(\tilde f\right) := \tilde f\left(\eta^{-1}(t,\cdot)\right) \quad \forall \tilde f \in W^{k,p}\left(\Omega_0\right)$$

for all $k = 0,1$ and $1 \leqslant p < \infty$ with an inverse

$$\Phi_{-t} : W^{k,p}\left(\Omega(t)\right) \to W^{k,p}\left(\Omega_0\right), \qquad \Phi_{-t}(f) := f\left(\eta(t,\cdot)\right) \quad \forall f \in W^{k,p}\left(\Omega(t)\right).$$

Both the mappings Φ_t and Φ_{-t} can be seen to be bounded uniformly with respect to t: Indeed, we first remark that, due to the energy bound in (A.7.5), the determinant of the deformation gradient is bounded away from zero uniformly with respect to $t \in [0,T]$, cf. Lemma A.7.1. Hence, for the boundedness of Φ_t we estimate

$$\left\|\Phi_t \tilde f\right\|_{L^p(\Omega(t))} = \left(\int_{\Omega(t)} \left|\tilde f\left(\eta^{-1}(t,\cdot)\right)\right|^p \, dx\right)^{\frac{1}{p}} = \left(\int_{\Omega_0} \det\left(\nabla_X \eta(t,\cdot)\right) \left|\tilde f\right|^p \, dX\right)^{\frac{1}{p}} \leqslant c \left\|\tilde f\right\|_{L^p(\Omega_0)}$$

and

$$\left\|\nabla\left(\Phi_t \tilde f\right)\right\|_{L^p(\Omega(t))} = \left(\int_{\Omega(t)} \left|\nabla \tilde f\left(\eta^{-1}(t,\cdot)\right)\right|^p \, dx\right)^{\frac{1}{p}} = \left(\int_{\Omega_0} \det\left(\nabla_X \eta(t,\cdot)\right) \left|\nabla_X \tilde f\left(\nabla_X \eta\right)^{-1}\right|^p \, dX\right)^{\frac{1}{p}} \leqslant c \left\|\nabla_X \tilde f\right\|_{L^p(\Omega_0)}$$

for almost all $t \in [0,T]$, for all $\tilde f \in W^{k,p}(\Omega_0)$ and a constant $c > 0$ independent of both t and $\tilde f$ thanks to the energy bound in (A.7.5). It follows that

$$\left\|\Phi_t \tilde f\right\|_{W^{k,p}(\Omega(t))} \leqslant c \left\|\tilde f\right\|_{W^{k,p}(\Omega_0)} \qquad \text{for almost all } t \in [0,T] \text{ and all } \tilde f \in W^{k,p}\left(\Omega_0\right). \tag{A.7.7}$$

For the boundedness of Φ_{-t} we estimate

$$\|\Phi_{-t} f\|_{L^p(\Omega_0)} = \left(\int_{\Omega_0} \left|f\left(\eta(t,\cdot)\right)\right|^p \, dX\right)^{\frac{1}{p}} = \left(\int_{\Omega(t)} \frac{1}{\det\left(\left[\nabla_X \eta(t,\cdot)\right]\left(\eta^{-1}(t,\cdot)\right)\right)} |f|^p \, dx\right)^{\frac{1}{p}} \leqslant c \|f\|_{L^p(\Omega(t))}$$

and

$$\begin{aligned} \|\nabla_X\left(\Phi_{-t} f\right)\|_{L^p(\Omega_0)} &= \left(\int_{\Omega_0} \left|\nabla_X f\left(\eta(t,\cdot)\right)\right|^p \, dX\right)^{\frac{1}{p}} \\ &= \left(\int_{\Omega(t)} \frac{1}{\det\left(\left[\nabla_X \eta(t,\cdot)\right]\left(\eta^{-1}(t,\cdot)\right)\right)} \left|\nabla f \left[\nabla_X \eta(t,\cdot)\right]\left(\eta^{-1}(t,\cdot)\right)\right|^p \, dx\right)^{\frac{1}{p}} \\ &\leqslant c \|\nabla f\|_{L^p(\Omega(t))} \end{aligned}$$

for almost all $t \in [0,T]$, for all $f \in W^{k,p}(\Omega(t))$ and a constant $c > 0$ independent of both t and $\tilde{f}$. Consequently it holds that

$$\|\Phi_{-t} f\|_{W^{k,p}(\Omega_0)} \leqslant c\,\|f\|_{W^{k,p}(\Omega(t))} \qquad \text{for almost all } t \in [0,T] \text{ and all } f \in W^{k,p}(\Omega(t)). \tag{A.7.8}$$

Moreover, from the $C([0,T]; C^1(\overline{\Omega_0}))$-regularity of η assumed in (A.7.5) it immediately follows that the mapping

$$t \mapsto \left\|\Phi_t \tilde{f}\right\|_{W^{k,p}(\Omega(t))}$$

is measurable. This, in combination with the bounds (A.7.7) and (A.7.8), implies the statement via [1, Theorem 2.4].

□

Finally, in order to pass to the limit in our approximate system in Chapter 5, we need convergence of compositions of the stray field and the external magnetic field with the deformation. While in general the convergence of compositions where the outer function is only integrable is a delicate issue, we can here prove the following lemma due to the good properties of the deformations.

Lemma A.7.4. *Let $E_0 > 0$, let $(\eta_j)_{j\in\mathbb{N}} \subset L^\infty(0,T;\mathcal{E})$ be a sequence of deformations such that $\tilde{E}_{\mathrm{el}}(\eta_j(t)) \leqslant E_0$ for almost all $t \in [0,T]$ and all $j \in \mathbb{N}$ and assume that*

$$\eta_j \overset{*}{\rightharpoonup} \eta \quad \text{in } L^\infty\left(0,T; W^{2,q}(\Omega_0)\right), \qquad \eta_j \to \eta \quad \text{in } C\left([0,T]; C^1\left(\overline{\Omega_0}\right)\right). \tag{A.7.9}$$

Further, let $1 \leqslant p < \infty$, $k = 0,1$ and let $(H_j)_{j\in\mathbb{N}} \subset L^p(0,T;W^{k,p}(\mathbb{R}^3))$ be a sequence of functions satisfying

$$H_j \to H \quad \text{in } L^p\left(0,T; W^{k,p}\left(\mathbb{R}^3\right)\right). \tag{A.7.10}$$

Then it holds that

$$H_j(\eta_j) \to H(\eta) \quad \text{in } L^p\left(0,T; W^{k,p}(\Omega_0)\right).$$

Proof

Due to the uniform bound of $\tilde{E}_{\mathrm{el}}(\eta_j)$, Lemma A.7.1 implies the existence of a constant $c > 0$, independent of j, such that

$$\det(\nabla_X \eta_j) \geqslant c \quad \text{in } (0,T) \times \Omega_0. \tag{A.7.11}$$

We write

$$\begin{aligned} &\|H_j(\eta_j) - H(\eta)\|_{L^p((0,T)\times\Omega_0)} \\ \leqslant &\|H_j(\eta_j) - H(\eta_j)\|_{L^p((0,T)\times\Omega_0)} + \|H(\eta_j) - H(\eta)\|_{L^p((0,T)\times\Omega_0)}. \end{aligned} \tag{A.7.12}$$

For the first term on the right-hand side of this inequality we see that

$$\begin{aligned} &\|H_j(\eta_j) - H(\eta_j)\|_{L^p((0,T)\times\Omega_0)} \\ =&\left(\int_0^T \int_{\eta_j(t,\Omega_0)} \frac{1}{\det\left([\nabla_X \eta_j(t,\cdot)]\left(\eta_j^{-1}(t,\cdot)\right)\right)} |H_j - H|^p \, dx dt\right)^{\frac{1}{p}} \\ \leqslant & c\,\|H_j - H\|_{L^p((0,T)\times\mathbb{R}^3)} \to 0 \end{aligned} \tag{A.7.13}$$

due to the bound (A.7.11) of $\det(\nabla_X \eta_j)$ away from zero and the convergence (A.7.10) of H_j. In order to check that also the second term on the right-hand side of the inequality (A.7.12) vanishes we choose a sequence of smooth functions $(H^n)_{n\in\mathbb{N}} \subset \mathcal{D}((0,T)\times\mathbb{R}^3)$ such that $H^n \to H$ in $L^2((0,T)\times\mathbb{R}^3)$. We estimate

$$\begin{aligned} &\|H(\eta_j) - H(\eta)\|_{L^p((0,T)\times\Omega_0)} \\ &\leqslant \|H(\eta_j) - H^n(\eta_j)\|_{L^p((0,T)\times\Omega_0)} + \|H^n(\eta_j) - H^n(\eta)\|_{L^p((0,T)\times\Omega_0)} + \|H^n(\eta) - H(\eta)\|_{L^p((0,T)\times\Omega_0)} . \end{aligned} \tag{A.7.14}$$

For the first term on the right-hand side we estimate

$$\begin{aligned} &\|H(\eta_j) - H^n(\eta_j)\|_{L^p((0,T)\times\Omega_0)} \\ &= \left(\int_0^T \int_{\eta_j(t,\Omega_0)} \frac{1}{\det\left([\nabla_X \eta_j(t,\cdot)] \left(\eta_j^{-1}(t,\cdot)\right)\right)} |H - H^n|^p \; dxdt \right)^{\frac{1}{p}} \\ &\leqslant c\,\|H - H^n\|_{L^p((0,T)\times\mathbb{R}^3)} \end{aligned}$$

with a constant $c > 0$ independent of j due to the uniform bound (A.7.11). Arguing similarly for the third term on the right-hand side of the inequality (A.7.14) we infer that

$$\|H(\eta_j) - H(\eta)\|_{L^p((0,T)\times\Omega_0)} \leqslant c\,\|H - H^n\|_{L^p((0,T)\times\mathbb{R}^3)} + \|H^n(\eta_j) - H^n(\eta)\|_{L^p((0,T)\times\Omega_0)} .$$

Letting first j and subsequently n tend to infinity we thus infer from the uniform convergence (A.7.9) and the smoothness of H^n that

$$\|H(\eta_j) - H(\eta)\|_{L^p((0,T)\times\Omega_0)} \to 0 \tag{A.7.15}$$

for $j \to \infty$. Applying the convergences (A.7.13) and (A.7.15) to the right-hand side of the inequality (A.7.12), we conclude that

$$H_j(\eta_j) \to H(\eta) \qquad \text{in } L^p((0,T)\times\Omega_0) .$$

By similar arguments we also see that

$$\nabla_X H_j(\eta_j) \to \nabla_X H(\eta) \qquad \text{in } L^p((0,T)\times\Omega_0) ,$$

which concludes the proof.

□

As a special case of the previous lemma we have the following corollary

Corollary A.7.1. *Let $E_0 > 0$, let $(\eta_j)_{j\in\mathbb{N}} \subset \mathcal{E}$ be a sequence of deformations such that $\tilde{E}_{\mathrm{el}}(\eta_j \leqslant E_0$ for all $j \in \mathbb{N}$ and assume that*

$$\eta_j \overset{*}{\rightharpoonup} \eta \quad \text{in } W^{2,q}(\Omega_0) , \qquad \eta_j \to \eta \quad \text{in } C^1\left(\overline{\Omega_0}\right) .$$

Further, let $1 \leqslant p < \infty$ and let $H \in L^p(\mathbb{R}^3)$. Then it holds that

$$H(\eta_j) \to H(\eta) \quad \text{in } L^p(\Omega_0) .$$

A.8 Variation of the stray field part

In Definition 5.1.1, the variational formulation of the model (1.3.30)–(1.3.32) of the evolution of a magnetoelastic material is presented in terms of the Fréchet derivatives of the energy potential (1.3.34) and the dissipation potential (1.3.37). These derivatives, which are written out explicitly in Remark 5.1.3, can be calculated in a straight forward way except for in the stray field part of the micromagnetic energy. In the following, we carry out the calculations in the stray field part precisely, i.e. we prove the identities (5.1.20) and (5.1.22). We begin with the equation (5.1.20), for which we follow the argumentation from [48, Section 2.7.1]. We fix some time $t \in [0,T]$, consider $\epsilon > 0$ and some arbitrary function $\chi \in \mathcal{D}((0,T) \times \Omega_0)$ and denote, in accordance with the notation introduced for the solution to the Poisson equation (5.1.6), by $\phi[\tilde{M}(t), \eta(t)) + \epsilon\chi(t)] \in \dot{H}^1(\mathbb{R}^3)$ the solution to the Poisson equation

$$\begin{aligned}\int_{\mathbb{R}^3} \nabla\phi\left[\tilde{M}(t), \eta(t) + \epsilon\chi(t)\right] \cdot \nabla\psi \, dx &= \int_{(\eta(t)+\epsilon\chi(t))(\Omega_0)} M_{\eta(t)+\epsilon\chi(t)}\left[\tilde{M}(t)\right] \cdot \nabla\psi \, dx \\ &= \int_{\Omega_0} \tilde{M}(t) \cdot \left[\left((\nabla_X(\eta(t) + \epsilon\chi(t)))^{-1}\right)^T \nabla_X\psi\left(\eta(t) + \epsilon\chi(t)\right)\right] dX\end{aligned} \tag{A.8.1}$$

for all $\psi \in \dot{H}^1(\mathbb{R}^3)$, where

$$M_{\eta(t)+\epsilon\chi(t)}\left[\tilde{M}(t)\right] = \frac{1}{\det\left(\left[\nabla_X\left(\eta(t) + \epsilon\chi(t)\right)\right]\left((\eta(t) + \epsilon\chi(t))^{-1}\right)\right)} \tilde{M}\left(t, (\eta(t) + \epsilon\chi(t))^{-1}\right)$$

is defined in accordance with the formula (5.1.7). Correspondingly we denote the stray field associated to the magnetization $\tilde{M}(t)$ and the deformation $\eta(t)$ by $H[\tilde{M}(t), \eta(t) + \epsilon\chi(t)] = -\nabla\phi[\tilde{M}(t), \eta(t) + \epsilon\chi(t)]$. The quantity we need to calculate reads

$$\begin{aligned}&\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \int_{\Omega_0} \frac{\mu}{2}\tilde{M}(t) \cdot H\left[\tilde{M}(t), \eta(t) + \epsilon\chi(t)\right](\eta(t) + \epsilon\chi(t)) \, dX \\ =&\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} - \int_{\Omega_0} \frac{\mu}{2}\tilde{M}(t) \cdot \left(\nabla\phi\left[\tilde{M}(t), \eta(t) + \epsilon\chi(t)\right](\eta(t) + \epsilon\chi(t))\right) dX. \\ =& - \int_{\Omega_0} \frac{\mu}{2}\tilde{M}(t) \cdot \nabla\left(\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \phi\left[\tilde{M}(t), \eta(t) + \epsilon\chi(t)\right](\eta(t))\right) dX \\ &- \int_{\Omega_0} \frac{\mu}{2}\left[\left(\nabla^2\phi\left[\tilde{M}(t), \eta(t)\right](\eta(t))\right)^T \tilde{M}(t)\right] \cdot \chi(t) \, dX.\end{aligned} \tag{A.8.2}$$

In order to calculate the first integral on the right-hand side of this identity we pick some arbitrary $\psi \in \dot{H}^1(\mathbb{R}^3) \bigcap H^2_{\mathrm{loc}}(\Omega(t))$ and differentiate the Poisson equation (A.8.1) with respect to ϵ. Under exploitation of the identity

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \left(\nabla_X\left(\eta(t) + \epsilon\chi(t)\right)\right)^{-1} = -\left(\nabla_X\left(\eta(t)\right)\right)^{-1} \nabla_X\chi(t) \left(\nabla_X\left(\eta(t)\right)\right)^{-1}$$

this leads to the relation

$$\begin{aligned}&\int_{\mathbb{R}^3} \nabla\left(\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \phi\left[\tilde{M}(t), \eta(t) + \epsilon\chi(t)\right]\right) \cdot \nabla\psi \, dx \\ =& - \int_{\Omega_0} \tilde{M}(t) \cdot \left[\left((\nabla_X\eta(t))^{-1} \nabla_X\chi(t) (\nabla_X\eta(t))^{-1}\right)^T \nabla_X\psi\left(\eta(t)\right)\right] dX \\ &+ \int_{\Omega_0} \tilde{M}(t) \cdot \left[\left((\nabla_X\eta(t))^{-1}\right)^T \nabla_X\left(\nabla\psi\left(\eta(t)\right) \cdot \chi(t)\right)\right] dX \\ =& - \int_{\Omega(t)} M(t) \cdot \left[\left(\nabla\chi\left(t, (\eta(t))^{-1}\right)\right)^T \nabla\psi\right] dx + \int_{\Omega(t)} M(t) \cdot \nabla\left(\nabla\psi \cdot \chi\left(t, (\eta(t))^{-1}\right)\right) dx \\ =& \int_{\Omega(t)} \left[\left(\nabla^2\psi\right)^T M(t)\right] \cdot \chi\left(t, (\eta(t))^{-1}\right) dx.\end{aligned} \tag{A.8.3}$$

We remark that the local H^2-regularity assumed for ψ is indeed sufficient for the right-hand side of this identity to be well-defined due to the compact support of χ. Next we test the Poisson equation satisfied by $\phi[\tilde{M}(t), \eta(t)]$ by the test function $\phi[\tilde{M}(t), \eta(t) + \epsilon\chi(t)]$ and differentiate with respect to ϵ

to see that

$$\int_{\mathbb{R}^3} \nabla\phi\left[\tilde{M}(t),\eta(t)\right]\cdot\nabla\left(\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\phi\left[\tilde{M}(t),\eta(t)+\epsilon\chi(t)\right]\right)\,dx$$
$$=\int_{\Omega(t)} M(t)\cdot\nabla\left(\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\phi\left[\tilde{M}(t),\eta(t)+\epsilon\chi(t)\right]\right)\,dx. \tag{A.8.4}$$

Due to the regularity (A.2.9), (A.2.10) of solutions to the Poisson equation given by Lemma A.2.3, it holds that $\phi[\tilde{M}(t),\eta(t)]\in\dot{H}^1(\mathbb{R}^3)\bigcap H^2_{\text{loc}}(\Omega(t))$ and so we can choose $\psi=\phi[\tilde{M}(t),\eta(t)]$ in the identity (A.8.3). Comparing the resulting equation to the equation (A.8.4) we infer that

$$\int_{\Omega_0}\tilde{M}(t)\cdot\nabla\left(\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\phi\left[\tilde{M}(t),\eta(t)+\epsilon\chi(t)\right](\eta(t))\right)\,dX$$
$$=\int_{\Omega(t)} M(t)\cdot\nabla\left(\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\phi\left[\tilde{M}(t),\eta(t)+\epsilon\chi(t)\right]\right)\,dx$$
$$=\int_{\Omega(t)}\left[\left(\nabla^2\phi\left[\tilde{M}(t),\eta(t)\right]\right)^T M(t)\right]\cdot\chi\left(t,(\eta(t))^{-1}\right)\,dx$$
$$=\int_{\Omega_0}\left[\left(\nabla^2\phi\left[\tilde{M}(t),\eta(t)\right](\eta(t))\right)^T\tilde{M}(t)\right]\cdot\chi(t)\,dX.$$

Applying this identity to the first integral on the right-hand side of the equation (A.8.2) we obtain

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\int_{\Omega_0}\frac{\mu}{2}\tilde{M}(t)\cdot H\left[\tilde{M}(t),\eta(t)+\epsilon\chi(t)\right](\eta(t)+\epsilon\chi(t))\,dX$$
$$=-\int_{\Omega_0}\mu\left[\left(\nabla^2\phi\left[\tilde{M}(t),\eta(t)\right](\eta(t))\right)^T\tilde{M}(t)\right]\cdot\chi(t)\,dX$$
$$=\int_{\Omega_0}\mu\left[\left(\left(\nabla_X H\left[\tilde{M}(t),\eta(t)\right](\eta(t))\right)(\nabla_X\eta(t))^{-1}\right)^T\tilde{M}(t)\right]\cdot\chi(t)\,dX,$$

i.e. the desired relation (5.1.20). For the derivation of the identity (5.1.22) we follow the arguments from [48, Section 2.7.2]. We fix $t\in[0,T]$, consider $\epsilon>0$ and an arbitrary function $\hat{M}\in L^\infty(0,T;H^1(\Omega(\cdot)))$. Using the Poisson equation (5.1.6), which defines both $H[\det(\nabla_X\eta(t))M(t),\eta(t)]$ and $H[\det(\nabla_X\eta(t))(M(t)+\epsilon\hat{M}),\eta(t)]$, we first see that

$$\int_{\Omega(t)}-\frac{\mu}{2}M(t)\cdot H\left[\det(\nabla_X\eta(t))\left(M(t)+\epsilon\hat{M}(t)\right),\eta(t)\right]\,dx$$
$$=\int_{\mathbb{R}^3}\frac{\mu}{2}H\left[\det(\nabla_X\eta(t))M(t),\eta(t)\right]\cdot H\left[\det(\nabla_X\eta(t))\left(M(t)+\epsilon\hat{M}(t)\right),\eta(t)\right]\,dx$$
$$=\int_{\Omega(t)}-\frac{\mu}{2}\left[M(t)+\epsilon\hat{M}(t)\right]\cdot H\left[\det(\nabla_X\eta(t))M(t),\eta(t)\right]\,dx.$$

Hence, under exploitation of the product rule,

$$
\begin{aligned}
&\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \int_{\Omega(t)} -\frac{\mu}{2}\left[M(t)+\epsilon\hat{M}(t)\right]\cdot H\left[\det\left(\nabla_X\eta(t)\right)\left(M(t)+\epsilon\hat{M}(t)\right),\eta(t)\right]\,dx \\
=&\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \int_{\Omega(t)} -\frac{\mu}{2}\left[M(t)+\epsilon\hat{M}(t)\right]\cdot H\left[\det\left(\nabla_X\eta(t)\right)M(t),\eta(t)\right]\,dx \\
&-\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \int_{\Omega(t)} \frac{\mu}{2}M(t)\cdot H\left[\det\left(\nabla_X\eta(t)\right)\left(M(t)+\epsilon\hat{M}(t)\right),\eta(t)\right]\,dx \\
=&\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \int_{\Omega(t)} -\mu\left[M(t)+\epsilon\hat{M}(t)\right]\cdot H\left[\det\left(\nabla_X\eta(t)\right)M(t),\eta(t)\right]\,dx \\
=&\int_{\Omega(t)} -\mu H\left[\tilde{M}(t),\eta(t)\right]\cdot\hat{M}(t)\,dX,
\end{aligned}
$$

which proves the desired relation (5.1.22).

Bibliography

[1] A. Alphonse, D. Caetano, A. Djurdjevac and C.M. Elliott, *Function spaces, time derivatives and compactness for evolving families of Banach spaces with applications to PDEs*, Journal of Differential Equations 353: 268–338 (2023)

[2] P. Angot, C.H. Bruneau, and P. Fabrie, *A penalization method to take into account obstacles in incompressible viscous flows*, Numer. Math. 81: 497–520 (1999)

[3] G. Ausanio, A.C. Barone, C. Hison, V. Iannotti, G. Mannara and L. Lanotte, *Magnetoelastic sensor application in civil buildings monitoring*, Sensors and Actuators A: Physical 123–124: 290–295 (2005)

[4] L. Bălilescu, J.S. Martín and T. Takahashi, *Fluid-rigid structure interaction system with Coulomb's law*, SIAM Journal on Mathematical Analysis 49 no. 6: 4625–4657 (2017)

[5] M. Barchiesi, D. Henao and C. Mora-Corral, *Local invertibility in Sobolev spaces with applications to nematic elastomers and magnetoelasticity*, Archive for Rational Mechanics and Analysis 224 no. 2: 743–816 (2017)

[6] B. Benešová, J. Forster, C. Liu, A. Schlömerkemper *Existence of weak solutions to an evolutionary model for magnetoelasticity*, SIAM Journal on Mathematical Analysis 50 no. 1: 1200–1236 (2018)

[7] B. Benešová, M. Kampschulte and S. Schwarzacher, *A variational approach to hyperbolic evolutions and fluid-structure interactions*, J. Eur. Math. Soc. (2023)

[8] B. Benešová, Š. Nečasová, J. Scherz and A. Schlömerkemper, *Fluid-rigid body interaction in an incompressible electrically conducting fluid*, SIAM Journal on Mathematical Analysis 55 no. 2: 929–965 (2023)

[9] W. Bielski and B. Gambin, *Relationship between existence of energy minimizers of incompressible and nearly incompressible magnetostrictive materials*, Reports on Mathematical Physics 66 no. 2: 147–157 (2010)

[10] A. Bieńkowski and R. Szewczyk, *New possibility of utilizing amorphous ring cores as stress sensor*, Physica Status Solidi (a) 189 no. 3: 787–790 (2002)

[11] A. Bieńkowski and R. Szewczyk, *The possibility of utilizing the high permeability magnetic materials in construction of magnetoelastic stress and force sensors*, Sensors and Actuators A: Physical 113 no. 3: 270–276 (2004)

[12] X. Blanc and B. Ducomet, *Weak and strong solutions of equations of compressible magnetohydrodynamics*, Handbook of Mathematical Analysis in Mechanics of Viscous Fluids: 2869–2925, Springer, Cham (2016)

[13] T. Bodnár, G.P. Galdi, Š. Nečasová, *Fluid-Structure Interaction and Biomedical Applications*, Birkhäuser, Basel (2014)

[14] M. Boulakia and S. Guerrero, *A regularity result for a solid-fluid system associated to the compressible Navier-Stokes equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire 26 no. 3: 777–813 (2009)

[15] C. Bost, G.H. Cottet and E. Maitre, *Convergence Analysis of a Penalization Method for the Three-Dimensional Motion of a Rigid Body in an Incompressible Viscous Fluid*, SIAM J. Numer. Anal. 48: 1313–1337 (2010)

[16] J.R. Brauer, *Magnetic Actuators and Sensors*, John Wiley & Sons, Inc., Hoboken, New Jersey (2006)

[17] M. Bresciani, E. Davoli and M. Kružík, *Existence results in large-strain magnetoelasticity*, Ann. Inst. H. Poincaré Anal. Non Linéaire 40 no. 3: 557–592 (2023)

[18] W.F. Brown, *Magnetoelastic Interactions*, Springer-Verlag, Berlin Heidelberg (1966)

[19] W.F. Brown, *Micromagnetics*, Interscience Publishers, New York (1963)

[20] H.-J. Bungartz, M. Schäfer: *Fluid-Structure Interaction: Modelling, Simulation, Optimisation*, Springer-Verlag, Berlin Heidelberg (2006)

[21] H. Cabannes: *Theoretical Magnetohydrodynamics*, Academic Press, New York (1970)

[22] G. Carbou, M.A. Efendiev and P. Fabrie, *Global weak solutions for the Landau-Lifschitz equation with magnetostriction*, Mathematical Methods in the Applied Sciences 34 no. 10: 1274–1288 (2011)

[23] M. Chipot, I. Shafrir, V. Valente, and G. Vergara Caffarelli, *On a hyperbolic-parabolic system arising in magnetoelasticity*, J. Math. Anal. Appl. 352 no. 1: 120–131 (2009)

[24] P.G. Ciarlet, J. Nečas, *Injectivity and self-contact in nonlinear elasticity*, Archive for Rational Mechanics and Analysis 97 no. 3: 171–188 (1987)

[25] C. Conca, J. San Martin and M. Tucsnak, *Existence of solutions for the equations modelling the motion of a rigid body in a viscous fluid*, Commun. Partial Differential Equations 25: 1019–1042 (2000)

[26] M. Coquerelle and G.H. Cottet, *A vortex level set method for the two-way coupling of an incompressible fluid with colliding rigid bodies*, J. Comput. Phys. 227: 9121–9137 (2008)

[27] R. Dautray and P.L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology. Volume 1: Physical Origins and Classical Methods*, Springer-Verlag, Berlin (2000)

[28] P.A. Davidson, *An Introduction to Magnetohydrodynamics*, Cambridge University Press, United Kingdom (2001)

[29] F. De Anna, J. Kortum, A. Schlömerkemper, *Struwe-like solutions for an evolutionary model of magnetoviscoelastic fluids*, Journal of Differential Equations 309: 455–507 (2022)

[30] E. De Giorgi, *New problems on minimizing movements*, Boundary Value Problems for PDE and Applications: 81–98 (1993)

[31] L. Del Pezzo and J.D. Rossi, *Traces for fractional Sobolev spaces with variable exponents*, Adv. Oper. Theory 2 no. 4: 435–446 (2017)

[32] A. DeSimone, R.V. Kohn, S. Müller and F. Otto, *Recent analytical developments in micromagnetics*, The science of Hysteresis: Physical Modeling, Micromagnetics and Magnetization Dynamics. Vol. 2: 269–381, Elsevier, Amsterdam (2006)

[33] B. Desjardins and M.J. Esteban, *Existence of weak solutions for the motion of rigid bodies in a viscous fluid*, Arch. Rational Mech. Anal. 146: 59–71 (1999)

[34] B. Desjardins and M.J. Esteban, *On Weak Solutions for Fluid-Rigid Structure Interaction: Compressible and Incompressible Models*, Comm. in Partial Differential Equations 25 no. 7–8: 1399–1413 (2000)

[35] R.J. DiPerna and P.L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math. 98: 511–547 (1989)

[36] M. Dreher and A. Jüngel, *Compact families of piecewise constant functions in Lp(0,T;B)*, Konstanzer Schriften in Mathematik 292 (2011)

[37] H. Du, Y. Shao and G. Simonett, *Well-posedness for magnetoviscoelastic fluids in 3D*, Nonlinear Analysis: Real World Applications 69 (2023)

[38] I. Ellahiani, E.–H. Essoufi and M. Tilioua, *Global existence of weak solutions to a three-dimensional fractional model in magneto-viscoelastic interactions*, Boundary Value Problems 122 (2017)

[39] A.C. Eringen and G.A. Maugin, *Electrodynamics of Continua II*, Springer-Verlag, New York (1990)

[40] L.C. Evans, *Partial Differential Equations*, American Mathematical Society, Providence, Rhode Island (1998)

[41] E. Feireisl, *Dynamics of viscous compressible fluids,* Oxford University Press, Oxford (2004)

[42] E. Feireisl, *On compactness of solutions to the compressible isentropic Navier-Stokes equations when the density is not square integrable,* Comment. Math. Univ. Carolin. 42 no. 1: 83–98 (2001)

[43] E. Feireisl, *On the motion of rigid bodies in a viscous compressible fluid*, Arch. Rational Mech. Anal. 167: 281–308 (2003)

[44] E. Feireisl, *On the motion of rigid bodies in a viscous incompressible fluid*, J. Evol. Equ. 3: 419–441 (2003)

[45] E. Feireisl and A. Novotný, *Singular Limits in Thermodynamics of Viscous Fluids*, Birkhäuser, Basel (2017)

[46] E. Feireisl, A. Novotný and H. Petzeltová, *On the domain dependence of solutions to the compressible Navier–Stokes equations of a barotropic fluid,* Math. Meth. Appl. Sci. 25: 1045–1073 (2002)

[47] E. Feireisl, A. Novotný and H. Petzeltová, *On the Existence of Globally Defined Weak Solutions to the Navier-Stokes Equations,* J. Math. Fluid Mech. 3: 359–392 (2001)

[48] J. Forster, *Variational approach to the modeling and analysis of magnetoelastic materials*, Ph.D. thesis, Universität Würzburg (2016), urn:nbn:de:bvb:20-opus-147226

[49] J. Franců, *Weakly continuous operators. Applications to differential equations*, Appl. Math. 39: 45–56 (1994)

[50] E.P. Furlani, *Magnetophoretic separation of blood cells at the microscale*, Journal of Physics D: Applied Physics 40 no. 5: 1313–1319 (2007)

[51] G. P. Galdi, *On the motion of a rigid body in a viscous liquid: A mathematical analysis with applications*, Handbook of Mathematical Fluid Dynamics, Vol. I: 655–791, Elsevier Sci, Amsterdam (2002)

[52] C.J. García-Cervera, *Numerical micromagnetics: a review*, Bol. Soc. Esp. Mat. Apl. SeMA 39: 103–135 (2007)

[53] H. Garcke, P. Knopf, S. Mitra and A. Schlömerkemper, *Strong well-posedness, stability and optimal control theory for a mathematical model for magneto-viscoelastic fluids*, Calc. Var. 61: 179 (2022)

[54] M. Geissert, K. Götze and M. Hieber, *Lp-theory for strong solutions to fluid-rigid body interaction in Newtonian and generalized Newtonian fluids*, Trans. Amer. Math. Soc. 365 no. 3: 1393–1439 (2013)

[55] J.F. Gerbeau, C. Le Bris and T. Lelièvre, *Mathematical Methods for the Magnetohydrodynamics of Liquid Metals*, Oxford University Press, Oxford (2006)

[56] N. Gigli and S.J.N. Mosconi, *A variational approach to the Navier–Stokes equations*, Bulletin des Sciences Mathématiques 136 no. 3: 256–276 (2012)

[57] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin (2001)

[58] T.S. Gregory, R. Cheng, G. Tang, L. Mao and Z.T.H. Tse, *The magnetohydrodynamic effect and its associated material designs for biomedical applications: a state-of-the-art review*, Adv. Funct. Mater. 26: 3942–3952 (2016)

[59] T.S. Gregory, K.J. Wu, J. Yu, J.B. Box, R. Cheng, L. Mao, G. Tang and Z.T.H. Tse, *Magnetohydrodynamic-Driven Design of Microscopic Endocapsules in MRI*, IEEE/ASME Transactions on Mechatronics 20 no. 6: 2691–2698 (2015)

[60] W. Greiner, *Classical Electrodynamics*, Springer-Verlag, New York (1998)

[61] D.J. Griffiths, *Introduction to Electrodynamics*, 3rd ed., Prentice Hall, Upper Saddle River (1999)

[62] C.A. Grimes, S.C. Roy, S. Rani and Q. Cai, *Theory, instrumentation and applications of magnetoelastic resonance sensors: a review*, Sensors 11 no.3: 2809–2844 (2011)

[63] J.L. Guermond and P.D. Minev, *Mixed Finite Element Approximation of an MHD Problem Involving Conducting and Insulating Regions: The 2D Case*, ESAIM: Mathematical Modelling and Numerical Analysis 136 no. 33: 517–536 (2002)

[64] J.L. Guermond and P.D. Minev, *Mixed Finite Element Approximation of an MHD Problem Involving Conducting and Insulating Regions: The 3D Case*, Numer. Meth. PDE 19: 709–731 (2003)

[65] M.D. Gunzburger, H. C. Lee and A. Seregin, *Global existence of weak solutions for viscous incompressible flow around a moving rigid body in three dimensions*, J. Math. Fluid Mech. 2: 219–266 (2000)

[66] B. H. Haak, D. Maity, T. Takahashi and M. Tucsnak, *Mathematical analysis of the motion of a rigid body in a compressible Navier-Stokes-Fourier fluid*, Math. Nachr. 292: 1972–2017 (2019)

[67] B. Halphen, Q.S. Nguyen, *Sur les matériaux standard généralisés*, Journal de Mécanique 14 no. 1: 39–63 (1975)

[68] T.J. Healey and S. Krömer, *Injective weak solutions in second-gradient nonlinear elasticity*, ESAIM: Control, Optimisation and Calculus of Variations 15 no. 4: 863–871 (2009)

[69] M.I. Herreros and S. Ligüérzana, *Rigid body motion in viscous flows using the finite element method*, Phys. Fluids 32 (2020)

[70] M. Hieber and M. Murata, *The Lp-approach to the fluid-rigid body interaction problem for compressible fluids*, Evol. Equ. Control Theory 4: 69–87 (2015)

[71] K.H. Hoffmann and V. N. Starovoitov, *On a motion of a solid body in a viscous fluid. Two-dimensional case*, Adv. Math. Sci. Appl. 9: 633–648, (1999)

[72] A. Hubert and R. Schäfer, *Magnetic Domains: The Analysis of Magnetic Microstructures*, Springer-Verlag, Berlin Heidelberg, 1998

[73] J.D. Jackson, *Classical Electrodynamics*, 3rd ed., John Wiley & Sons, United States (1998)

[74] S. Jiang and F. Li, *Convergence of the complete electromagnetic fluid system to the full compressible magnetohydrodynamic equations*, Asymptotic Analysis 95: 161–185 (2015)

[75] S. Jiang and F. Li, *Rigorous derivation of the compressible magnetohydrodynamic equations from the electromagnetic fluid system*, Nonlinearity 25: 1735–1752 (2012)

[76] M. Kalousek, J. Kortum and A. Schlömerkemper, *Mathematical analysis of weak and strong solutions to an evolutionary model for magnetoviscoelasticity*, Discrete Contin. Dyn. Syst. Ser. S 14 no. 1, 17–39 (2021)

[77] J. Kortum, *Existence of a solution for an evolutionary magnetostrictive model*, Master thesis, Universität Würzburg (2017)

[78] J. Kortum, *Global existence and uniqueness results for nematic liquid crystal and magnetoviscoelastic flows*, Ph.D. thesis, Universität Würzburg (2021), urn:nbn:de:bvb:20-opus-278271

[79] O. Kreml, Š. Nečasová and T. Piasecki, *Weak-strong uniqueness for the compressible fluid-rigid body interaction*, J. Differential Equations 268 no. 8: 4756–4785 (2020)

[80] M. Kružík and A. Prohl, *Recent developments in the modeling, analysis, and numerics of ferromagnetism*, SIAM Rev 48 no. 3: 439–483 (2006)

[81] M. Kružík and T. Roubíček, *Mathematical Methods in Continuum Mechanics of Solids*, Springer-Verlag, Cham (2019)

[82] M. Kružík, U. Stefanelli and J. Zeman, *Existence results for incompressible magnetoelasticity*, Discrete and Continuous Dynamical Systems 35 no. 6: 2615–2623 (2015)

[83] A.G. Kulikovskiy and G.A. Lyubimov, *Magnetohydrodynamics*, Addison Wesley, Reading, Massachussets (1965)

[84] L.D. Landau and E.M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon Press, Oxford (1960)

[85] L.D. Landau and E.M. Lifshitz, *Theory of Elasticity*, Pergamon Press, Oxford (1970)

[86] P.L. Lions, *Mathematical Topics in Fluid Mechanics: Volume 1: Incompressible Models*, Oxford University Press, Oxford (1996)

[87] P.L. Lions, *Mathematical Topics in Fluid Mechanics: Volume 2: Compressible Models*, Oxford University Press, Oxford (1998)

[88] R. Moreau, *Magnetohydrodynamics*, Springer, Dordrecht (1990)

[89] B. Muha, Š. Nečasová and A. Radošević, *A uniqueness result for 3D incompressible fluid-rigid body interaction problem*, J. Math. Fluid Mech. 23 no. 1: (2021)

[90] Š. Nečasová, J. Ogorzały, J. Scherz, *The compressible Navier-Stokes equations with slip boundary conditions of friction type*, Z. Angew. Math. Phys. 74 no. 5: 188 (2023)

[91] Š. Nečasová, M. Ramaswamy, A. Roy and A. Schlömerkemper, *Motion of a rigid body in a compressible fluid with Navier-slip boundary condition*, J. Differential Equations 338: 256–320 (2022)

[92] Š. Nečasová, T. Takahashi and M. Tucsnak, *Weak Solutions for the Motion of a Self-propelled Deformable Structure in a Viscous Incompressible Fluid*, Acta Appl. Math. 116: 329–352 (2011)

[93] Q.S. Nguyen, *Stability and Nonlinear Solid Mechanics*, Wiley, Chichester (2000)

[94] A. Novotný and I. Straškraba, *Introduction to the mathematical theory of compressible flow*, Oxford University Press, Oxford (2004)

[95] D. O'Regan, *Existence Theory for Nonlinear Ordinary Differential Equations*, Springer-Verlag, Dordrecht (1997)

[96] H. Pauly and H.P. Schwan, *Dielectric properties and ion mobility in erythrocytes*, Biophys J. 6 no. 5: 621–639 (1966)

[97] M. Pokorný, *Navier-Stokes equations*, Lecture notes, Charles University in Prague (2022), URL: `https://www.karlin.mff.cuni.cz/~pokorny/LectureNotes/NavierandStokes_eng.pdf` (Accessed: 22 September 2023)

[98] D. Praetorius, *Analysis of the operator Δ^{-1} div arising in magnetic models*, Zeitschrift für Analysis und ihre Anwendungen 23 no. 3: 589–605 (2004)

[99] T. Roubíček, *Nonlinear Partial Differential Equations with Applications*, Birkhäuser Verlag, Basel, Boston, Berlin (2005)

[100] A. Roy and T. Takahashi, *Stabilization of a rigid body moving in a compressible viscous fluid*, Journal of Evolution Equations 21: 167–200 (2020)

[101] P. Rybka and M. Luskin, *Existence of energy minimizers for magnetostrictive materials*, SIAM Journal on Mathematical Analysis 36 no. 6: 2004–2019 (2005)

[102] A.H. Sacks and E.G. Tickner, *The compressibility of blood*, Biorheology 5 no. 4: 271–274 (1968)

[103] J.A. San Martin, V. Starovoitov and M. Tucsnak, *Global weak solutions for the two dimensional motion of several rigid bodies in an incompressible viscous fluid*, Arch. Rational Mech. Anal. 161: 93–112 (2002)

[104] R. Sart, *Existence of finite energy weak solutions for the equations MHD of compressible fluids*, Applicable Analysis 19 no. 3: 357–379 (2009)

[105] J. Scherz, *Fluid-rigid body interaction in a compressible electrically conducting fluid*, Math. Nachr. (2023)

[106] A. Schlömerkemper, J. Žabenský, *Uniqueness of solutions for a mathematical model for magneto-viscoelastic flows*, Nonlinearity 31 no. 6: 2989–3012 (2018)

[107] B. Schweizer: *On Friedrichs inequality, Helmzholtz decomposition, vector potentials, and the div-curl lemma*, Trends on Applications of Mathematics to Mechanics - Springer INdAM series: 65–79, Springer, Cham (2018)

[108] T.W. Secomb and A.R. Pries, *Blood viscosity in microvessels: Experiment and theory*, Comptes Rendus Physique 14 no. 6: 470–478 (2013)

[109] D. Serre, *Chute libre d'un solide dans un fluide visqueux incompressible. Existence*, Jap. J. Appl. Math. 4: 99–110 (1987)

[110] R.L. Snyder, V.Q. Nguyen and R.V. Ramanujan, *The energetics of magnetoelastic actuators is analogous to phase transformations in materials*, Acta Materialia 58 no. 17: 5620–5630 (2010)

[111] T. Takahashi, *Analysis of strong solutions for the equations modeling the motion of a rigid-fluid system in a bounded domain*, Adv. Differential Equations 8: 1499–1532 (2003)

[112] R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*, North-Holland Publishing Company, Amsterdam (1979)

[113] C. Wang, *Strong solutions for the fluid-solid systems in a 2-D domain*, Asymptot. Anal. 89 no. 3–4: 263–306 (2014)

[114] Z. Yang and L. Zhang, *Magnetic actuation systems for miniature robots: a review*, Advanced Intelligent Systems 2 no. 13 (2020)

[115] E. Zeidler, *Nonlinear Functional Analysis and its Applications I: Fixed-Point Theorems*, Springer Verlag, New York (1986)

[116] Y. Zhou, X. Zhao, J. Xu, Y. Fang, G. Chen, Y. Song, S. Li and J. Chen, *Giant magnetoelastic effect in soft systems for bioelectronics*, Nature Materials 20 no. 12: 1670–1676 (2021)

Acknowledgements

First and foremost I would like to express my deepest gratitude to my supervisors Prof. Dr. Anja Schlömerkemper and Dr. Barbora Benešová as well as my advisor Prof. Dr. Šárka Nečasová. I would like to thank all of them for their continuous support and guidance all throughout my PhD project and for sacrificing their time for countless hours of discussions - wether in person or online -, helping me to find the right path in many scientific problems.
I would further like to thank Anja Schlömerkemper for initiating my PhD project by making me aware of the possibility of a Cotutelle de Thèse between the University of Würzburg and the Charles University in Prague while I was still writing my master's thesis. She also motivated me to increase my mathematical knowledge by inviting me to several of her lecture series and working groups as well as by introducing me to mathematical modeling.
I would like to thank Barbora Benešová for welcoming me to Prague and helping me a lot with the organization of my stays there, which greatly facilitated my first time living in a foreign city. She furthermore motivated me to supplement my PhD thesis by some work in the calculus of variations and acquainted me with the mathematical theory of magnetoelasticity. She had useful advice for any mathematical question I managed to come up with.
Moreover, I would like to express my gratefulness to Šárka Nečasová for helping me - to the same extent - to feel welcome and arrange my life in Prague. It was also her who introduced me to the theory of fluid-structure interaction during her Giovanni Prodi Professorship in Würzburg back in 2018 when I was still a master's student. This laid the foundation for my PhD thesis and all throughout the project I was able to greatly benefit from her large knowledge and experience in this research field.
Also I would like to thank the whole team of Anja Schlömerkemper at the Chair of Mathematics in the Sciences for the good working climate and the supportive environment during my time in Würzburg. Additionally I would like to thank Joshua Kortum for several helpful scientific discussions and for proofreading parts of the thesis
Finally, I acknowledge the financial support by the Czech Science Foundation (GAČR) through the projects GC22–08633J and 3–04766S, the financial support by Praemium Academiæ of Šárka Nečasová and the financial support by the Charles University in Prague via the project PRIMUS/19/SCI/01.