Topological Properties of Quasiconformal Automorphism Groups

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Topological Properties of Quasiconformal Automorphism Groups

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Für meine Familie

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Nomenclature

Symbol	Short description	Page
$\mathbb{N}$	$:= \{1, 2, 3, \dots\}$, the set of natural numbers	
$\mathbb{N}_0$	$:= \mathbb{N} \cup \{0\}$, the set of non–negative integers	
$\mathbb{Z}$	The set of integers	
$\mathbb{Q}$	The set of rational numbers	
$\mathbb{R}$	The set of real numbers	
$\mathbb{R}^+$	$:= \{x \in \mathbb{R} \mid x > 0\}$	
$\mathbb{R}_0^+$	$:= \{x \in \mathbb{R} \mid x \geq 0\}$	
$\mathbb{C}$	The set of complex numbers	
$\overline{\mathbb{C}}$	$:= \mathbb{C} \cup \{\infty\}$, the Riemann sphere	
$\mathbb{D}$	$:= \left\{z \in \mathbb{C} \,\middle|\, \lvert z\rvert < 1\right\}$, the open unit disk in $\mathbb{C}$	
$\varnothing$	The empty set	
$\square$	The end of a proof	
$\subseteq$	The subset relation, i.e. for sets A, B, it is $A \subseteq B :\Longleftrightarrow \forall x \in A : x \in B$	
$\subsetneqq$	The proper subset relation, i.e. $A \subsetneqq B :\Longleftrightarrow (A \subseteq B$ and $A \neq B)$	
G	A domain in $\mathbb{C}$, i.e. a non–empty, open and connected subset of $\mathbb{C}$	
$\mathcal{JD}$	The class of bounded Jordan domains in $\mathbb{C}$	1
$\mathcal{P}(G)$	The set of prime ends of a domain G	2
$\mathcal{P}_1(G)$	The set of prime ends of the first kind of a domain G	2
$f_z, f_{\overline{z}}$	The Wirtinger derivatives of a differentiable function $f : G \longrightarrow \mathbb{C}$	
μ_f	The complex dilatation of a differentiable function $f : G \longrightarrow \mathbb{C}$	10
ω_g	The modulus of continuity of a function $g : X \longrightarrow Y$ between metric spaces $(X, d_X), (Y, d_Y)$	1
$C(X)$	The set of continuous functions $f : X \longrightarrow \mathbb{C}$ of a topological space X into $\mathbb{C}$	
$C_b(X)$	The set of bounded continuous functions $f : X \longrightarrow \mathbb{C}$ of a topological space X into $\mathbb{C}$	34
$\mathcal{H}(X, Y)$	The function space of homeomorphisms between metric spaces X, Y	26
$\mathcal{H}(X)$	The homeomorphism group of a (metric or topological) space X	25
$\mathcal{H}^+(D)$	The subgroup of orientation–preserving homeomorphisms of a domain or its closure in $\mathbb{C}$ onto itself	25
$\mathcal{M}(X, Y)$	The set of all monotone mappings between compact metric spaces X, Y	25
$\mathcal{M}(X)$	The set of all monotone self–mappings of a compact metric space X	25

Symbol	Short description	Page
$L^\infty(\Omega)$	The set of essentially bounded measurable functions $f : \Omega \longrightarrow \mathbb{C}$ of a measure space Ω into $\mathbb{C}$	10
$\mathbb{B}_{L^\infty}(G)$	The open unit ball in the Banach space $(L^\infty(G), \|\cdot\|_{L^\infty(G)})$	11
$Q(G)$	The set of all quasiconformal automorphisms of a domain G, either as group or metric space	1
$Q(\overline{G})$	The set of all homeomorphic extensions of quasiconformal automorphisms of a Jordan domain $G \in \mathcal{JD}$ to $\overline{G}$	12
$Q_K(G)$	The subset of all K–quasiconformal automorphisms of a domain G, as a metric (sub)space	2
$Q_{K,\mathrm{fix}(z_0)}(G)$	The subset of all K–quasiconformal automorphisms of a domain G having $z_0 \in G$ as a fixed point, as a metric (sub)space	50
$\Sigma(G)$	The set of all conformal automorphisms of a domain G, either as (sub)group (of $Q(G)$) or metric (sub)space (of $Q(G)$)	2
$\mathrm{Diff}^\infty(G)$	The set of all C^∞–diffeomorphisms of a domain G onto itself	36
$C^\infty Q(G)$	The set of all quasiconformal C^∞–automorphisms of a domain G	36
$HQ(\mathbb{D})$	The set of all harmonic quasiconformal automorphisms of the unit disk	68
id_A	The identity mapping of a set A onto itself	1
Φ	Group isomorphism between two quasiconformal automorphism groups, called the conjugation mapping	2
$\|\mu\|_{L^\infty(G)}$	The essential supremum norm of a function $\mu \in L^\infty(G)$	10
$d_{\mathrm{sup}}(f,g)$	The distance between two mappings f, g in the supremum metric	1
$d_{\mathrm{sym}}(f,g)$	The distance between two mappings f, g in the symmetric supremum	30
Δ	The Laplace operator	68
$\mathfrak{P}$	The Poisson transformation	68
$\mathfrak{H}$	The (periodic) Hilbert transformation	72

Introduction

QUASICONFORMAL MAPPINGS of domains in the complex plain and the corresponding automorphism groups formed by this particular class of mappings are at the very heart of the matter this thesis is concerned with. This introductory chapter is devoted to establish general notations and to give a brief overview of the historical development of quasiconformal mappings and the corresponding automorphism groups in $\mathbb{C}$.

Central object of investigation and general prerequisites

Henceforth, the following general prerequisites will be used throughout this thesis, unless the contrary is explicitly stated.
If $(X, \mathcal{T})$ is a topological space and $A \subseteq X$ any subset, then A will always be endowed with the *subspace topology* inherited from X. All topological operations and notions of subsets of $\mathbb{C}$ will be considered with respect to the standard topology induced by the Euclidean norm $|\cdot|$. A *domain* in $\mathbb{C}$ is a non–empty, open and connected subset of the complex plain, usually denoted by G. The class consisting of all bounded Jordan domains in $\mathbb{C}$ will be denoted by $\mathcal{JD}$.
If $(H, *)$ is a group and $U \subseteq H$ is a (normal) subgroup, this circumstance will be denoted by $U \leq H$ ($U \trianglelefteq H$). The subgroup generated by $S \subseteq H$ in H is denoted by $\langle S \rangle$. Let A, B be any two sets and $f : A \longrightarrow B$ a mapping which extends to a mapping $\widehat{f} : A' \longrightarrow B'$ on a superset $A' \supseteq A$. If no misunderstanding is possible, then the extension $\widehat{f}$ of f to A' will usually be denoted by the same letter. The identity mapping on A sending each element onto itself will be denoted by id_A.
If $(X, d_X), (Y, d_Y)$ are metric spaces and $g : X \longrightarrow Y$ is a mapping, then for $t \geq 0$ the expression

$$\omega_g(t) := \sup_{d_X(x,x') \leq t} d_Y\left(g(x), g(x')\right) \tag{0.1}$$

with $x, x' \in X$ denotes the *modulus of continuity* of g.

The central object of investigation this work is concerned with is

$$\boxed{Q(G) := \left\{ f : G \longrightarrow G \;\middle|\; f \text{ is a quasiconformal mapping of } G \text{ onto itself} \right\}} \tag{0.2}$$

for a bounded, simply connected domain $G \subsetneq \mathbb{C}$, called the **quasiconformal automorphism group** of G. These sets of mappings – surely being non–empty due to the easily verified fact that the identity mapping $\mathrm{id}_G : G \longrightarrow G$ is always an element of $Q(G)$ – allow for rich mathematical structure, as on the one hand, $Q(G)$ naturally carries the structure of a group by endowing it with the canonical composition of mappings, denoted by $\circ$, in which the identity mapping id_G plays the prominent role of the group's neutral element. On the other hand, as a family of mappings, $Q(G)$ may be equipped with the *supremum metric*

$$d_{\sup}(f, g) := \sup_{z \in G} |f(z) - g(z)| \tag{0.3}$$

for $f, g \in Q(G)$, thus turning $Q(G)$ into a metric (and therefore topological) space[1]. When speaking of $Q(G)$, it is always referred to (at least) one of these mathematical structures – the group $(Q(G), \circ)$ or the metric space $(Q(G), d_{\sup})$ – in which it will be clear from the respective context which particular structure is to be meant. As is well–known, the supremum metric induces a special topology, the *topology of uniform convergence*, also referred to as the *uniform topology.* A particularly important subset of $Q(G)$ is given by

$$\Sigma(G) := \left\{ f \in Q(G) \;\middle|\; f \text{ is conformal} \right\} \tag{0.4}$$

which turns out being not merely a subset, but also a *subgroup*, i.e. $\Sigma(G) \leq Q(G)$. More generally, one can consider the subsets

$$Q_K(G) := \left\{ f \in Q(G) \;\middle|\; K(f) \leq K \right\} \tag{0.5}$$

for $K \in [1, +\infty)$, yielding the canonical decomposition

$$Q(G) = \bigcup_{K \geq 1} Q_K(G) \tag{0.6}$$

with $Q_K(G) \subseteq Q_{K'}(G)$ for $K \leq K'$. In this notation, it is $\Sigma(G) = Q_1(G)$.
As there are, by definition, uncountably many different instances of $Q(G)$, the question arises as to which extent the properties of one quasiconformal automorphism group $Q(G)$ carry over to another object of this kind, say, $Q(G')$ for two bounded, simply connected domains $G, G' \subsetneqq \mathbb{C}$. The classical Riemann Mapping Theorem assures the existence of a conformal mapping $F : G \longrightarrow G'$ with corresponding inverse $F^{-1} : G' \longrightarrow G$. These two transition mappings now give rise for considering the *conjugation mapping*

$$\Phi : Q(G) \longrightarrow Q(G'),\ f \longmapsto \Phi(f) := F \circ f \circ F^{-1} \tag{0.7}$$

A simple calculation shows that Φ is bijective with inverse mapping $\Phi^{-1}(h) = F^{-1} \circ h \circ F$ for $h \in Q(G')$. Hence, from a purely set–theoretical point of view, all quasiconformal automorphism groups are identical. However, due to the various mathematical structures being present on $Q(G)$ as expounded above, the metric, topological and group–theoretical properties of these sets of mappings may vary heavily depending on the underlying domain G. In particular, the nature of the domain's boundary has a major effect on the behaviour of the conformal transition mappings F and F^{-1}, respectively, and consequently to the mapping properties of the conjugation mappings Φ and Φ^{-1}. In this regard, especially the topological properties of the quasiconformal automorphism groups are of great interest, since it turns out that the ability of the conjugation mappings to transfer certain topological data depends intimately on the boundary regularity of the involved domains. This, in turn, traces back to the boundary behaviour of the conformal mappings F and F^{-1}, which can be described by the sophisticated machinery of the theory of *prime ends*, established at the beginning of the 20^{th} century by Constantin Carathéodory (1873 – 1950) in [Car13]. An account of this theory can e.g. be found in [Pom75] and [Pom92]. Given a bounded, simply connected domain $G \subsetneqq \mathbb{C}$, the set of prime ends of G will be denoted by $\mathcal{P}(G)$, while $\mathcal{P}_1(G)$ will refer to the (sub)set of *prime ends of the first kind* according to the prime end classification found by Carathéodory.

[1] Of course, also other topologies on $Q(G)$ are reasonable, e.g. the topology of locally uniform convergence.

Historical overview

Quasiconformal mappings

The historical development of quasiconformal mappings is commonly considered to begin with Herbert Grötzsch's investigation of "*most nearly conformal mappings*" in 1928 (see [Ahl06, p. 5]), in which he considered two conformally inequivalent rectangles in the plane and asked for a diffeomorphic mapping between these sets that behaves "as conformal as possible". Consequently, Grötzsch had to introduce a possibility to measure the deviation of a diffeomorphism f to a conformal mapping, a task he solved by using a tool that turned out to be the *dilatation quotient*

$$D_f = \frac{\max_\alpha |\partial_\alpha f|}{\min_\alpha |\partial_\alpha f|} = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|}$$

where $\partial_\alpha f$ denotes the directional derivative of f in the direction α (see [Leh87, p. 19]). The term *quasiconformal mapping* was then coined by Lars V. Ahlfors in 1935, and numerous further mathematicians studied the properties of this class of mappings and its versatile connections to further areas of mathematics in the following year intensively, such as Bers, Beurling, Bojarski, Gehring, Lavrentiev, Lehto and Teichmüller.
Starting in the late 1950s, the higher–dimensional setting was brought into focus, i.e. the development of an analogous theory of quasiconformal mappings in spaces $\mathbb{R}^n$ with $n \geq 3$. Among others, driving forces behind this development were eminent names as Gehring, Martio, Reshetnyak, Rickman and Väisälä. Similar to the situation of holomorphic functions in one and several[2] complex variables, many problems turned out to be significantly more delicate in the higher–dimensional setting, among others due to the following two major reasons:

- There exists no direct counterpart to the classical Riemann Mapping Theorem in several (complex) variables (see e.g. [RS07, Abschnitt 8.3.6, p. 186]);
- In comparison with the planar case, there are only "few" conformal mappings in Euclidean spaces $\mathbb{R}^n$ with $n \geq 3$ by a famous result of Liouville, stating that conformal mappings in these spaces are exactly the higher–dimensional versions of (restrictions of) Möbius transformations (see e.g. [GMP17, Section 3.8, pp. 64–75]).

These topics concerning the higher–dimensional setting of quasiconformal mappings and their general theory are discussed in substantially more detail in [GMP17] and the classical monograph [Väi71]. Moreover, see [IM01, Chapter 1, pp. 1–31] and [Leh84] as well as the references cited therein for further information on the historical development of quasiconformal mapping theory.

Finally, a theory of quasiconformal mappings in an even more general setting was sought, such as metric spaces. This goal – typical for the development in mathematical research – was, among others, motivated by the fact that quasiconformal mappings in $\mathbb{C}$ (or more generally, in Euclidean spaces $\mathbb{R}^n$ for $n \geq 2$) can be equivalently defined in terms of the so–called *linear dilatation* of a homeomorphism f, which is given by

$$H_f(x) = \limsup_{r \to 0} \frac{L_f(x,r)}{l_f(x,r)}$$

with $L_f(x,r) = \max_{|h|=r} |f(x+h) - f(x)|$ and $l_f(x,r) = \min_{|h|=r} |f(x+h) - f(x)|$ for x in the domain of f and $r > 0$ chosen sufficiently small; see [GMP17, Subsection 6.4.1, pp. 229–238] for details.

[2] For a good overview of these mentioned – often fundamental – differences between holomorphic functions in one and several (possibly infinitely many) complex variables, see the numerous *"Outlook"*–sections in [RS07].

This definition of H_f can be generalized to homeomorphic mappings between appropriate metric spaces (X, d_X) and (Y, d_Y) via

$$H_f(x) = \limsup_{r \to 0} \frac{\sup\limits_{d_X(x,y) \leq r} d_Y(f(x), f(y))}{\inf\limits_{d_X(x,y) \geq r} d_Y(f(x), f(y))}$$

for $x \in X$ and $r > 0$, as defined in [HK98, p. 1]. However, in order for this definition to create a theory of quasiconformality in this general context that is "similar" to the quasiconformal theory in $\mathbb{C}$ and $\mathbb{R}^n$, respectively, certain constraints on the geometry of the spaces X and Y are to be requested, as pointed out in [HK98, pp. 22–25] (see also the general assumptions made in [HK98, Section 2]). Moreover, other generalizations of quasiconformal mapping theory in further directions were introduced by Tukia–Väisälä (for arbitrary metric spaces, see also [AIM08, p. 50]) and Väisälä (for infinite–dimensional Banach spaces), as stated in [HK98, p. 2].

Conformal automorphism groups and groups of quasiconformal mappings

The investigation of $\Sigma(G)$ and its properties, both as a group and a topological space (in which the topology is induced by the supremum metric $d_{\sup}$), was initiated by Gaier in [Gai84]. In the following years, a multitude of further research papers concerning this topic – especially with emphasis on the topological structure of $\Sigma(G)$ in connection with the boundary of the domain G – were published, e.g. [Lau95], [Lau99], [Sch86], [Sch92], [Vol92] and [LSV00]. Moreover, the Ph.D. thesis [Lau94] was concerned with topological properties of conformal automorphism groups of simply connected domains in the plane, also containing a comprehensive overview of the results obtained in the previously mentioned publications. In sharp contrast to the conformal special case, however, the more general situation of groups of quasiconformal automorphisms and their topological properties doesn't seem to have drawn much attention yet. A few scattered publications concerning or at least partially touching these groups or certain subsets of them are to be mentioned, for example:

- Gehring and Palka studied *K–quasiconformal groups* of domains in extended Euclidean spaces $\overline{\mathbb{R}}^n$ for $n \geq 3$ in [GP76], addressing the problem of *quasiconformal homogeneity*. A K–quasiconformal group Γ is a subgroup of the quasiconformal automorphism group of a domain with the property that every element of Γ is a K–quasiconformal mapping, and a *quasiconformal group* is a K–quasiconformal group for some fixed K (see [GP76, p. 173]). In this paper, Gehring and Palka also raise the question of whether every quasiconformal group Γ is necessarily a quasiconformally conjugated group of conformal mappings (see [GP76, p. 197]), i.e.
$$\Gamma = f \circ \Gamma_0 \circ f^{-1}$$
for some quasiconformal mapping f (with inverse mapping f^{-1}) and a group Γ_0 of conformal mappings. This problem that remained unsolved for several years until Sullivan [Sul81] and Tukia [Tuk80] answered this question affirmatively for the case $n = 2$ (i.e. for quasiconformal mappings defined in subsets of $\mathbb{C} \cong \mathbb{R}^2$); see also [AIM08, Subsection 10.3.2, pp. 285–287]. The higher–dimensional case for $n \geq 3$, however, is to be answered negatively, as demonstrated by Tukia in [Tuk81] (see also [GM87, p. 331]).

- In [GM87], Gehring and Martin consider *discrete groups* of quasiconformal mappings. In this context, the term *discrete* refers to the topological notion of a discrete space, in which the topology of compact convergence is used: A group of homeomorphisms of a domain in $\mathbb{R}^n$ onto itself is called *discrete* if it contains no infinite sequence of distinct elements converging compactly to a limit element which is itself a member of the group (see [GM87, p. 332]).

Typical for research questions in Geometric Function Theory, the mode of convergence considered by Gehring and Martin is the uniform convergence on compact subsets.

- The question for the *Hilbert–Smith conjecture* in the context of quasiconformal mappings has been addressed increasingly in recent years. This conjecture, originally formulated by David Hilbert as part of his famous list of *Hilbert's 23 Problems* and extended by Paul A. Smith, poses the following question (see e.g. [Mar99, p. 67]):

  If a locally compact topological group acts effectively on a finite–dimensional topological manifold, is this group necessarily a Lie group?

  In 1999, Martin answered this question in the affirmative in the case that the group under consideration consists of quasiconformal mappings acting on a Riemannian manifold, see [Mar99, Theorem 1.2, p. 67]. A few years later, Gong solved this problem for the (more elementary) situation that a group of K–quasiconformal automorphisms acts on a domain in the extended Euclidean space $\overline{\mathbb{R}}^n$, see [Gon10, Theorem 3, p. 509]. As in the case of the previously mentioned studies of Gehring and Martin on discrete quasiconformal groups, the topological structure used in [Mar99] and [Gon10] is the topology of compact convergence.

- In [MNP98], the authors investigate the *quasiconformal homogeneity* of compact subsets of the Riemann sphere $\overline{\mathbb{C}}$. In order to define this property, set

  $$Q(\overline{\mathbb{C}}, E) := \left\{ f \in Q(\overline{\mathbb{C}}) \,\middle|\, f(E) = E \right\}$$

  for a (not necessarily compact) set $E \subseteq \overline{\mathbb{C}}$. Consequently, E is called *quasiconformally homogeneous* if the canonically induced group[3] action of $Q(\overline{\mathbb{C}}, E)$ on E via $f \cdot z = f(z)$ for $f \in Q(\overline{\mathbb{C}}, E), z \in E$ is transitive, i.e. for all $a, b \in E$ there exists a mapping $f \in Q(\overline{\mathbb{C}}, E)$ such that $f(a) = b$. In particular, MacManus et. al. are concerned with and utilize the group of all quasiconformal automorphisms of the Riemann sphere, $Q(\overline{\mathbb{C}})$. After defining the notion of quasiconformal homogeneity, the authors derive several interesting results for compact subsets of $\overline{\mathbb{C}}$ in connection with this property. However, the group $Q(\overline{\mathbb{C}})$ and its subsets/subgroups are merely tools in order to formulate and prove these results rather than central objects of investigation in their own right.

- Finally, Yagasaki investigates in [Yag99] the topological structure of $Q(S)$ in terms of infinite–dimensional manifolds with respect to the topology of compact convergence, where S is a connected Riemann surface[4].

Common for all of the previously mentioned research work is the fact that – as soon as topological questions are studied – the underlying topological structure is exclusively the topology induced by compact convergence (i.e. locally uniform convergence, or the compact–open topology, which agree in the special case of domains in $\mathbb{C}$ or $\mathbb{R}^n$ – see [Gon10, Corollary 1, p. 512] and also [RS02, pp. 84–85]). The topology induced by *uniform convergence* in connection with groups of quasiconformal automorphisms, however, seems as if it has not been considered so far in Geometric Function Theory. This thesis tries to fill this apparent gap to a small and humble part by studying quasiconformal automorphisms of bounded (simply and multiply connected) domains in $\mathbb{C}$ and the induced automorphism groups endowed with the supremum metric d_{sup}.

[3] It is clear and can easily be seen that $Q(\overline{\mathbb{C}}, E)$ forms a subgroup of $Q(\overline{\mathbb{C}})$, thus the usage of the term "group action" is justified.

[4] Naturally, in order to study the mentioned properties of $Q(S)$, one needs to define the notion of a quasiconformal mapping between Riemann surfaces. This is done in the usual "chart–wise" manner, see e.g. [BF14, p. 35] or [GL00, p. 18].

Outline and structure of this thesis

The aim and purpose of this thesis is to investigate topological and algebraic (i.e. group–theoretic) properties of the quasiconformal automorphism groups of (bounded) simply and multiply connected domains in $\mathbb{C}$, in which the topology is canonically induced by the supremum metric defined on the respective domains, and the group structure is impressed on quasiconformal mappings with common preimage domain and range by their fundamental mappings properties.
More precisely, questions for topological attributes such as compactness, path–connectedness, separability or group–theoretic properties such as subgroups and generating sets are focused. Besides this, by combining these two canonical structures on $Q(G)$ – topological space and group – the quasiconformal automorphisms groups are studied from the point of view of topological groups. The content of Table 2 on page 7 summarizes the central topological properties of $Q(G)$ for bounded, simply connected domains studied in this thesis and compares it to the corresponding situation with the conformal automorphism group $\Sigma(G)$. Furthermore, in order to visualize the geometric mapping behaviour of quasiconformal automorphisms of domains with sufficiently controllable boundary structure, several concrete examples of such objects are constructed and analyzed in detail.
In addition, on the basis of a recent publication, the thesis at hand introduces the principle idea of applying quasiconformal unit disk automorphisms to a highly relevant topic in modern mathematics and computer sciences, namely cryptographic algorithms and encryption schemes.

Chapter 1 introduces the most important terminology and results for quasiconformal mappings in $\mathbb{C}$ required in the remainder of the thesis. Apart from existence, boundary extension and convergence results for this distinguished class of homeomorphism, several known facts about $Q(G)$ and $\Sigma(G)$ are presented.
In Chapter 2, one of the most important metric properties of $Q(G)$ and some of its subspaces is studied, namely completeness. As will be demonstrated, $Q(G)$ is always incomplete. Consequently, the question for the corresponding completion arises, which is also investigated. Also, a new metric defined on quasiconformal automorphism groups, closely related to incompleteness and *Polish groups*, is studied.
The topological structure of $Q(G)$ for bounded, simply connected domains $G \subsetneq \mathbb{C}$ is the central object of investigation in Chapter 3. Many of the most important properties of topological spaces are studied for $Q(G)$, such as separability, local compactness and path–connectedness. Furthermore, the case of multiply connected domains in $\mathbb{C}$ is focused in terms of topological properties of the corresponding quasiconformal automorphism groups.
Chapter 4 treats miscellaneous further topics in connection with $Q(\mathbb{D})$, starting with harmonic quasiconformal unit disk automorphisms. Furthermore, a rather unexpected construction method related to classical *Cesàro summation* is studied, based on a particular class of quasiconformal mappings of $\mathbb{D}$. Finally, a potential future application of quasiconformal automorphisms in the context of cryptographic systems and encryption algorithms is presented.

The thesis closes with a collection of the most important open questions that arose throughout the development of this text.

Property	$Q(G)$	$\Sigma(G)$
Compactness	Never compact (Proposition 1.3.1(i))	Never compact (Proposition 1.2.1(i))
Completeness	Always incomplete and never completely metrizable (Theorem 2.3.3 & Corollary 3.3.7)	Always complete (Proposition 1.2.1(i))
Baire space	Never a Baire space (Theorem 3.3.4)	Always a Baire space (Proposition 1.2.1(i))
Separability	Separable if and only if $\mathcal{P}(G) = \mathcal{P}_1(G)$ (Theorems 3.1.2 & 3.1.4)	Separable if and only if $\mathcal{P}(G) = \mathcal{P}_1(G)$ (Proposition 1.2.1(iii))
Local compactness	Never locally compact (Theorem 3.3.6)	Local compactness depending on ∂G, e.g. locally compact if $\mathcal{P}(G) = \mathcal{P}_1(G)$ ([Lau94, Tabelle II.2.1, p. 37])
Path–connectedness	Path–connected if $\mathcal{P}(G) = \mathcal{P}_1(G)$ (Theorem 3.4.1) Converse statement unknown	Path–connected if and only if $\mathcal{P}(G) = \mathcal{P}_1(G)$ (Proposition 1.2.1(iv))
Discreteness	Never discrete (Corollary 3.2.6)	Discreteness depending on ∂G, e.g. discrete for comb domains of the first kind ([Gai84, Satz 9, p. 254])
Topological group	Topological group if $G \in \mathcal{JD}$ (Proposition 1.3.3(ii)) If $Q(G)$ topological group, then $\mathcal{P}(G) = \mathcal{P}_1(G)$ or $\Sigma(G)$ discrete (Lemma 1.3.4 & Remark 1.3.5)	Topological group if and only if $\mathcal{P}(G) = \mathcal{P}_1(G)$ or if $\Sigma(G)$ discrete ([Lau94, Satz I.1.1, p. 14])

Table 2: Similarities and differences between certain topological properties of $Q(G)$ and $\Sigma(G)$ for bounded, simply connected domains $G \subsetneqq \mathbb{C}$.

Chapter 1

Preliminaries

This first chapter introduces in a moderate way the basic definitions and results which are needed in the remainder of the thesis at hand.

The central term of this work is the notion of a quasiconformal mapping in $\mathbb{C}$, to be defined precisely in the first Section 1.1, together with the most important existence and representation results. From the extensive research work on quasiconformal mappings carried out in the past, several equivalent definitions are known for this class of mappings, whereas for this thesis, the analytic definition is used (see Definition 1.1.1). A cornerstone in quasiconformal mapping theory in the plane is the Measurable Riemann Mapping Theorem, to be presented in Proposition 1.1.2 together with further representation and regularity results. Many of the statements of this work intimately depend on the boundary of the domains under consideration and the possibility to extend quasiconformal mappings appropriately to these boundary curves, respectively. Therefore certain boundary extension properties are required in order to formulate and prove the mentioned statement. Finally, from the essential theory of quasiconformal mappings, several results on convergent sequences of these mappings and the corresponding limit mappings are provided. This convergence theory is presented due to the fact that the overall analytic situation of this thesis – automorphism groups of quasiconformal mappings in the topology of uniform convergence – is settled in the context of metric spaces.

As already mentioned in the introductory chapter, the investigation of $Q(G)$ and its properties was motivated by the initial work of Gaier on conformal automorphism groups, followed by numerous further research activities. Thus it may seem reasonable to relate certain aspects of $Q(G)$ with the corresponding situation in $\Sigma(G)$. Some of the relevant information about conformal automorphism groups and their properties is therefore summarized in Section 1.2.

In a similar spirit as mentioned previously, the contents of the current chapter's final Section 1.3 are organized as follows: Several seminal results on the properties of quasiconformal automorphism groups of domains in $\mathbb{C}$, obtained in the Master Thesis [Bie17], are presented. In particular, questions for $Q(G)$ being a topological group, properties of the conjugation mapping Φ (see (0.7)) and relations to the subgroup $\Sigma(G)$ are studied.

1.1 Quasiconformal mappings in $\mathbb{C}$

1.1.1 Existence theory, factorization and dependence on parameters

Since this thesis is concerned with quasiconformal mappings in $\mathbb{C}$, it is only natural to begin with the definition of this particular class of mappings. Quasiconformal mappings admit various ways for their precise definition, one of these given by (see e.g. [BF14, Definition 1.11, p. 24], [GL00, Definition 2, p. 5] and [Leh87, pp. 20–23])

Definition 1.1.1 (Quasiconformal mapping, analytic definition)**.**
*A homeomorphism $f = u+iv : G \longrightarrow G'$ between domains $G, G' \subseteq \mathbb{C}$ is called K–**quasiconformal** if*

*(i) f is **absolutely continuous on lines (ACL)** on G, i.e. for each closed rectangle $\{x+iy \mid a \le x \le b, c \le y \le d\} \subsetneq G$, the mapping $x \longmapsto u(x+iy)$ is absolutely continuous on $[a,b]$ for almost every $y \in [c,d]$, and the mapping $y \longmapsto u(x+iy)$ is absolutely continuous on $[c,d]$ for almost every $x \in [a,b]$; likewise for the imaginary part v of f;*

(ii) the Wirtinger derivatives of f satisfy $|f_{\overline{z}}| \le k|f_z|$ almost everywhere in G, where $k := \frac{K-1}{K+1}$.

*The mapping f is called **quasiconformal** if it is K–quasiconformal for some $K \in [1, +\infty)$. The smallest constant K such that (ii) holds is called the **maximal dilatation**[1] of f, denoted by $K(f)$.*

Definition 1.1.1(i) implies two important facts: On the one hand, it follows that a quasiconformal mapping $f : G \longrightarrow G'$ is differentiable almost everywhere in G. On the other hand, one concludes that the Jacobian determinant $J_f(z) = |f_z(z)|^2 - |f_{\overline{z}}(z)|^2$ of f is positive and $f_z(z) \neq 0$ for almost every $z \in G$. In consequence, f is orientation–preserving[2]. Moreover, the expression

$$\mu_f := \frac{f_{\overline{z}}}{f_z} \tag{1.1}$$

can be considered almost everywhere in G, called the **complex dilatation**[3] of f. This expression defines a measurable function on G with

$$\left|\mu_f(z)\right| = \left|\frac{f_{\overline{z}}(z)}{f_z(z)}\right| \le k = \frac{K-1}{K+1} < 1$$

for almost every $z \in G$ by Definition 1.1.1(ii). Passing to the essential supremum given by

$$\|\mu_f\|_{L^\infty(G)} := \operatorname*{ess\,sup}_{z\in G} \left|\mu_f(z)\right| = \inf_{\substack{N\subseteq G \\ \lambda(N)=0}} \sup_{z\in G\setminus N} \left|\mu_f(z)\right| \tag{1.2}$$

(λ denoting the Lebesgue measure in $\mathbb{C}$) in the previous inequality yields an element of the open unit ball in the complex Banach space $L^\infty(G)$ of all essentially bounded measurable functions[4]

[1]This admittedly slight abuse of language is due to the classical *Geometric Definition* of quasiconformal mappings, see for example [Leh87, p. 12] and also [SS11, Definition 2.3.1, p. 61], where the same naming convention and notation is used.

[2]This is a small but important detail in the definition of quasiconformal mappings: The analytic definition of quasiconformality as stated in Definition 1.1.1 *implies* that the homeomorphism is orientation–preserving. The geometric definition, however, being based on the conformal modulus (see e.g. [LV73, Definition, p. 16]), needs to *require* the mapping's orientation–preservation in order to yield an equivalent notion; see also [BF14, p. 32]. A simple, but nevertheless very illuminating example in this direction is given by the (orientation–reversing) mapping $z \longmapsto \overline{z}$, which is clearly an ACL homeomorphism that fails to be quasiconformal by Definition 1.1.1(ii).

[3]More precisely, the quotient μ_f defined by (1.1) is called the *first* complex dilatation of f. Consequently, there is a closely related expression, called the *second complex dilatation*, given by $\nu_f := \overline{f_{\overline{z}}}/f_z$, see [Dur04, p. 5].

[4]As usual, the members of $L^\infty(G)$ will be considered as functions rather than equivalence classes of functions with respect to the equivalence relation "$f \sim g :\Longleftrightarrow f - g \equiv 0$ almost everywhere", see [RF10, pp. 394–395].

on G, i.e.

$$\mu_f \in \mathbb{B}_{L^\infty}(G) := \left\{ g \in L^\infty(G) \;\middle|\; \|g\|_{L^\infty(G)} < 1 \right\} \subsetneq L^\infty(G) \tag{1.3}$$

The elements of $\mathbb{B}_{L^\infty}(G)$ are also called Beltrami coefficients on G. Equation (1.1) may also be considered from a different point of view, namely written in the form

$$f_{\overline{z}} = \mu \cdot f_z \tag{1.4}$$

which can be considered as a partial differential equation with respect to z and $\overline{z}$ when given a prescribed function $\mu \in \mathbb{B}_{L^\infty}(G)$. This differential equation (1.4) is known as the *Beltrami equation*, named for the Italian mathematician Eugenio Beltrami (1835 – 1900). Every quasiconformal mapping f solves a Beltrami equation with $\mu = \mu_f$ as Beltrami coefficient. One central aspect in the theory of quasiconformal mappings is given by a certain counterpart of this statement, made precise in (see e.g. [LV73, Existence/Mapping Theorem, p. 194])

Proposition 1.1.2 (Measurable Riemann Mapping Theorem).

(I) Existence Theorem: Let $G \subseteq \mathbb{C}$ be a domain and $\mu \in \mathbb{B}_{L^\infty}(G)$ be a Beltrami coefficient on G. Then there exists a quasiconformal mapping on G whose complex dilatation coincides with μ almost everywhere in G.

(II) Mapping Theorem: Let $G, G' \subseteq \mathbb{C}$ be conformally equivalent simply connected domains and $\mu \in \mathbb{B}_{L^\infty}(G)$ be a Beltrami coefficient on G. Then there exists a quasiconformal mapping $f : G \longrightarrow G'$ whose complex dilatation coincides with μ almost everywhere in G. Furthermore, f is uniquely determined by μ up to post–composition with a conformal automorphism of G'.

Definition 1.1.3 (Normalized quasiconformal mapping, [BF14, pp. 42–43]).
*Let $G \subsetneq \mathbb{C}$ be a simply connected domain with $z_1, z_2 \in G$ distinct. A quasiconformal mapping $f : G \longrightarrow \mathbb{D}$ is called **normalized** if $f(z_1) = 0$ and $f(z_2) \in \mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$.*

Combining with the classical Riemann Mapping Theorem, Proposition 1.1.2(II) implies that for every Beltrami coefficient $\mu \in \mathbb{B}_{L^\infty}(G)$, there exists a unique normalized quasiconformal mapping $f : G \longrightarrow \mathbb{D}$ with $\mu_f = \mu$ almost everywhere in G. It is therefore well–defined and reasonable to speak of *the* normalized quasiconformal mapping for a given Beltrami coefficient, which will be done henceforth. Among other results, the following factorization statement for quasiconformal mappings may be deduced from Proposition 1.1.2 (see [AIM08, Theorem 5.6.2, p. 185] and [Leh87, Section 4.7, pp. 29–30]):

Proposition 1.1.4 (Factorization with small dilatation).
Let $f : G \longrightarrow G'$ be a K–quasiconformal mapping and let $\epsilon > 0$. Then f can be written as $f = f_1 \circ f_2 \circ \cdots \circ f_n$ where each mapping $f_j : G_j \longrightarrow G'_j$ is an $(1+\epsilon)$–quasiconformal mapping between intermediate domains $G_j, G'_j \subseteq \mathbb{C}$ for $j = 1, \ldots, n$.

Moreover, one is often interested not only in the sheer existence, but also in the dependence of solutions of the Beltrami equation if the corresponding Beltrami coefficients depend on a certain parameter (see [BF14, Theorem 1.30(b), p. 43]):

Proposition 1.1.5 ((Pointwise) Continuous dependence on parameters).
Let Λ be an open subset of $\mathbb{R}$, $G \in \mathcal{JD}$ and $(\mu_t)_{t\in\Lambda}$ be a family in $\mathbb{B}_{L^\infty}(G)$. Suppose $t \longmapsto \mu_t(z)$ is continuous for every fixed $z \in G$ (whenever defined). Moreover, assume there exists $k < 1$ such that $\|\mu_t\|_{L^\infty(G)} \leq k$ for all $t \in \Lambda$, and denote by $f_t : G \longrightarrow \mathbb{D}$ the normalized quasiconformal mapping with $\mu_{f_t} = \mu_t$ almost everywhere in G. Then $t \longmapsto f_t(z)$ is continuous for every fixed $z \in G$.

In fact, Proposition 1.1.5 can be sharpened to differentiable, real–analytic and holomorphic dependence, see e.g. [AIM08, pp. 185–189] and [BF14, pp. 42–43].

1.1.2 Boundary extension of quasiconformal mappings

A well–known extension result for quasiconformal mappings is ([AIM08, Corollary 5.9.2, p. 193])

Proposition 1.1.6 (Homeomorphic boundary extension).
Let $G, G' \in \mathcal{JD}$ be Jordan domains and $f : G \longrightarrow G'$ a quasiconformal mapping. Then f extends to a homeomorphism $\widehat{f} : \overline{G} \longrightarrow \overline{G'}$.

Proposition 1.1.6 states in particular that quasiconformal automorphisms of a Jordan domain extend to homeomorphisms of the corresponding closure of the domain. Since many topics in this thesis will be concerned with the possible boundary extension of quasiconformal mappings, the following definition is reasonable and useful:

Definition 1.1.7.
For $G \in \mathcal{JD}$, let

$$Q(\overline{G}) := \left\{ f \in C(\overline{G}) \;\middle|\; f\big|_G \in Q(G) \right\} \tag{1.5}$$

denote the set of all homeomorphic extensions of the quasiconformal automorphisms of G to $\overline{G}$.

Boundary extension of quasiconformal mappings was studied to a huge extent, in particular for the case of higher–dimensional mappings, i.e. $f : D \longrightarrow D'$ with $D, D' \subseteq \mathbb{R}^n$ for $n \geq 3$, see [GMP17, Section 6.5, pp. 251–271] and [Väi71, Section 17, pp. 51–63]. Especially in the higher–dimensional setting, the circumstances for extending quasiconformal mappings continuously or even homeomorphically to the boundary are unequally more delicate than in the planar situation (comparable to many major differences between complex analysis in one and several variables). In the classical conformal case, the boundary extension problem was comprehensively solved by Carathéodory using prime ends. A nearby idea was to extend the planar conformal prime end theory to the higher–dimensional quasiconformal case, which has been carried out by Näkki in [Näk72] and [Näk79] (see also [Väi71, Remark 17.24.5, p. 63]). The main results of Näkki's work for quasiconformal mappings in $\mathbb{C}$ are summarized in

Theorem 1.1.8.
Let $G \subsetneqq \mathbb{C}$ be a bounded, simply connected domain and $f : \mathbb{D} \longrightarrow G$ a quasiconformal mapping. Then the following statements are mutually equivalent:

(i) f can be extended to a continuous mapping $f : \overline{\mathbb{D}} \longrightarrow \overline{G}$.

(ii) G has only prime ends of the first kind, i.e. $\mathcal{P}(G) = \mathcal{P}_1(G)$.

*(iii) G is **finitely connected** on the boundary, i.e. every boundary point has arbitrarily small neighborhoods U such that $U \cap G$ consists of a finite number of connected components.*

Proof. By [Näk72, Lemma 2.5, p. 5] and Proposition 1.1.6, it is sufficient to consider $\mathbb{D}$ as the preimage domain. The equivalence of (i) and (iii) is stated in [Näk72, Theorem 3.1, p. 6], and the equivalence of (i) and (ii) is demonstrated in [Näk79, Section 8.1, p. 30]. □

1.1.3 Convergent sequences of quasiconformal mappings

Sequences of quasiconformal mappings and their limit mappings will play a crucial role in this thesis. The first statement in this direction is similar to a well–known result on convergent sequences of conformal mappings (see e.g. [LV73, Theorem 5.5, p. 78]):

Proposition 1.1.9 (Hurwitz–type Theorem for quasiconformal mappings).
Let $G \subseteq \mathbb{C}$ be a domain possessing at least two boundary points (in $\overline{\mathbb{C}}$) and $(f_n)_{n \in \mathbb{N}}$ be a sequence of K–quasiconformal mappings of G onto a fixed domain $G' \subseteq \mathbb{C}$, i.e. for every $n \in \mathbb{N}$ it is $f_n : G \longrightarrow G'$ and $K(f_n) \leq K < +\infty$. If $(f_n)_n$ converges in G to a limit mapping f, then f is either a K–quasiconformal mapping of G onto G' or a constant mapping of G onto a boundary point of G'.

Naturally, different types of convergence occur with sequences of continuous mappings, e.g. pointwise or (locally) uniform convergence. Of course, uniform convergence implies pointwise convergence, but the converse statement is false in general. However, there are certain situations in which these notions are equivalent. These circumstances are also present in the context of quasiconformal mappings, as shown in (see [NP73, Corollary 4.4, p. 432])

Proposition 1.1.10 (Näkki–Palka)**.**
Let $G' \subseteq \mathbb{C}$ be a domain with finitely many boundary components which is finitely connected on the boundary. Furthermore, let $(f_n)_{n\in\mathbb{N}}$ be a sequence of K–quasiconformal mappings of a domain G onto G' converging pointwise in G to a homeomorphism f. Then the sequence $(f_n)_n$ converges uniformly on G to f.

The question for the relationship between convergence of a sequence of quasiconformal mappings $(f_n)_n$ and the corresponding sequence of complex dilatations $(\mu_{f_n})_n$ – initially without strict attention to the particular mode of convergence – is a delicate one, see [Leh87, Section 4.6]. In general, convergence of the mappings f_n does not imply the convergence of the sequence μ_{f_n}, as shown in Example[5] 1.1.11 given below:

Example 1.1.11.
This example is based on a classical construction in plain quasiconformal mapping theory given in [LV73]. Let

$$R = \left\{ z = x + iy \in \mathbb{C} \,\middle|\, 0 < x, y < 1 \right\}$$

be the open unit square in $\mathbb{C}$ and let $(\epsilon_m)_{m\in\mathbb{N}}$ be the sequence of real numbers given by $\epsilon_m = \frac{1}{m+1}$. Then by [LV73, p. 186], for each $m \in \mathbb{N}$, there exists a sequence[6] $(f_{m,n})_{n\in\mathbb{N}}$ of quasiconformal automorphisms of R, i.e. $f_{m,n} \in Q(R)$ for all $n \in \mathbb{N}$, with the properties $d_{\sup}(f_{m,n}, \mathrm{id}_R) \overset{n\to\infty}{\longrightarrow} 0$ and

$$\left|\mu_{f_{m,n}}(z)\right| = 1 - \epsilon_m = \frac{m}{m+1}$$

for almost every $z \in R$. In other words, the sequence $(f_{m,n})_n$ converges uniformly on R to the identity mapping, whereas the absolute values of the corresponding complex dilatations (whenever defined) remain constant with modulus $\frac{m}{m+1}$. Consequently, by using a diagonal argument, there exists a subsequence $(n_m)_{m\in\mathbb{N}}$ such that the resulting diagonal sequence $(f_{m,n_m})_m$ converges uniformly to id_R as well by construction. However, the sequence of the corresponding complex dilatations $\mu_{f_{m,n_m}}$ converges almost everywhere in R to 1 in absolute value, thus the maximal dilatation of the f_{m,n_m} in fact diverges. Accordingly, despite the fact that the diagonal sequence $(f_{m,n_m})_m$ converges in $Q(R)$ to the identity which is even a conformal mapping, thus "very regular" from the point of view of $Q(R)$, the complex dilatation diagonal sequence is not at all convergent with regard to quasiconformal mapping theory.

1.2 Results on conformal automorphism groups

This section collects some results concerning conformal automorphism groups of bounded, simply connected domains in $\mathbb{C}$ as obtained by Gaier, Schmieder and Volynec. Several important metric and topological properties of $\Sigma(G)$ are collected in

[5] The author would like to thank Ikkei Hotta and Ken–ichi Sakan for pointing out the idea of this example.

[6] The mappings $f_{m,n}$ are constructed using a composition of a conformal mapping and its inverse (provided by the classical Riemann Mapping Theorem) together with a certain quasiconformal unit disk automorphism, a so–called *monomial–like radial stretching*, to be introduced in (2.3) on page 22.

Proposition 1.2.1 (Metric and topological properties of $\Sigma(G)$).
(i) The space $\Sigma(G)$ is always complete, but never compact. ([Gai84, Satz 1, p. 229]). In particular, $\Sigma(G)$ is always a Baire space.
(ii) The space $\Sigma(G)$ is locally compact if $\mathcal{P}(G) = \mathcal{P}_1(G)$. ([Gai84, Zusatz, p. 230])
(iii) The space $\Sigma(G)$ is separable if and only if $\mathcal{P}(G) = \mathcal{P}_1(G)$. ([Vol92, Theorem 3, p. 201])
(iv) The space $\Sigma(G)$ is path–connected if and only if $\mathcal{P}(G) = \mathcal{P}_1(G)$. ([Sch86, Corollary, p. 199])

In view of the structure of topological groups, the following characterization was shown in [Gai84, Satz 5, p. 235] and [Vol92, Theorem 1, p. 196] (see also [Lau94, Satz I.1.1, p. 14]):

Proposition 1.2.2 (Topological group property).
$\Sigma(G)$ is a topological group $\iff \mathcal{P}(G) = \mathcal{P}_1(G)$ or $\Sigma(G)$ is discrete.

There has been considerably more research on $\Sigma(G)$: Lauf [Lau99] and Schmieder [Sch92] constructed explicit examples of domains having non–locally compact conformal automorphism group. In [Lau95], Lauf succeeded in characterizing all locally compact spaces $\Sigma(G)$. Related investigations have been carried out by Rodin [Rod84], Pommerenke–Rodin [PR85] and Rogers [Rog93] in terms of intrinsic rotations of simply connected domains in $\mathbb{C}$, thereby partially studying the famous *Siegel Problem* (see [PR85, p. 224] and the references therein as well as [Lau94, p. 70]) in Complex Dynamics:

Is a Siegel disk of a rational function (of degree at least 2) always a Jordan domain?

Beyond that, in more recent years there has been research on conformal automorphism groups of domains in $\mathbb{C}^n$ for $n \geq 1$. The focus in these studies mainly was on $\Sigma(G)$ endowed with the topology of compact convergence. An overview of the planar case $n = 1$ can be found in [Kra06, Chapter 12], the case $n \geq 2$ is treated in [KK05], [IK99] and [SV18] (see also the references in the cited literature).

1.3 Results on quasiconformal automorphism groups

In [Bie17], the investigation of $Q(G)$ was initiated. Some of the obtained results are stated in the following. Naturally, several properties of $Q(G)$ and $\Sigma(G)$, as presented in the previous Section 1.2, are intimately related. An example of this situation is shown in

Proposition 1.3.1 (Metric and topological properties).
(i) The space $Q(G)$ is never compact. ([Bie17, Theorem 2.48, p. 77])
(ii) The subspace $\Sigma(G)$ is always closed in $Q(G)$. ([Bie17, Theorem 2.27, p. 64])

As for the algebraic structure of $Q(G)$, the following results were obtained

Proposition 1.3.2 (Algebraic properties of $Q(G)$).
(i) The conjugation mapping $\Phi : Q(G) \longrightarrow Q(G')$ is a group isomorphism. ([Bie17, Theorem 2.8, p. 40])
(ii) $\Sigma(G)$ is always a proper subgroup, but never a normal subgroup of $Q(G)$. ([Bie17, Theorem 2.11/2.14, p. 43/48])

The question for $Q(G)$ being a topological group has a characterizing answer, which at the same time is the corresponding result to Gaier's investigation concerning $\Sigma(G)$ (see [Gai84, Satz 4, p. 235]):

Proposition 1.3.3 (Topological Groups)**.**

(i) $Q(G)$ forms a topological group if and only if the following condition holds ([Bie17, Theorem 3.4, p. 88]):

$$\forall f \in Q(G)\, \forall (\varphi_n)_{n\in\mathbb{N}} \subseteq Q(G) : \left(\varphi_n \overset{n\to\infty}{\longrightarrow} \mathrm{id}_G \Longrightarrow f \circ \varphi_n \overset{n\to\infty}{\longrightarrow} f\right) \tag{1.6}$$

(ii) $Q(G)$ forms a topological group if $G \in \mathcal{JD}$ or if id_G is isolated in $Q(G)$. ([Bie17, Theorem 3.5/Corollary 3.7, p. 89/91])

Regarding the question under which circumstances $Q(G)$ actually forms a topological group, the characterizing answer of Gaier and Volynec for the corresponding situation in the conformal special case, as stated in Proposition 1.2.2, yields the following necessary criterion:

Lemma 1.3.4.
If $Q(G)$ is a topological group, then $\mathcal{P}(G) = \mathcal{P}_1(G)$ or $\Sigma(G)$ is a discrete space.

Proof. Suppose $Q(G)$ is a topological group. Then the subgroup $\Sigma(G)$ is also a topological group in the subspace topology ([Sin19, p. 276]). Now Proposition 1.2.2 yields the claim. □

Remark 1.3.5.
The contraposition of Lemma 1.3.4 yields:

If G has not only prime ends of the first kind and $\Sigma(G)$ is not discrete, then $Q(G)$ is no topological group.

By utilizing knowledge about the topology of $\Sigma(G)$ for specialized domains G, one can for example draw the following conclusion (see [Gai84, Satz 10, p. 256] and [Lau94, Tabelle II.2.1, p. 37]):

$Q(G)$ is no topological group if G is a comb domain of the second kind.

A particularly useful property of $Q(G)$ is stated in the following (see [Bie17, Lemma 2.5, p. 37])

Proposition 1.3.6 ($d_{\sup}$–isometry of right multiplication)**.**
For each $g \in Q(G)$, the right multiplication $R_g : Q(G) \longrightarrow Q(G)$, $h \longmapsto R_g(h) := h \circ g$ is a bijective isometry, i.e.

$$d_{\sup}(R_g(h_1), R_g(h_2)) = d_{\sup}(h_1, h_2) \tag{1.7}$$

for every $h_1, h_2 \in Q(G)$. In particular, $d_{\sup}(f, h) = d_{\sup}(f \circ h^{-1}, \mathrm{id}_G)$ and $d_{\sup}(h, \mathrm{id}_G) = d_{\sup}(h^{-1}, \mathrm{id}_G)$ for $f, h \in Q(G)$.

The left multiplication, however, is in fact never an isometric mapping, even in the case $\Sigma(\mathbb{D})$ as shown by Gaier in [Gai84, p. 234], not to mention in $Q(G)$. Furthermore, Proposition 1.3.6 is valid for every space $Q(G)$, regardless of the particular boundary structure of G.

Chapter 2

Incompleteness and completion of $Q(G)$

A central property of metric spaces is completeness. This chapter is intended to study several aspects of $Q(G)$ concerning this metric attribute. Unless the contrary is explicitly stated, *G will always refer to a bounded and simply connected domain* in $\mathbb{C}$ throughout the entire chapter.

As in the conformal special case and as a direct consequence of the Measurable Riemann Mapping Theorem 1.1.2, the central reference object of investigation in this situation is given by $Q(\mathbb{D})$. Therefore, it is of particular interest whether the conjugation mapping $\Phi : Q(\mathbb{D}) \longrightarrow Q(G)$ and its inverse transformation Φ^{-1} transport topological properties and to which extent Φ preserves convergent sequences and Cauchy sequences. Section 2.1 is devoted to studying this question.
In Section 2.2, the subspaces $Q_K(G)$ are investigated. Among others, it is shown in Theorem 2.2.1 that these subspaces are complete in the supremum metric, but – just as in the special case $Q_1(G) = \Sigma(G)$ studied by Gaier – lack the compactness property. This allows for a conclusion on $Q(G)$ regarding its topological structure in the Banach space $C_b(G)$, in which it is canonically embedded (Corollary 2.2.2).
The major result presented in Section 2.3 is the incompleteness of the metric space $(Q(G), d_{\text{sup}})$. In order to prove this statement, formulated in Theorem 2.3.3, an extraordinary important and useful class of quasiconformal automorphisms of the unit disk is introduced, the so–called *(general) radial stretchings* of $\mathbb{D}$. This class of quasiconformal mappings will be used throughout this thesis for deriving numerous further results and to visualize the geometric mapping behaviour of concrete quasiconformal automorphisms, respectively. As for the incompleteness of the space $Q(G)$, a sequence of radial stretchings is constructed that converges uniformly on G to a non–injective limit mapping, thereby implying that $Q(G)$ cannot be complete for it is not closed in the ambient Banach space $C_b(G)$.
As a direct consequence of the incompleteness of $Q(G)$, the question for the completion of this metric space arises. This topic is the central concern of Section 2.4. As an important step towards the solution of this problem, the class of *monotone mappings* between metric spaces is introduced in Definition 2.4.4. From convergence results for homeomorphisms, it will become apparent that monotone mappings form an "upper bound" for all possible limit mappings of convergent sequences of quasiconformal automorphisms. Approximation theorems for monotone mappings in terms of homeomorphisms and statements on the algebraic structure of the set $\mathcal{M}(\overline{G})$ of all monotone mappings on the closure of the domain G then lead to the formulation of the main result in Theorem 2.4.18.
The incompleteness of the metric spaces $Q(G)$ and $Q(\overline{G})$ is focused once again in Section 2.5, where the question is asked whether $Q(G)$ is *completely metrizable*, i.e. if there exists a metric that induces the uniform topology on $Q(G)$ as well and at the same time turns it into a complete

metric space. Furthermore, due to the fact that $Q(G)$ forms a topological group for $G \in \mathcal{JD}$ by Proposition 1.3.3(ii), one may consider this question from the point of view of *Polish groups.* Therefore, a new metric is introduced that is closely related to d_{sup}, the *symmetric supremum*, and the resulting properties of $Q(G)$ and $Q(\overline{G})$ are studied.

2.1 Continuous mappings between quasiconformal automorphism groups

By (0.7), for every domain $G \subsetneq \mathbb{C}$, the conjugation mapping

$$\Phi : Q(\mathbb{D}) \longrightarrow Q(G),\ h \longmapsto \Phi(h) = F^{-1} \circ h \circ F$$

may be considered with conformal $F : G \longrightarrow \mathbb{D}$. Being a bijective mapping between metric spaces, the question for continuity of Φ and Φ^{-1} arises. The restrictions of these mappings to the corresponding subgroups of conformal automorphisms were already studied by Gaier in [Gai84, Satz 2, p. 231]:

Proposition 2.1.1 (Gaier)**.**
The mapping $\Phi^{-1}\big|_{\Sigma(G)}$ is always continuous. The mapping $\Phi\big|_{\Sigma(\mathbb{D})}$ is continuous if and only if G has only prime ends of the first kind, i.e. $\mathcal{P}(G) = \mathcal{P}_1(G)$.

Gaier's proof of Proposition 2.1.1 is essentially based on two foundations:

(1) Algebraic transformations in the groups $\Sigma(\mathbb{D})$ and $\Sigma(G)$, or rather using the conjugation mappings Φ and Φ^{-1}: For $\sigma \in \Sigma(\mathbb{D})$ and $h \in \Sigma(G)$, it is $h = F^{-1} \circ \sigma \circ F$, and accordingly $\sigma = F \circ h \circ F^{-1}$;
(2) Convergence properties of sequences of conformal mappings and their boundary behaviour.

In view of (1), the algebraic transformations done by Gaier are immediately transferable to $Q(G)$, since the only requirement used is the group structure. Concerning (2), since quasiconformal mappings in $\mathbb{C}$ possess very similar properties concerning convergence and boundary extension as their conformal special cases, Gaier's arguments may be transferred as follows (see also [BL23, Theorem 1]):

Theorem 2.1.2.
The mapping $\Phi^{-1} : Q(G) \longrightarrow Q(\mathbb{D})$ is continuous if $G \in \mathcal{JD}$. The mapping $\Phi : Q(\mathbb{D}) \longrightarrow Q(G)$ is continuous if and only if $\mathcal{P}(G) = \mathcal{P}_1(G)$.

Proof. As for the continuity of Φ^{-1}, let $(h_n)_n$ be a sequence in $Q(G)$ converging to $h \in Q(G)$, and denote by $g_n = \Phi^{-1}(h_n)$ and $g = \Phi^{-1}(h)$ the respective images in $Q(\mathbb{D})$. For arbitrary $\epsilon > 0$ and all $w \in \mathbb{D}$, it is

$$\begin{aligned} |g_n(w) - g(w)| &= \left|(F \circ h_n \circ F^{-1})(w) - (F \circ h \circ F^{-1})(w)\right| = |(F \circ h_n)(z) - (F \circ h)(z)| \\ &\le \omega_F\left(|h_n(z) - h(z)|\right) \le \omega_F\left(d_{\text{sup}}(h_n, h)\right) \end{aligned} \tag{2.1}$$

where ω_F denotes the modulus of continuity of F and $z = F^{-1}(w) \in G$. Since G is a Jordan domain, F is uniformly continuous on G by Proposition 1.1.6. Now this uniform continuity implies $\omega_F(d_{\text{sup}}(h_n, h)) < \epsilon$ for sufficiently large n, since $d_{\text{sup}}(h_n, h)$ becomes arbitrarily small for these indices. Switching to the supremum over all $w \in \mathbb{D}$ in the above inequality chain yields the continuity of Φ^{-1}.
The second claim can be seen as follows: If Φ is a continuous mapping on $Q(\mathbb{D})$, also its restriction to the subspace $\Sigma(\mathbb{D})$ is continuous. By Proposition 2.1.1, this is equivalent to $\mathcal{P}(G) = \mathcal{P}_1(G)$. On the contrary, assume $\mathcal{P}(G) = \mathcal{P}_1(G)$, then F^{-1} extends continuously to $\overline{\mathbb{D}}$ and is therefore uniformly continuous on $\mathbb{D}$ by Theorem 1.1.8. Thus the analogous reasoning as in (2.1) applies to F^{-1}, showing the continuity of Φ. □

Remark 2.1.3.

(i) *The continuity of the inverse conjugation mapping* Φ^{-1} *is shown in Theorem 2.1.2 for Jordan domains. However, when restricting* Φ^{-1} *to* $Q_K(G)$ *for a fixed* $K \in [1, \infty)$, *one arrives in fact at a continuous mapping* $\Phi^{-1} : Q_K(G) \longrightarrow Q_K(\mathbb{D})$ *for arbitrary domains:*

Let $(g_n)_n$ *be a sequence in* $Q_K(G)$ *converging uniformly on* G *to* $g \in Q_K(G)$ *and* $w \in \mathbb{D}$. *Then*

$$\left|(F \circ g_n \circ F^{-1})(w) - (F \circ g \circ F^{-1})(w)\right| = |(F \circ g_n)(z) - (F \circ g)(z)|$$

with $z = F^{-1}(w) \in G$. *Thus, by construction, it is* $(F \circ g_n)_n$ *a sequence of* K*–quasiconformal mappings* $F \circ g_n : G \longrightarrow \mathbb{D}$ *with image domain* $\mathbb{D}$ *being a Jordan domain; in particular,* $\mathbb{D}$ *is finitely connected on the boundary (see Theorem 1.1.8). This sequence converges pointwise in* G *to the* K*–quasiconformal mapping* $F \circ g : G \longrightarrow \mathbb{D}$, *since* $g_n \longrightarrow g$ *uniformly, hence*

$$|(F \circ g_n)(z) - (F \circ g)(z)| \overset{n\to\infty}{\longrightarrow} 0$$

due to the continuity of F. *Therefore, all assumptions are fulfilled in order to apply the Näkki–Palka Theorem 1.1.10, yielding that in fact* $(F \circ g_n)_n$ *converges uniformly to* $F \circ g$ *on* G. *This concludes in*

$$d_{\sup}(\Phi^{-1}(g_n), \Phi^{-1}(g)) = \sup_{z\in G} |(F \circ g_n)(z) - (F \circ g)(z)| \overset{n\to\infty}{\longrightarrow} 0$$

(ii) *By setting* $K = 1$, *part (i) of this remark yields a result of Gaier shown in [Gai84, Satz 2a, p. 231] as a special case due to* $Q_1(G) = \Sigma(G)$. *Nevertheless, it is unknown whether the (unrestricted) inverse conjugation mapping* $\Phi^{-1} : Q(G) \longrightarrow Q(\mathbb{D})$ *is in fact always continuous, regardless of the boundary of the domain* G.

In the development of mathematical analysis, different notions of continuity were introduced, among them uniform continuity, which has the advantage of preserving Cauchy sequences. As for the mapping Φ in connection with uniform continuity, one has the additional and pleasant

Theorem 2.1.4.
If $\mathcal{P}(G) = \mathcal{P}_1(G)$, *the mapping* $\Phi : Q(\mathbb{D}) \longrightarrow Q(G)$ *is uniformly continuous.*

Proof. The conformal mapping $F^{-1} : \mathbb{D} \longrightarrow G$ is uniformly continuous on $\mathbb{D}$, thus its modulus of continuity $\omega_{F^{-1}} : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ is a continuous, non–decreasing mapping with $\omega_{F^{-1}}(0) = 0$ (see for example [Bie17, Lemma 2.43, p. 74]). Let $\epsilon > 0$, then there exists $\delta > 0$ such that $\omega_{F^{-1}}(t) < \epsilon$ if $0 \le t < \delta$. For arbitrary $g, h \in Q(\mathbb{D})$ with $d_{\sup}(g, h) < \delta$ and $z \in G$, it is

$$\begin{aligned} |\Phi(g)(z) - \Phi(h)(z)| &= \left|(F^{-1} \circ g \circ F)(z) - (F^{-1} \circ h \circ F)(z)\right| = \left|(F^{-1} \circ g)(w) - (F^{-1} \circ h)(w)\right| \\ &\le \omega_{F^{-1}}\left(|g(w) - h(w)|\right) \le \omega_{F^{-1}}\left(d_{\sup}(g, h)\right) < \epsilon \end{aligned}$$

Switching to the supremum over all $z \in G$ yields that Φ is uniformly continuous on $Q(\mathbb{D})$. □

Remark 2.1.5.

(i) *The converse statement of Theorem 2.1.4 is also valid, more precisely:*

If the mapping $\Phi : Q(\mathbb{D}) \longrightarrow Q(G)$ *is uniformly continuous, then* $\mathcal{P}(G) = \mathcal{P}_1(G)$.

This follows from Theorem 2.1.2 and the fact that uniform continuity implies (ordinary) continuity.

(ii) *Even though the occurrence of uniform continuity of* $\Phi : Q(\mathbb{D}) \longrightarrow Q(G)$ *is clarified to the full extent by Theorem 2.1.4 and part (i) of this remark, the question for the uniform continuity of the inverse conjugation mapping* $\Phi^{-1} : Q(G) \longrightarrow Q(\mathbb{D})$ *must be left unanswered.*

2.2 Completeness and generating set property of $Q_K(G)$

By (0.5) and (0.6), the quasiconformal automorphism groups can be written as

$$Q(G) = \bigcup_{K\geq 1} Q_K(G)$$

with the subspaces

$$Q_K(G) = \left\{ f \in Q(G) \,\middle|\, K(f) \leq K \right\}$$

The properties of these sets $Q_K(G)$ are therefore of interest for the study of quasiconformal automorphism groups. In this context, classical results for quasiconformal mappings yield

Theorem 2.2.1.
For each $K \geq 1$, the subspace $Q_K(G)$ is always complete, but never compact. In particular, $Q_K(G)$ is always closed in $Q(G)$.

Proof. In view of completeness, it is sufficient to prove that $Q_K(G)$ is closed in $C_b(G)$. To this end, let $(h_n)_{n\in\mathbb{N}}$ be a sequence in $Q_K(G)$ uniformly converging to $h \in C_b(G)$. Since $K(h_n) \leq K$ for all $n \in \mathbb{N}$, the Hurwitz–type Theorem 1.1.9 implies that h is either a K–quasiconformal automorphism of G or a constant mapping onto a boundary point $z_0 \in \partial G$. The latter case would lead to

$$|h_n(z) - h(z)| = |h_n(z) - z_0| < \epsilon$$

for every $z \in G$ and sufficiently large n, which is impossible since h_n is bijective for all $n \in \mathbb{N}$ (see also [Gai84, Proof of Satz 1b, p. 230]). Hence it is $h \in Q_K(G)$, and $Q_K(G)$ is complete.
As for the non–compactness of $Q_K(G)$, it suffices to note that $\Sigma(G) \subseteq Q_K(G)$ by definition, thus Gaier's non–compactness result for $\Sigma(G)$ (Proposition 1.2.1(i)) yields the claim. ☐

Due to $Q_1(G) = \Sigma(G)$, the first part of Theorem 2.2.1 is the direct generalization of Gaier's result for $\Sigma(G)$ concerning completeness and compactness, see Proposition 1.2.1(i). The second part concerning the closedness of $Q_K(G)$ in $Q(G)$ could as well be deduced directly from the Hurwitz–type Theorem 1.1.9 for sequences of K–quasiconformal mappings, but also follows from the completeness of $Q_K(G)$ and the fact that complete metric subspaces of metric spaces are closed (in the ambient space). Furthermore, the previous result allows for the deduction of the following

Corollary 2.2.2.
*The subset $Q(G)$ is an F_σ–**set** in $C_b(G)$, i.e. the countable union of closed subsets.*

Proof. Clearly, $Q(G)$ is the countable union of the sets $Q_N(G)$ with $N \in \mathbb{N}$. The proof of Theorem 2.2.1 shows that each set $Q_N(G)$ is closed in $C_b(G)$. ☐

Since, by construction, it is $Q_K(G) \subseteq Q_{K'}(G)$ for $K \leq K'$, Corollary 2.2.2 implies that $Q(G)$ is actually even an F_σ–set of increasing closed subsets of $C_b(G)$.

In order to investigate an interesting group–theoretic property of $Q_K(G)$, recall that in a group $(H, *)$, a **generating set** of H is a subset $E \subseteq H$ such that every $h \in H$ can be written as a finite product of elements of E and $E^{-1} := \{\varepsilon^{-1} \,|\, \varepsilon \in E\}$, i.e.

$$h = \varepsilon_1 * \varepsilon_2 * \cdots * \varepsilon_n$$

with $n \in \mathbb{N}_0$ and $\varepsilon_j \in E \cup E^{-1}$ for all $j = 1, \ldots, n$. A generating set E of H is called **symmetric** if $\varepsilon \in E \iff \varepsilon^{-1} \in E$. Now for $K \in (1, +\infty)$, the factorization property of quasiconformal mappings (Proposition 1.1.4) states that every $f \in Q(G)$ can be written as a finite composition $f = f_1 \circ \cdots \circ f_n$

with "intermediate" mappings $f_j : G_j \longrightarrow G'_j$ and $K(f_j) \le K$ for $j = 1, \dots, n$. By applying the Riemann Mapping Theorem iteratively, each f_j can be composed with conformal mappings to a mapping h_j in order to map G onto itself without altering the maximal dilatations. Consequently, it is $h_j \in Q_K(G)$ for all j, and due to $K(g) = K(g^{-1})$ for every quasiconformal mapping, it follows that $Q_K(G)$ is closed under inversion, i.e. $Q_K(G)^{-1} = Q_K(G)$. These considerations yield

Theorem 2.2.3.
For every $K \in (1, +\infty)$, the set $Q_K(G)$ forms a symmetric generating set of the group $Q(G)$.

Remark 2.2.4.

(i) The group $Q(G)$, even though containing uncountably many generating systems as shown above, can actually never be finitely generated. This is denied by the principal obstacle in group theory given by the fact that a finitely generated group is necessarily (at most) countable. However, $Q(G)$ is clearly uncountable, as can be seen in numerous ways, for example by checking that the function

$$\mathrm{dil} : Q(G) \longrightarrow \mathbb{B}_{L^\infty}(G),\ f \longmapsto \mu_f$$

is surjective (which in turn is essentially the statement of the Measurable Riemann Mapping Theorem 1.1.2). Another possibility is related to an argument of Gaier: $\Sigma(G)$ is always a proper subgroup of $Q(G)$ (Proposition 1.3.2(ii)), and from a purely set–theoretical point of view, $\Sigma(G)$ is equivalent to the torus $\mathbb{D} \times \partial\mathbb{D}$ which is clearly an uncountable set, see [Gai84, pp. 228–230]. Hence $\Sigma(G)$ and therefore $Q(G)$ is uncountable.

(ii) The statement of Theorem 2.2.3 obviously becomes wrong for $K = 1$, for in this situation, it is $Q_1(G) = \Sigma(G)$, and $\Sigma(G)$ is a subgroup of $Q(G)$ (rather than merely a subset). Therefore it is closed under composition, making it impossible to express any $f \in Q(G)$ with $K(f) > 1$ as a (finite) composition of elements of $\Sigma(G)$.

2.3 Incompleteness of $Q(G)$

Completeness is, in general, a desirable property of metric spaces, for example due to the fact that in complete metric spaces the powerful machinery of Baire's Category Theorem is at one's disposal, often allowing for elegant reasoning. Gaier's completeness result for $\Sigma(G)$ may lead to the conjecture that $Q(G)$ is also a complete metric space. However, this assumption will be disproved in the following. To this end, the following mappings defined on and valued in $\mathbb{D}$ are of central importance:

Definition 2.3.1.
Let $\rho \in C([0,1])$ be a strictly increasing function of the interval $[0,1]$ onto itself. A mapping of the form

$$f_\rho : \mathbb{D} \longrightarrow \mathbb{D},\ z = re^{i\varphi} \longmapsto f_\rho(z) := \rho(r)e^{i\varphi} \tag{2.2}$$

is called a ***(general) radial stretching*** *of $\mathbb{D}$.*

Note that radial stretchings may also be written in the form $f_\rho(z) = \rho(|z|)\frac{z}{|z|}$ for $z \neq 0$ and setting $f_\rho(0) := 0$, a convention that will be used tacitly in the following. The mapping ρ, which can be thought of "controlling" the radial dilation of f_ρ, satisfies $\rho(x) = x$ for $x \in \{0, 1\}$ by definition and is differentiable almost everywhere on $[0,1]$. The importance of general radial stretchings is given by the following result (see for example [IM08, p. 7] and [LV73, p. 220] as well as [AIM08, Section 2.6, pp. 28–29]):

Lemma 2.3.2.
For each mapping $\rho \in C([0,1])$ as in Definition 2.3.1 such that ρ is a piecewise C^1–mapping on $[0,1]$, the corresponding general radial stretching f_ρ is a quasiconformal automorphism of $\mathbb{D}$ with complex dilatation

$$\mu_{f_\rho}(z) = \frac{z}{\bar{z}} \cdot \frac{|z|\rho'(|z|) - \rho(|z|)}{|z|\rho'(|z|) + \rho(|z|)}$$

for almost every $z \in \mathbb{D}$, where ρ' denotes the (almost everywhere existing) derivative of ρ.

An example of a general radial stretching is visualized in Figure 2.1, in which the piecewise–differentiable radial dilation mapping $\rho(x) = \begin{cases} \sqrt{\frac{x}{2}} & , x \in [0, \frac{1}{2}) \\ 2(x - \frac{1}{2})^2 + \frac{1}{2} & , x \in [\frac{1}{2}, 1] \end{cases}$ is used.

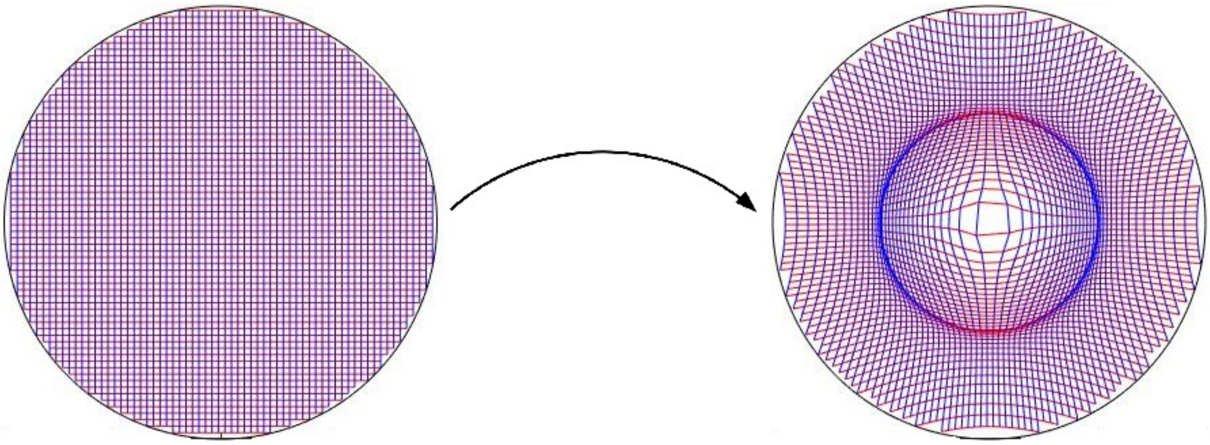

Figure 2.1: Visualization of a general radial stretching on a Cartesian grid in $\mathbb{D}$.

An important special case of radial stretchings is given by setting $\rho_K(x) = x^K$ for $K \in \mathbb{R}^+$, then the resulting quasiconformal automorphism of $\mathbb{D}$ is

$$f_K(z) = \rho_K(|z|)\frac{z}{|z|} = z|z|^{K-1} \tag{2.3}$$

The mappings f_K are called **monomial–like radial stretchings**, due to the special form of the radial mappings ρ_K, and represent in fact $\max\{K, \frac{1}{K}\}$–quasiconformal mappings of $\mathbb{D}$ onto itself (see [AIM08, p. 29]). Figure 2.2 shows the monomial–like radial stretching f_3 modeled on a Cartesian mesh. An eye–catching attribute of the mapping behaviour of f_3 is that it "pulls" the Cartesian grid towards the origin in $\mathbb{D}$, a fact shared by every mapping f_K with $K > 1$ (this particular effect is also examined in Section 4.2, see Figure 4.2). On the contrary, for $K \in (0,1)$, such a mapping f_K would "push" the grid away from the origin and in the direction of $\partial\mathbb{D}$.
As to return to the question for completeness of $Q(G)$, the announced result can be deduced (see also [BL23, Theorem 5]):

Theorem 2.3.3.
The space $Q(G)$ is always incomplete.

Proof. First, the case $G = \mathbb{D}$ will be treated. For $n \in \mathbb{N}$ and $x \in [0,1]$, consider the function

$$\rho_n(x) := \begin{cases} 2x, & x \in [0, \frac{1}{4}] \\ \frac{1}{2}\left(\frac{x}{n} + 1 - \frac{1}{4n}\right), & x \in (\frac{1}{4}, \frac{3}{4}] \\ \left(2 - \frac{1}{n}\right)x - 1 + \frac{1}{n}, & x \in (\frac{3}{4}, 1] \end{cases} \tag{2.4}$$

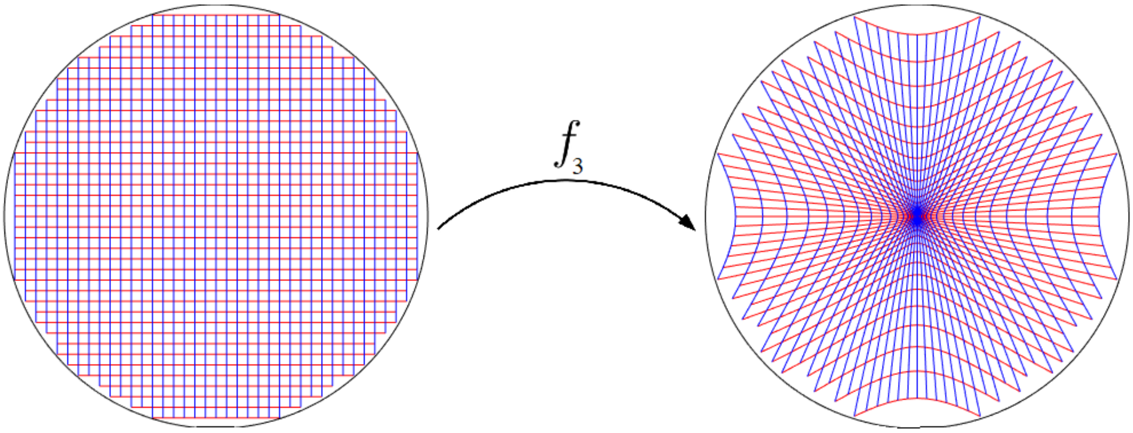

Figure 2.2: Visualization of the monomial–like radial stretching $f_3(z) = z|z|^2 = z^2\overline{z}$ on a Cartesian grid in the unit disk. The grid is deformed by f_3 by "pulling" it towards the origin.

which is a strictly increasing, continuous and piecewise C^1–mapping of the interval $[0,1]$ onto itself. It follows that the sequence $(\rho_n)_{n\in\mathbb{N}}$ converges uniformly on $[0,1]$ to the non–injective limit mapping

$$\widetilde{\rho}(x) := \begin{cases} 2x, & x \in [0, \frac{1}{4}] \\ \frac{1}{2}, & x \in (\frac{1}{4}, \frac{3}{4}] \\ 2x-1, & x \in (\frac{3}{4}, 1] \end{cases}$$

Now this situation is lifted from $[0,1]$ to the unit disk by defining the general radial stretching $f_n(z) := \rho_n(|z|)\frac{z}{|z|}$ for $z \in \mathbb{D}$ and $n \in \mathbb{N}$, yielding a sequence of quasiconformal automorphisms $(f_n)_{n\in\mathbb{N}}$ of $\mathbb{D}$ according to Lemma 2.3.2. Due to the uniform convergence of the sequence $(\rho_n)_n$, it is

$$d_{\sup}(f_n, f_{\widetilde{\rho}}) = \sup_{z\in\mathbb{D}} |f_n(z) - f_{\widetilde{\rho}}(z)| = \sup_{z\in\mathbb{D}} \left|\rho_n(|z|)\frac{z}{|z|} - \widetilde{\rho}(|z|)\frac{z}{|z|}\right| = \sup_{x\in[0,1)} |\rho_n(x) - \widetilde{\rho}(x)| \overset{n\to\infty}{\longrightarrow} 0$$

with $f_{\widetilde{\rho}}(z) := \widetilde{\rho}(|z|)\frac{z}{|z|}$ and $x = |z| \in [0,1)$. Hence, the sequence $(f_n)_n$ converges with respect to $d_{\sup}$ and is therefore a Cauchy sequence in $Q(\mathbb{D})$ whose non–injective limit $f_{\widetilde{\rho}}$ is clearly not contained in $Q(\mathbb{D})$. Thus $Q(\mathbb{D})$ is incomplete.

For a general domain G, let $z_0 \in G$ be a fixed inner point and let $B \subsetneq G$ be an open ball about z_0 in G. Clearly the sequence $(f_n)_n$ of the proof's first part together with its limit function $f_{\widetilde{\rho}}$ can be transferred to B via conformal equivalence, denoted by g_n and g_ρ, respectively. Then, define quasiconformal automorphisms h_n of G by

$$h_n(z) = \begin{cases} g_n(z), & z \in B \\ \mathrm{id}_G(z), & z \in G\backslash B \end{cases}$$

and likewise for h_ρ and g_ρ. Now, the sequence $(h_n)_{n\in\mathbb{N}}$ converges uniformly to the non-injective limit function h_ρ on G. □

2.4 On the completion of $Q(G)$

Immediately following the incompleteness of $Q(G)$ shown in Theorem 2.3.3, the question for the completion of these spaces and the corresponding sets $Q(\overline{G})$ arises, to be studied in the following.

2.4.1 Completion of metric spaces and closure of $Q(G)$ in $C(\overline{G})$

Since $Q(G)$ is a proper subset of the Banach space $C_b(G)$, the completion of $Q(G)$ certainly is a subset of $C_b(G)$ (or of $C(\overline{G})$, in case the extended mappings are considered). Furthermore, the following partial result on convergent sequences of homeomorphisms on compact spaces holds:

Lemma 2.4.1.
Let (X, d_X) be a compact metric space and $(f_n)_{n\in\mathbb{N}}$ be a sequence of self–homeomorphisms of X converging uniformly to a mapping $f : X \longrightarrow X$. Then f is continuous and surjective.

Proof. The fact that f is continuous is a standard result in analysis. In order to show the surjectivity of f, let $y \in X$ be given. Then for every $n \in \mathbb{N}$, there exists a uniquely determined $x_n \in X$ such that $f_n(x_n) = y$, yielding a sequence $(x_n)_n$ in X. By compactness of X, there exists a convergent subsequence $(x_{n_j})_j$ with limit $\widetilde{x} \in X$. This concludes in

$$f(\widetilde{x}) = \lim_{j\to\infty} f_{n_j}\left(x_{n_j}\right) = \lim_{j\to\infty} y = y$$

where in the first equality the fact has been used that uniform convergence implies continuous convergence[1] in the present situation. Hence $f(\widetilde{x}) = y$, showing that f is surjective. □

Remark 2.4.2.
In general, however, a uniformly convergent sequence of self–homeomorphisms is not injective. An example for this situation is provided by the sequence $(f_n)_n$ which was utilized in the proof of Theorem 2.3.3 in order to show the incompleteness of $Q(\mathbb{D})$.

Turning towards the completion of $Q(G)$, in order to be able to use Lemma 2.4.1 it is required to work in the context of compact spaces, i.e. compact preimage domains of sequences of (appropriately extended) quasiconformal mappings. This can be achieved by assuming the additional requirement $G \in \mathcal{JD}$ and applying Proposition 1.1.6, thus embedding $Q(G)$ into the Banach space $C(\overline{G})$ in form of the set $Q(\overline{G})$ (see Definition 1.1.7). Then Lemma 2.4.1 implies that the completion is quite restricted by the necessary requirement that it may only contain continuous, surjective mapping of $\overline{G}$ onto itself. Therefore, in the following, $Q(G)$ will be interpreted as a subset of the space $C(\overline{G})$ via $Q(\overline{G})$ and vice versa, if necessary.

As a consequence of Theorem 2.3.3, the set $Q(G)$ (or more precisely, the set $Q(\overline{G})$) is not closed in the Banach space $C(\overline{G})$, therefore its closure $\overline{Q(\overline{G})}$ in the ambient space $C(\overline{G})$ may be considered. Since the latter space is complete, the closed set $\overline{Q(\overline{G})}$ is complete as well and $Q(\overline{G})$ is obviously dense in $\overline{Q(\overline{G})}$. This leads to

Lemma 2.4.3.
Let $G \in \mathcal{JD}$, then up to isometry, the closure $\overline{Q(\overline{G})} \subsetneq C(\overline{G})$ is the completion of $Q(\overline{G})$.

Hence, if one is to find the completion $\overline{Q(\overline{G})}$, by definition this means studying the set

$$\left\{f : \overline{G} \longrightarrow \mathbb{C} \;\middle|\; f \text{ is uniform limit of a sequence } (f_n)_{n\in\mathbb{N}} \text{ in } Q(\overline{G})\right\} \subsetneq C(\overline{G}) \tag{2.5}$$

Consequently, this brings up the question which mappings can occur as uniform limits of sequences in $Q(\overline{G})$, to be studied in the following.

[1] A sequence of mappings $f_n : X \longrightarrow Y$ between metric spaces is called *continuously convergent* in X if for every convergent sequence $(x_n)_n$ in X with limit $a \in X$, the limit $f_n(x_n)$ exists in Y; in this situation, there is a limit mapping $f : X \longrightarrow Y$ and it is $f(a) = \lim_{n\to\infty} f_n(x_n)$, see [RS02, pp. 87–89].

2.4.2 Monotone mappings and locally connected spaces

The construction in the proof of Theorem 2.3.3 yields that the uniform limit of a sequence of quasiconformal mappings is in general no bijective mapping, and in particular need not be a homeomorphism. Put differently, the closure of $Q(\overline{G})$ in $C(\overline{G})$ is no subset of the homeomorphism group

$$\mathcal{H}(\overline{G}) := \left\{ h \in C(\overline{G}) \;\middle|\; h \text{ is a homeomorphism of } \overline{G} \text{ onto itself} \right\} \tag{2.6}$$

or the orientation–preserving homeomorphism (sub)group

$$\mathcal{H}^+(\overline{G}) := \left\{ h \in \mathcal{H}(\overline{G}) \;\middle|\; h \text{ is orientation–preserving} \right\} \leq \mathcal{H}(\overline{G}) \tag{2.7}$$

which of course properly contains $Q(\overline{G})$. In turn, this raises the question for the class of mappings that can actually occur as the mentioned limit mappings. In this regard, the following class of mappings will be useful in this context (see [Why42, (4.1), p. 127]):

Definition 2.4.4 (Monotone mapping)**.**
Let X, Y be metric spaces. A continuous surjective mapping $f : X \longrightarrow Y$ is called ***monotone*** *if for each $y \in Y$, the preimage $f^{-1}(\{y\})$ is a continuum in X, i.e. a connected compact subset. Denote by $\mathcal{M}(X, Y)$ the class of all monotone mappings from X to Y, and set $\mathcal{M}(X) := \mathcal{M}(X, X)$.*

Remark 2.4.5.

(i) In Definition 2.4.4, it is no loss of generality to assume that f is surjective, for if $y \in Y$ is not contained in $f(X)$, then $f^{-1}(\{y\}) = \varnothing$, which by definition is a continuum in X.

(ii) The definition of a monotone mapping is not necessarily restricted to metric spaces and could equivalently be formulated for topological spaces as well. In the same manner, the mapping f needn't be continuous in order for the definition to be reasonable. Due to this situation, there are several slightly different definitions for monotone mappings to be found in the literature (see e.g. [AIM08, Definition 20.1.2, p. 531]).

(iii) In particular, it follows immediately from Definition 2.4.4 that every homeomorphism between metric spaces is a monotone mapping.

The notion of a monotone mapping was originally coined by C.B. Morrey in [Mor35]. In the case of compact metric spaces, the following characterization was shown (see [Why42, (2.2), p. 138]):

Proposition 2.4.6.
A continuous surjective mapping $f : X \longrightarrow Y$ between compact metric spaces X, Y is monotone if and only if $f^{-1}(C)$ is connected in X for every connected set $C \subseteq Y$.

The class of monotone mappings between certain metric spaces obeys a persistence property for uniformly convergent sequences, similarly to continuous mappings (see [IO17, Theorem 3.3, p. 484]):

Proposition 2.4.7.
Let X, Y be compact Hausdorff spaces and Y be locally connected. Furthermore, let $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{M}(X, Y)$ be a sequence of monotone mappings converging uniformly on X to a mapping $f : X \longrightarrow Y$. Then f is monotone, i.e. $f \in \mathcal{M}(X, Y)$.

Since the topic at hand is concerned with the more concrete situation of domains in $\mathbb{C}$ rather than abstract general topological spaces, the question for local connectedness of such domains and their closures arises. A useful result in this situation is (see [Sin19, p. 75])

Proposition 2.4.8.
Let X be a locally connected topological space and $A \subseteq X$. If ∂A is locally connected, then $\overline{A}$ is locally connected as well.

Corollary 2.4.9.
Let $G \in \mathcal{JD}$, then ∂G and $\overline{G}$ are locally connected.

Proof. The topological space $X = \mathbb{C}$ with its standard topology is clearly locally connected. Let $A = G \in \mathcal{JD}$, then ∂G is locally connected as the homeomorphic image of the locally connected space $\partial\mathbb{D}$ (see [Sin19, Theorem 3.4.3, p. 70]). The claim follows from Proposition 2.4.8. □

Corollary 2.4.10.
Let $G \in \mathcal{JD}$ and $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{H}(\overline{G})$ a sequence of homeomorphisms of $\overline{G}$ onto itself converging uniformly on $\overline{G}$ to a mapping $f : \overline{G} \longrightarrow \overline{G}$. Then $f \in \mathcal{M}(\overline{G})$. In particular, this statement applies if $(f_n)_n$ is a uniformly convergent sequence of (the homeomorphic extensions to $\overline{G}$ of) quasiconformal automorphisms of G.

Proof. By definition, a homeomorphism is a monotone mapping, see Remark 2.4.5(iii). Lemma 2.4.1 implies that the limit mapping f is continuous and surjective. The claim follows from Proposition 2.4.7 and Corollary 2.4.9 by setting $X = Y = \overline{G}$. □

Corollary 2.4.10 yields in particular the following important partial result in view of the current section's central topic:

The completion of $Q(\overline{G})$ is contained in $\mathcal{M}(\overline{G})$

2.4.3 Uniform approximation of monotone mappings by homeomorphisms

In order to be able to properly formulate the statements required in the remainder of this section, the following terminology is needed (see [Lee11, pp. 38–45]):

Definition 2.4.11.
*Let $n \in \mathbb{N}$. An n–**manifold** is a separable metric space M in which every point of M has an open neighborhood homeomorphic to $\mathbb{R}^n$. A separable metric space M in which every point has an open neighborhood homeomorphic either to an open subset of $\mathbb{R}^n$ or to an open subset of the **closed** n–**dimensional upper half space***

$$\mathbb{H}^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \,\big|\, x_n \geq 0\}$$

*endowed with the subspace topology of $\mathbb{R}^n$, is called an n–**manifold with boundary**. An n–manifold is called **compact** if the corresponding metric space is compact.*

For homeomorphic compact metric spaces X, Y (not necessarily n–manifolds), let

$$\mathcal{H}(X,Y) := \big\{f : X \longrightarrow Y \,\big|\, f \text{ is homeomorphism}\big\} \tag{2.8}$$

denote the set of all homeomorphic mappings between X and Y endowed with the supremum metric $d_{\sup}$; consequently, $\mathcal{H}(X) = \mathcal{H}(X,X)$ then yields the homeomorphism group of X. Obviously, it is $\mathcal{H}(X,Y) \subseteq \mathcal{M}(X,Y)$, see Remark 2.4.5(iii). Moreover, Proposition 2.4.7 implies $\overline{\mathcal{H}(X,Y)} \subseteq \mathcal{M}(X,Y)$ for compact metric spaces X, Y with Y being locally connected, i.e. the closure of $\mathcal{H}(X,Y)$ in the uniform topology is a subspace of $\mathcal{M}(X,Y)$. The question raises whether this subspace relation is in fact an equality of sets. This question has been studied intensively, and the following affirmative answer for certain 2–manifolds has been given in [Rad45, Approximation Theorem 2.17, p. 435] and [You48, Approximation Theorem, p. 92], as stated in (see also [IO15, p. 490])

Proposition 2.4.12.
Let X, Y be homeomorphic compact 2–manifolds with boundary, $\epsilon > 0$ and $f \in \mathcal{M}(X,Y)$. Then there exists $h \in \mathcal{H}(X,Y)$ with $d_{\sup}(f,h) < \epsilon$. In particular, this statement holds if X and Y are closures of Jordan domains in $\mathbb{C}$.

Consequently, in view of Corollary 2.4.10 and Proposition 2.4.12, the following central conclusion can be drawn immediately:

Corollary 2.4.13.
Let $G \in \mathcal{JD}$ and $f : \overline{G} \longrightarrow \overline{G}$ a mapping. Then f is monotone if and only if it is the uniform limit of homeomorphisms of $\overline{G}$ onto itself.

Remark 2.4.14.

(i) The notation in [You48] is somewhat confusing. In the introduction, his main result, i.e. the Approximation Theorem, is announced in full generality for compact 2–manifolds with or without boundary. However, in Section 3.9, Youngs formulates his result only for (compact) closed 2–manifolds, i.e. 2–manifolds with empty manifold boundary. The important case of non–empty manifold boundary is not proved until the very last Section 3.10, without restating the corresponding version of the Approximation Theorem again.

(ii) The previously mentioned Corollary 2.4.13 is also stated in [IO16, pp. 160–163], even though the authors formulate them in a slightly different manner in order to serve their purposes; see also [IO15, p. 490].

2.4.4 Algebraic structure and continuous structure–preserving maps of $\mathcal{M}(\overline{G})$

This subsection is concerned with the study of $\mathcal{M}(\overline{G})$ for $G \in \mathcal{JD}$ in terms of its algebraic structure[2]. First of all, it is $\mathrm{id}_{\overline{G}} \in \mathcal{M}(\overline{G})$, in particular this set is not empty. Moreover, the following closedness property with respect to composition of mappings holds:

Lemma 2.4.15.
Let $G \in \mathcal{JD}$ and $f, g \in \mathcal{M}(\overline{G})$, then $(f \circ g) \in \mathcal{M}(\overline{G})$.

Proof. Clearly, $h := f \circ g$ is a surjective continuous mapping of $\overline{G}$ onto itself. In order to show that h is monotone, let $C \subseteq \overline{G}$ be connected. Then it is

$$h^{-1}(C) = (f \circ g)^{-1}(C) = g^{-1}(f^{-1}(C))$$

Since f is monotone and $\overline{G}$ is compact, the set $C' := f^{-1}(C)$ is connected by Proposition 2.4.6. The same argument shows that $g^{-1}(C')$ is connected as well, thus $h^{-1}(C)$ is connected. Applying Proposition 2.4.6 once again yields the claim. ☐

Corollary 2.4.16.
Let $G \in \mathcal{JD}$, then $(\mathcal{M}(\overline{G}), \circ)$ is a monoid with neutral element $\mathrm{id}_{\overline{G}}$.

Proof. The identity mapping on $\overline{G}$ serves as the neutral element in $\mathcal{M}(\overline{G})$. Lemma 2.4.15 shows that the composition of monotone mappings is again monotone. ☐

Naturally, both, the homeomorphism group $\mathcal{H}(\overline{G})$ and the set $Q(\overline{G})$ are submonoids of $\mathcal{M}(\overline{G})$. Now, since $\mathcal{M}(\overline{G})$ carries algebraic structure, the persistence of this property under certain mappings can be studied, especially when considering the before–mentioned submonoid $Q(\overline{G})$ and the corresponding conjugation mapping Φ (in which the involved conformal mappings are homeomorphically extended to the closures of the respective domains). The nearby and pleasant answer to this question is stated in

[2]Many of the results of this subsection hold in much greater generality. However, for the sake of readability and simplicity, the case of Jordan domains and their closures in $\mathbb{C}$ is exclusively studied here.

Theorem 2.4.17.
Let $G, G' \in \mathcal{JD}$ and $F : G \longrightarrow G'$ be conformal, its homeomorphic extension to $\overline{G}$ denoted by the same letter. Then the conjugation mapping

$$\Phi : \mathcal{M}(\overline{G}) \longrightarrow \mathcal{M}(\overline{G'}),\ g \longmapsto \Phi(g) := F \circ g \circ F^{-1} \tag{2.9}$$

is a monoid isomorphism, i.e. bijective, homomorphic and maps the neutral element onto the neutral element. Furthermore, if these sets are endowed with the supremum metric $d_{\sup}$, the mapping Φ defined in (2.9) is a homeomorphism.

Proof. By Lemma 2.4.15, Φ is well–defined since F and F^{-1} are monotone, therefore $\Phi(g) \in \mathcal{M}(\overline{G'})$ for every $g \in \mathcal{M}(\overline{G})$. It is obvious that $\Phi(\mathrm{id}_{\overline{G}}) = \mathrm{id}_{\overline{G'}}$, and also the bijectivity and homomorphism property follow immediately. For proving the second claim, endow the monoids' underlying sets with the corresponding supremum metrics, let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}(\overline{G})$ converging uniformly on $\overline{G}$ to $g \in \mathcal{M}(\overline{G})$ and $\epsilon > 0$. Since F is uniformly continuous on $\overline{G}$, it follows that

$$|F(g_n(z)) - F(g(z))| < \epsilon$$

for sufficiently large $n \geq N(\epsilon)$, due to $|g_n(z) - g(z)| < \delta$ for such indices n, all $z \in G$ and a certain $\delta > 0$. Switching to the supremum of $z = F^{-1}(w) \in \overline{G}$ in the previous inequality concludes in

$$d_{\sup}(\Phi(g_n), \Phi(g)) = \sup_{w\in\overline{G'}} |\Phi(g_n)(w) - \Phi(g)(w)| = \sup_{z\in\overline{G}} |F(g_n(z)) - F(g(z))| \leq \epsilon$$

Thus Φ is continuous. The same reasoning provides the continuity of Φ^{-1}. □

In particular, Φ as in (2.9) maps convergent sequences in $\mathcal{M}(\overline{G})$ to convergent sequences in $\mathcal{M}(\overline{G'})$ and $Q(\overline{G})$ bijectively to $Q(\overline{G'})$.

2.4.5 Proof and discussion of the main statement

This final subsection will now combine the previously established theory of the foregoing subsections for $G \in \mathcal{JD}$. First of all, *the uniform limit of homeomorphisms of $\overline{G}$ onto itself is a monotone mapping* by Corollary 2.4.10. Conversely, *every monotone mapping on $\overline{G}$ is uniform limit of homeomorphisms on $\overline{G}$* by Corollary 2.4.13. Consequently, the set $\mathcal{H}(\overline{G})$ is dense in $\mathcal{M}(\overline{G})$. Together with $Q(\overline{G}) \subseteq \mathcal{H}(\overline{G})$, one has the following inclusion chains of sets:

$$Q(\overline{G}) \subseteq \mathcal{H}(\overline{G}) \subseteq \mathcal{M}(\overline{G})$$

By Proposition 2.4.7, the set $\mathcal{M}(\overline{G})$ is closed in $C(\overline{G}, \overline{G})$, the space of continuous mappings of $\overline{G}$ into itself. When switching to the closures, one arrives at

$$\overline{Q(\overline{G})} \subseteq \overline{\mathcal{H}(\overline{G})} = \mathcal{M}(\overline{G}) \tag{2.10}$$

Since the overall goal of this section is to consider all possible limits of uniformly convergent sequences of homeomorphically extended quasiconformal automorphisms and due to the fact that these mappings are orientation–preserving (i.e. $Q(\overline{G}) \subseteq \mathcal{H}^+(\overline{G})$), it is beyond that reasonable and necessary to take into account only those limits that arise from orientation–preserving mappings. These arguments conclude in

Theorem 2.4.18.
Let $G \in \mathcal{JD}$. Then the closure of $Q(\overline{G})$ in $C(\overline{G})$ is contained in

$$\mathfrak{M} := \left\{ f \in \mathcal{M}(\overline{G}) \;\middle|\; f \text{ is uniform limit of orientation–preserving homeomorphisms of } \overline{G} \right\}$$

In other words, the completion of $Q(\overline{G})$ is a subset of the class of monotone mappings of $\overline{G}$ onto itself arising as uniform limits of sense–preserving homeomorphisms of $\overline{G}$ onto itself:

$$\overline{Q(\overline{G})} \subseteq \mathfrak{M} \tag{2.11}$$

Remark 2.4.19.
In the first place, the statement of Theorem 2.4.18 is non–trivial inasmuch as the general restriction to the class of monotone mappings, i.e. elements of $\mathcal{M}(\overline{G})$, is explicitly taken into account. Furthermore, the argumentation before Theorem 2.4.18 show that each mapping in $\mathcal{M}(\overline{G})$ can be generated as the uniform limit of homeomorphic mappings (see (2.10)), but the orientation–preservation of the latter mappings is not necessarily guaranteed (since there exist orientation–reversing homeomorphisms as well). Thus the additional limitation in the definition of $\mathfrak{M}$ is to be considered, since quasiconformal mappings are always orientation–preserving; This situation becomes particularly obvious in the related one–dimensional case: For a compact interval $I = [a, b]$ in $\mathbb{R}$, the set of monotone mappings of I onto itself basically consists of two subclasses of surjective mappings, namely the increasing ones and the decreasing ones[3]*. Clearly, the decreasing monotone mappings are uniform limits of elements of $\mathcal{H}(I)$ as well*[4]*, but they can never be realized as uniform limits of the one–dimensional counterparts of quasiconformal mappings, the so–called **quasisymmetric mappings** on I (see [LV73, p. 88]), which are strictly increasing (= orientation–preserving) by definition. Theorem 2.4.18 is formulated in the very same spirit: A monotone mapping of $\overline{G}$ onto itself can arise as the uniform limit of homeomorphisms, but these mappings are not necessarily orientation–preserving, thus may not necessarily be quasiconformal. Furthermore, it follows from the discussion of the current section that the completion of $Q(\overline{G})$ basically consists of three "parts":*

- *The homeomorphically extended quasiconformal automorphisms themselves;*
- *The homeomorphisms of $\overline{G}$ onto itself which are not quasiconformal (i.e. homeomorphisms with unbounded maximal dilatation);*
- *The strict monotone mappings which are not injective on $\overline{G}$.*

Finally, the results of this section naturally lead to ask the

Question 2.4.20.
If G is no Jordan domain, is the corresponding metric space $Q(\overline{G})$ incomplete as well? If so, what is its completion?

2.5 Incompleteness revisited: The symmetric supremum on $Q(\overline{G})$

In the following, it will be shown that – beyond the incompleteness in the supremum metric – the space $Q(\overline{G})$ of homeomorphic extensions to $\overline{G}$ (see Definition 1.1.7) also possesses a certain kind of incompleteness when endowed with a different, yet very similar metric structure, called the *symmetric supremum.* To begin with, the required terminology is introduced in

Definition 2.5.1.

*(i) A topological space $(X, \mathcal{T})$ is called **metrizable** if there exists a metric d on X such that $\mathcal{T}$ is the topology of the metric space (X, d), i.e. the metric d induces the topology $\mathcal{T}$. In this case, the metric d is called **compatible** with $\mathcal{T}$; (see [Kec95, p. 3])*

*(ii) A topological space $(X, \mathcal{T})$ is called **completely metrizable** if there exists a compatible metric d on X such that the metric space (X, d) is complete. A separable completely metrizable space is called **Polish**; (see [Kec95, Definition 3.1, p. 13])*

*(iii) A topological group $(X, \mathcal{T}, *)$ is called a **Polish group** if the topological space $(X, \mathcal{T})$ is Polish. (see [Kec95, Definition 9.2, p. 58])*

[3]The probably most simple examples of such mappings are $\mathrm{id}_I(x) = x$ and $y(x) = -x + a + b$ for $x \in I$.
[4]In this regard, see also Radó's remark on *"closed linear intervals"* in [Rad45, p. 437].

In other words, a topological space is (completely) metrizable if and only if it is homeomorphic to a (complete) metric space (see [Wil70, p. 176]). Occasionally, the term *topologically complete* is used instead of completely metrizable, see e.g. [Sin19, Definition 10.1.3, p. 220]. A simple example of a completely metrizable space is given by the interval $(0,1)$: Endowed with the standard Euclidean metric d of $\mathbb{R}$, the space $((0,1),d)$ is obviously incomplete; However, via the mapping $x \longmapsto \tan(\pi x - \frac{\pi}{2})$, the topological space $\left((0,1), \mathcal{T}_d\big|_{(0,1)}\right)$ is homeomorphic to the space $(\mathbb{R}, \mathcal{T}_d)$, which is complete as a metric space (see [Wil70, p. 176]).
For the sake of readability, no exact distinction between a metrizable space and the corresponding metric space is made if no misunderstanding is to be suspected.

In order to switch the focus to the situation of $Q(G)$, is is trivial to observe that the *topological* space $Q(G)$ is metrizable due to the fact that the *metric* space $(Q(G), d_{\text{sup}})$ is considered in this thesis. Theorem 2.3.3 states that $Q(G)$ is incomplete, thus the question raises whether $Q(G)$ is completely metrizable. Furthermore, due to the fact that for $G \in \mathcal{JD}$, $Q(G)$ is separable (Theorem 3.1.4) and a topological group (Proposition 1.3.3(ii)), this question is equivalent to asking for whether $Q(G)$ is a Polish group. In view of this, consider the following mapping:

$$d_{\text{sym}} : Q(G) \times Q(G) \longrightarrow \mathbb{R},\ (f,g) \longmapsto d_{\text{sym}}(f,g) := d_{\text{sup}}(f,g) + d_{\text{sup}}(f^{-1}, g^{-1}) \tag{2.12}$$

The mapping d_{sym} is easily seen to be a metric on $Q(G)$ (see also [Kec95, p. 58]), called the **symmetric supremum**[5]. Unlike the supremum metric d_{sup}, the symmetric supremum is not right–invariant anymore, i.e. d_{sym} is no isometry with respect to right multiplication in $Q(G)$. However, d_{sym} is isometric with respect to inversion, as follows immediately from (2.12). Obviously, the metric d_{sym} can (and will in the following) be considered on the extended space $Q(\overline{G})$ as well. More generally, the metric d_{sym} can be considered on any homeomorphism group $\mathcal{H}(X)$ of a compact metric space (X,d), i.e.

$$d_{\text{sym}}(f,g) = d_{\text{sup}}(f,g) + d_{\text{sup}}(f^{-1}, g^{-1}) \quad \text{for} \quad f,g \in \mathcal{H}(X)$$

where the supremum metric is of course induced by the given metric d, see e.g. [Mel16, p. 100] or [Ros08, p. 350]. On $Q(G)$, the metric d_{sym} now gives rise to consider the space $(Q(G), d_{\text{sym}})$. A first important observation to be made in this regard is stated in

Lemma 2.5.2.
For $G \in \mathcal{JD}$, the spaces $(Q(G), d_{\text{sup}})$ and $(Q(G), d_{\text{sym}})$ are homeomorphic.

Proof. Trivially, the claimed homeomorphism will be given by the identity mapping $\text{id} = \text{id}_{Q(G)}$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $Q(G)$ converging to $f \in Q(G)$ with respect to the symmetric supremum d_{sym}. Then clearly

$$d_{\text{sup}}(f_n, f) \leq d_{\text{sym}}(f_n, f) \overset{n\to+\infty}{\longrightarrow} 0$$

hence id is continuous in the one direction[6]. In the other direction, let $(f_n)_n$ converge to f with respect to d_{sup}. Since $Q(G)$ is a topological group due to $G \in \mathcal{JD}$, inversion on $Q(G)$ is continuous, thus f_n^{-1} converges to f^{-1} in the space $(Q(G), d_{\text{sup}})$ as well. This concludes in

$$d_{\text{sym}}(f_n, f) = d_{\text{sup}}(f_n, f) + d_{\text{sup}}(f_n^{-1}, f^{-1}) \overset{n\to+\infty}{\longrightarrow} 0$$

showing that $\text{id}^{-1} = \text{id}$ is also continuous, and the claim follows. □

[5]This terminology was chosen by the author of this thesis due to the apparent symmetry in (2.12) regarding the supremum metric together with the mappings f and g and their inverses. Apart from this, to the best of the author's knowledge, this name does not seem to have been used in the respective mathematical literature so far.

[6]This observation is even independent of the domain G, i.e. it is irrelevant that G is a Jordan domain.

The proof of Lemma 2.5.2 shows that the metrics d_{sup} and d_{sym} induce the same topology on $Q(G)$, a property that is sometimes called *(topological) equivalence* of metrics (see [Sin19, Definition 1.4.7, p. 21]). Now turning the focus towards $Q(\overline{G})$, the statement of the Lemma 2.5.2 is in fact a special case of the following result concerning Polish group structure of certain homeomorphism groups (see [Kec95, Example 8, p. 60] and [Mel16, p. 100]):

Proposition 2.5.3.
Let (X, d) be a compact metric space. Then its homeomorphism group $\mathcal{H}(X)$ is a Polish group in the uniform topology with compatible metric d_{sym}. In particular, $\mathcal{H}(X)$ is a separable topological group whose topology is (equivalently) induced by the supremum metric d_{sup}.

Hence, the statement of Proposition 2.5.3 clearly applies to $X = \overline{G}$ for $G \in \mathcal{JD}$. Consequently, via the subspace topology, the set $Q(\overline{G}) \leq \mathcal{H}(\overline{G})$ becomes a topological group as well in the topology of uniform convergence (see e.g. [Hus66, p. 54] or [Sin19, p. 276]). In view of subgroups, Polish groups possess certain structural hereditary properties, as stated in (see [Mel16, Theorem 2.16, p. 95])

Proposition 2.5.4.
Let X be a Polish group and $U \leq X$ be a subgroup of X (in the algebraic sense). Then U, endowed with the subspace topology, is a Polish group if and only if U is closed in X.

These preparatory steps are now used in order to derive the following statements:

Theorem 2.5.5.
For $G \in \mathcal{JD}$, the space $Q(\overline{G})$ is not closed in $\mathcal{H}(\overline{G})$.

Proof. In the first place, the case $G = \mathbb{D}$ is considered, for which Corollary[7] 4.1.25 provides exactly the claimed statement. The general case $G \in \mathcal{JD}$ follows via the extended conjugation mapping $\Phi : \mathcal{H}(\overline{\mathbb{D}}) \longrightarrow \mathcal{H}(\overline{G})$; to be more precise, the conformal mappings $F : G \longrightarrow \mathbb{D}$ and its inverse F^{-1} in

$$\Phi(f) = F^{-1} \circ f \circ F, \ f \in Q(\overline{\mathbb{D}}),$$

are extended (via Proposition 1.1.6) to the closures of G and $\mathbb{D}$, respectively, thus yielding a homeomorphic mapping Φ (see Theorem 2.1.2) between $Q(\overline{\mathbb{D}})$ and $Q(\overline{G})$. Furthermore, since F and F^{-1} are homeomorphisms (on either preimage set), the mapping Φ extends to a homeomorphism between $\mathcal{H}(\overline{\mathbb{D}})$ and $\mathcal{H}(\overline{G})$. Since $Q(\overline{\mathbb{D}})$ is not closed in $\mathcal{H}(\overline{\mathbb{D}})$, the corresponding homeomorphic image $Q(\overline{G}) = \Phi(Q(\overline{\mathbb{D}}))$ is not closed in $\mathcal{H}(\overline{G})$ as well. □

Corollary 2.5.6.
For $G \in \mathcal{JD}$, the topological group $Q(\overline{G})$ is not Polish in the uniform topology induced by d_{sym}.

Proof. Proposition 2.5.3 implies that $\mathcal{H}(\overline{G})$ is in fact a Polish group in the uniform topology by means of the symmetric supremum d_{sym}. By Propositions 2.5.4 and 2.5.5, the topological (sub)group $Q(\overline{G}) \leq \mathcal{H}(\overline{G})$ is not Polish in the uniform topology, since it is not closed in $\mathcal{H}(\overline{G})$. □

Remark 2.5.7.

(i) *The statement of Corollary 2.5.6 can be interpreted as follows: For $G \in \mathcal{JD}$, the topological group $Q(\overline{G})$ is no Polish group in the subspace topology of $\mathcal{H}(\overline{G})$, which is the uniform topology. Consequently, the topological space $Q(\overline{G})$ in the uniform topology is not Polish by Definition 2.5.1(ii). In turn, this means by definition that $Q(\overline{G})$ is either not separable or not completely metrizable (or both). But since $Q(\overline{G})$ is a separable space – this follows from the proof of Theorem 3.1.4 (and the fact that d_{sup} and d_{sym} both induce the same topology) – the complete metrizability must be violated. Hence, every metric defined on $Q(\overline{G})$ that induces the uniform topology will inevitably result in an incomplete metric space.*

[7] For reasons of readability and chapter arrangement of this thesis, the mentioned Corollary 4.1.25 is stated and proved only after being used in the proof of Theorem 2.5.5.

*(ii) There exists a vast literature on Polish groups and related topics, see [Kec95, Chapter I, Section 9], [Man16], [Mel16] and [Ros08] as well as the references contained therein. Among many others, the following result – called the **automatic continuity property** – was shown in this context by Rosendal (see [Ros08, Theorem 1.1, p. 351]):*

Let M be a compact 2–manifold[8] and $\pi : \mathcal{H}(M) \longrightarrow H$ be a group homomorphism into a separable topological group H. Then π is continuous when $\mathcal{H}(M)$ is endowed with the topology of uniform[9] convergence.

Furthermore, the cited statement of Rosendal was later shown by Mann to be valid for compact manifolds of any (finite) dimension, possibly with boundary, see [Man16, Theorem 1.2, p. 3034]. This result combined with Proposition 2.5.3 yields for example that the extended mapping $\Phi : \mathcal{H}(\overline{\mathbb{D}}) \longrightarrow \mathcal{H}(\overline{G})$ – which is clearly a group isomorphism – in the proof of Theorem 2.5.5 is automatically a homeomorphism without utilizing any further continuity considerations.

[8] In this context, a manifold is always assumed to be a metric space, see [Man16, pp. 3035–3036] and [Ros08, p. 354].

[9] Actually, the cited statement is formulated for the compact–open topology, but consider the abstract and the remark on p. 350 in [Ros08].

Chapter 3

Topological properties of $Q(G)$

This chapter is concerned with various topological properties of the space $Q(G)$ under the general prerequisite that the underlying domain $G \subsetneq \mathbb{C}$ is bounded and simply connected. Hence, throughout the current chapter, G *will always refer to a bounded and simply connected domain* in $\mathbb{C}$, unless the contrary is explicitly stated.

Many of the results to be shown in this chapter are formulated for the class of domains G possessing sufficiently well–behaved boundary structure, in particular for the "most regular" case $G \in \mathcal{JD}$, or – more generally – domains with solely prime ends of the first kind, i.e. $\mathcal{P}(G) = \mathcal{P}_1(G)$. As will become clear, numerous of the presented results are similar or even equivalent to the case of $\Sigma(G)$ presented in Section 1.2, but also striking differences will become evident, e.g. regarding local compactness of $Q(G)$. Moreover, some of the results to be presented yield immediate conclusions and applications for $\Sigma(G)$, for example compactness criteria for certain subsets of $Q(G)$. In greater detail, the following topological properties of $Q(G)$ will be treated in the current chapter:

(1) Section 3.1 is concerned with the question for when the topological space $Q(G)$ is separable. Using a helpful statement on the hereditary property of separability in the context of metric spaces, one necessary and one sufficient condition is derived, thereby establishing the analogous characterization of separability for $Q(G)$ as it was found for the (sub)space $\Sigma(G)$.
(2) Section 3.2 of the current chapter treats the question for the existence of dense subsets and the discreteness of $Q(G)$, providing an answer to this question in Theorem 3.2.5.
(3) The Baire space property and local compactness of the space $Q(G)$ are the subjects of investigation in Section 3.3. Except for the "regular case" of G having only prime ends of the first kind, local compactness used to be challenging to identify for the spaces $\Sigma(G)$. By examining the topological properties of the subsets $Q_K(G)$ regarding their inner points, a terminal answer can be given for local compactness and the Baire space property of $Q(G)$.
(4) Utilizing the continuous dependence of solutions of the Beltrami equation on certain parameter values stated in Proposition 1.1.5, the path–connectedness of the space $Q(G)$ for $\mathcal{P}(G) = \mathcal{P}_1(G)$ is established in Section 3.4. Moreover, a certain "transfer" of path–connectedness from the subspaces $Q_K(G)$ for $K > 1$ to the ambient space $Q(G)$ is shown.
(5) The Section 3.5 is then concerned with compact subsets of $Q(G)$ and the related concept of σ–compactness. One necessary and one sufficient condition for $M \subsetneq Q(G)$ to be compact are derived. Moreover, a compactification procedure related to the subsets $Q_K(G)$ is shown, and the σ–compactness of $Q(G)$ is proved. All of these results are subject to the regularization assumptions $G \in \mathcal{JD}$ or $\mathcal{P}(G) = \mathcal{P}_1(G)$.
(6) Finally, Section 3.6 focuses on multiply connected domains and several of the before–mentioned topological properties of the corresponding quasiconformal automorphism groups.

3.1 Separability

A separable space may be thought of being "small" from a certain topological point of view. One particularly useful property in the context of separable metric spaces is that the space's dense subsets are passed through to subspaces (see e.g. [Wil70, p. 114]):

Proposition 3.1.1.
If (X, d) is a separable metric space and $A \subseteq X$, then the subspace $(A, d\big|_A)$ is separable as well.

Proposition 3.1.1 will be used several times throughout this thesis, therefore it is cited as a Proposition in its own right for the sake of completeness, even though its statement is an elementary result of General Topology.

3.1.1 Necessary condition

Proposition 1.2.1(iii) states that separability of the metric space $\Sigma(G)$ is equivalent to G possessing solely prime ends of the first kind. This knowledge combined with Proposition 3.1.1 immediately yields (see also [BL23, Theorem 6])

Theorem 3.1.2.
If $Q(G)$ is separable, then $\mathcal{P}(G) = \mathcal{P}_1(G)$.

3.1.2 Sufficient condition

In order to derive a sufficient condition for the separability of $Q(G)$, the following result related to Functional Analysis and General Topology will be helpful (see [Con90, Theorem 6.6, p. 140]):

Proposition 3.1.3.
Let X be a completely regular topological space and denote by $C_b(X)$ the Banach space of all bounded continuous (real– or complex–valued) functions endowed with the supremum norm $\|\cdot\|_{\sup}$. Then $C_b(X)$ is separable if and only if X is a compact metric space.

Here, a topological space X is called *completely regular* if it is a Hausdorff space with the additional property that for every closed set $A \subseteq X$ and every point $x \in X \backslash A$, there exists a continuous function $f : X \longrightarrow \mathbb{R}$ such that $f(x) = 1$ and $f\big|_A \equiv 0$ (see [Sin19, Definition 8.3.1, p. 196]); a completely regular space is occasionally also called a *Tychonoff space*. Metric spaces are always completely regular, thus Proposition 3.1.3 applies to domains in $\mathbb{C}$. The Banach space $C_b(G)$ contains $Q(G)$ as a subset, hence the statement of Proposition 3.1.3 in combination with Proposition 3.1.1 paves the way for deriving the announced sufficient separability criterion for $Q(G)$ (see also [BL23, Theorem 6]):

Theorem 3.1.4.
If G has only prime ends of the first kind, then $Q(G)$ is separable.

Proof. First, the case $G = \mathbb{D}$ will be treated. By Proposition 1.1.6, each $f \in Q(\mathbb{D})$ extends homeomorphically to $\overline{\mathbb{D}}$, therefore $Q(\mathbb{D})$ embeds into $C(\overline{\mathbb{D}})$ via $Q(\overline{\mathbb{D}})$ (see Definition 1.1.7). Since $\overline{\mathbb{D}}$ is a compact metric space, Proposition 3.1.3 gives that $C(\overline{\mathbb{D}}) = C_b(\overline{\mathbb{D}})$ is separable. Due to $Q(\overline{\mathbb{D}}) \subsetneq C(\overline{\mathbb{D}})$, Proposition 3.1.1 implies that the metric subspace $Q(\overline{\mathbb{D}})$ is separable. Now the separability of $Q(\mathbb{D})$ can be concluded by restricting each extended mapping $f \in Q(\overline{\mathbb{D}})$ back to $\mathbb{D}$. In the general case for G with $\mathcal{P}(G) = \mathcal{P}_1(G)$, the separability of $Q(G)$ follows from Theorem 2.1.2, since $Q(G)$ is the continuous image under Φ of the separable space $Q(\mathbb{D})$. □

Remark 3.1.5.
The combination of the statements of Theorem 3.1.2 and Theorem 3.1.4 yields exactly the same characterization for the separability of $Q(G)$ as it was found for $\Sigma(G)$ by Volynec in [Vol92], see Proposition 1.2.1(iii).

Since the theoretical question for the occurrence of separability of $Q(G)$ is answered in its entirety by Theorems 3.1.2 and 3.1.4, the more constructive question for concrete countable subsets lying dense in the corresponding quasiconformal automorphism groups arises. In the conformal special case, i.e. for $\Sigma(G)$, this question can be answered fairly easily: The space $\Sigma(\mathbb{D})$ is clearly separable (see Proposition 1.2.1(iii)), and a concrete countable dense subset is given by

$$S = \left\{ \mathbb{D} \ni z \longmapsto e^{i\alpha} \frac{z-a}{1-\overline{a}z} \;\middle|\; \alpha \in [0,2\pi] \cap \mathbb{Q},\, a \in \mathbb{D} \cap (\mathbb{Q} + i\mathbb{Q}) \right\} \subsetneq \Sigma(\mathbb{D})$$

which is ultimately due to Proposition 1.1.10 combined with the fact that $\mathbb{Q}$ is countable and dense in $\mathbb{R}$. A countable dense subset for $\Sigma(G)$ with $\mathcal{P}(G) = \mathcal{P}_1(G)$ is then given by the image of S under the (restriction of the) conjugation mapping $\Phi : \Sigma(\mathbb{D}) \longrightarrow \Sigma(G)$ (see Proposition 2.1.1). As for the situation of $Q(G)$, this approach is not possible since no explicit countable dense subset of $Q(\mathbb{D})$ is known, leading to ask the

Question 3.1.6.
For G with solely prime ends of the first kind, what are the dense countable subsets of $Q(G)$?

3.2 Discreteness and dense subsets

This section is concerned with one of the central questions Gaier studied in his paper [Gai84] for $\Sigma(G)$, namely for discreteness of the spaces under consideration. In this regard, approximation results for quasiconformal mappings are applied and dense subsets of quasiconformal automorphism groups will be studied.

3.2.1 Discreteness in $Q(G)$

The following result characterizes the discreteness of $Q(G)$ by using its group structure (see [Bie17, Theorem 2.52, p. 80]):

Proposition 3.2.1.
The space $Q(G)$ is discrete if and only if id_G is an isolated element in $Q(G)$, i.e. there exists $\epsilon > 0$ such that $Q(G) \cap B_\epsilon(\mathrm{id}_G) = \{\mathrm{id}_G\}$, where $B_\epsilon(\mathrm{id}_G) := \{f \in Q(G) \mid d_{\sup}(\mathrm{id}_G, f) < \epsilon\}$ denotes the open ϵ–ball about id_G in $Q(G)$.

The statement of Proposition 3.2.1 is of course also valid in the subspace $\Sigma(G)$. The following result is an first necessary condition for $Q(G)$ in order to be discrete:

Lemma 3.2.2.
If $Q(G)$ is a discrete space, then $\Sigma(G)$ is also discrete.

Proof. By Proposition 3.2.1, the identity mapping is isolated in $Q(G)$. In particular, there exists an open ball around id_G containing no non–identity conformal automorphisms. Hence, $\Sigma(G)$ is discrete as well. □

Discreteness of $\Sigma(G)$ was studied by several authors, see e.g. [Gai84, Satz 7b, p. 237] and [Sch86, Corollary, p. 202], hence – similar to the situation concerning local compactness in $Q(G)$, see Theorem 3.3.1 – the statement of Lemma 3.2.2 provides a simple but effective tool in order to decide whether $Q(G)$ can be discrete. For example, it is immediately clear that $Q(G)$ is not discrete if $\mathcal{P}(G) = \mathcal{P}_1(G)$.

3.2.2 Diffeomorphic approximation of quasiconformal mappings

In the remainder of the current section, the following classical notions will be utilized: Let $G, G' \subseteq \mathbb{C}$ be domains. A homeomorphism $f : G \longrightarrow G'$ is called *(continuously) differentiable* at $z \in G$ if the real and imaginary parts of f are (continuously) differentiable at z in the common sense (see [LV73, Section 1.6, p. 9]). A *diffeomorphism* is a (orientation–preserving) homeomorphism $f : G \longrightarrow G'$ such that both, f and its inverse mapping f^{-1}, are continuously differentiable throughout their respective domains (see [Leh87, Section 3.1, p. 18]). A diffeomorphism f is called C^∞*–diffeomorphism* if f and its inverse f^{-1} are differentiable infinitely often. Denote by

$$\mathrm{Diff}^\infty(G) := \left\{ f : G \longrightarrow G \;\middle|\; f \text{ is a } C^\infty\text{–diffeomorphism of } G \text{ onto itself} \right\} \tag{3.1}$$

the set of all C^∞–diffeomorphisms of G onto itself, i.e. the *diffeomorphism group* of G. The following remarkable result of Kiikka is concerned with the approximation of quasiconformal mappings by C^∞–diffeomorphisms which are at the same time quasiconformal mappings (see [Kii83, Theorem 1, p. 252]):

Proposition 3.2.3.
Let $G, G' \subseteq \mathbb{C}$ be domains and let $f : G \longrightarrow G'$ be a K–quasiconformal mapping. Furthermore, suppose $\epsilon : G \longrightarrow (0, +\infty)$ is continuous. Then there exists $\widetilde{K} \geq 1$ and a $\widetilde{K}$–quasiconformal C^∞–diffeomorphism $\widetilde{f} : G \longrightarrow G'$ such that $\left| f(z) - \widetilde{f}(z) \right| < \epsilon(z)$ for every $z \in G$.

Remark 3.2.4.
The formulation of Proposition 3.2.3 is already adapted to the case of quasiconformal mappings in $\mathbb{C}$. The original statement in Kiikka's paper [Kii83] deals with the situation of quasiconformal mappings in the real Euclidean spaces $\mathbb{R}^2 \cong \mathbb{C}$ and $\mathbb{R}^3$. In the case of $\mathbb{R}^2$, however, the definition of quasiconformality is equivalent to the usual definition for this class of mappings in $\mathbb{C}$ given in Definition 1.1.1. Furthermore, the original statement in Kiikka's paper uses a somewhat strange "order" of prerequisites: First, she fixes a maximal dilatation $K \geq 1$, and then claims the existence of a corresponding number $\widetilde{K} \geq 1$. Afterwards, she specifies the domains G, G', the K–quasiconformal mapping $f : G \longrightarrow G'$ and the continuous mapping $\epsilon : G \longrightarrow (0, +\infty)$, and not until then the existence of the quasiconformal C^∞–mapping $\widetilde{f}$ is announced. This type of formulation is a bit misleading, since of course the domains G and G' as well as the mapping ϵ are by no means related to the fixture of a maximal dilatation K, the corresponding mapping f or the existence of another maximal dilatation $\widetilde{K}$. Therefore, the formulation given in Proposition 3.2.3 was chosen for this thesis.

3.2.3 Denseness of $C^\infty Q(G)$ in $Q(G)$ and application to quasiregular mappings

In view of $Q(G)$, the following central result can be concluded from Kiikka's above–mentioned approximation statement in Proposition 3.2.3:

Theorem 3.2.5.
Let $G \subseteq \mathbb{C}$ be a bounded, simply connected domain. Then the set

$$C^\infty Q(G) := Q(G) \cap \mathrm{Diff}^\infty(G) \tag{3.2}$$

is dense in the space $Q(G)$.

Proof. Let $f \in Q(G)$ and $\epsilon > 0$. The constant mapping

$$\widetilde{\epsilon} : G \longrightarrow (0, +\infty),\; z \longmapsto \widetilde{\epsilon}(z) := \frac{\epsilon}{2}$$

is clearly continuous. By Proposition 3.2.3, there exists a number $\widetilde{K} \geq 1$ and a $\widetilde{K}$–quasiconformal homeomorphism $\widetilde{f} \in Q(G)$ which is at the same time a C^∞–diffeomorphism satisfying

$$\left|f(z) - \widetilde{f}(z)\right| < \widetilde{\epsilon}(z) = \frac{\epsilon}{2}$$

for all $z \in G$. In particular, it is $\widetilde{f} \in C^\infty Q(G)$. Switching to the supremum over all $z \in G$ yields

$$\sup_{z \in G} \left|f(z) - \widetilde{f}(z)\right| = d_{\sup}(f, \widetilde{f}) \leq \frac{\epsilon}{2} < \epsilon$$

This implies that the open ball $B_\epsilon(f)$ about f in $Q(G)$ contains an element of $C^\infty Q(G)$, namely $\widetilde{f}$, which is equivalent to the denseness of this set in the space $Q(G)$. □

This result now immediately implies (see also [BL23, Theorem 12])

Corollary 3.2.6 ($Q(G)$ is never discrete)**.**
The identity mapping id_G *is never isolated in* $Q(G)$*. In particular,* $Q(G)$ *is never discrete for every bounded, simply connected domain* $G \subsetneq \mathbb{C}$*.*

Proof. Let $f \in Q(G)$. By Theorem 3.2.5 there exists a sequence $(\widetilde{f}_n)_{n \in \mathbb{N}} \subseteq C^\infty Q(G)$ of quasiconformal C^∞–diffeomorphisms of G converging uniformly on G to f; without loss of generality, assume that f itself is not a C^∞–diffeomorphism, for otherwise the sequence $(\widetilde{f}_n)_n$ can be chosen (eventually) constant. Thus the sequence $(\widetilde{f}_n)_n$ is not (eventually) constant. By the isometry of right multiplication in $Q(G)$ (see Proposition 1.3.6), the sequence $(\widetilde{f}_n \circ f^{-1})_n \subseteq Q(G)$ converges uniformly on G to the identity mapping id_G:

$$d_{\sup}(\widetilde{f}_n \circ f^{-1}, \mathrm{id}_G) = d_{\sup}(\widetilde{f}_n, f) \overset{n\to\infty}{\longrightarrow} 0$$

Therefore id_G is not isolated in $Q(G)$. Proposition 3.2.1 yields that $Q(G)$ is not discrete. □

Remark 3.2.7.

(i) In the results of the current subsection, actually no restrictions were made concerning the connectivity or the boundary of the domain G except for the proof of Corollary 3.2.6, where the $d_{\sup}$*–isometry of right multiplication in* $Q(G)$ *was used – but this property does not depend on the quasiconformality of the elements of* $Q(G)$*, and is valid for arbitrary elements of* $\mathcal{H}(G)$*, as can be seen as in [Gai84, Proof of Hilfssatz 4, p. 234]. Thus, the results of the current subsection are not restricted to simply connected domains and their boundary regularity. This is a somewhat unexpected discovery when considering the corresponding situation in the conformal case. Among other results for* $\Sigma(G)$*, it is known that*

- $\Sigma(G)$ *is always finite if G has connectivity* ≥ 3*, and is therefore always discrete, see [Gai84, § 4, Bemerkung 2];*
- $\Sigma(G)$ *is discrete if G is a so–called "comb domain of the first kind", see [Gai84, Satz 9, p. 254].*

Thus, domains of the above–mentioned types give concrete examples for the situation that the metric (sub)space $\Sigma(G)$ *is discrete, whereas the metric (super)space* $Q(G)$ *is not discrete. Furthermore, Corollary 3.2.6 states that the situation described in Lemma 3.2.2 can in fact never occur – in other words, this necessary condition is actually no condition at all.*

(ii) In a certain sense, Corollary 3.2.6 sets the final point to the problem possed by Gaier in [Gai84, p. 227] of whether $\Sigma(G)$ *is always non–discrete. As already shown by Gaier himself, the space* $\Sigma(G)$ *can be discrete ([Gai84, Satz 9, p. 254]), therefore his original question stated in the conformal setting is surely to be answered negatively. However, Corollary 3.2.6 demonstrates that in the quasiconformal setting, this situation cannot occur - the space* $Q(G)$ *is always capable of approximating* id_G *arbitrarily well, regardless of the nature of* ∂G*.*

In order to give an application of the non–discreteness of $Q(G)$ to quasiregular mappings in the context of universality, a few notions and results are required:

(1) Denote by $\mathrm{Hol}(G)$ the set of all holomorphic functions on G. If $\mathrm{Hol}(G)$ is endowed with the topology of locally uniform convergence, the setting in which it is usually studied in Complex Analysis, it is a metrizable topological space.
(2) A mapping $f \in C(G)$ is called **quasiregular** if it can be represented as $f = g \circ \phi$ with $g \in \mathrm{Hol}(G)$, $g \not\equiv$ const., and $\phi \in Q(G)$ (see [LV73, Definition, p. 239][1]). Let $QR(G)$ denote the set of all quasiregular mappings on G.
(3) A function $f \in \mathrm{Hol}(G)$ is called **universal** if the set

$$\left\{ f \circ \sigma \,\middle|\, \sigma \in \Sigma(G) \right\}$$

is dense in $\mathrm{Hol}(G)$ w.r.t. locally uniform convergence. For basic information on universal functions in $\mathbb{C}$, see [Poh19, pp. 9–11] and the references cited therein.
(4) Classical results show that there exists a universal function for every simply connected domain G in $\mathbb{C}$, i.e. there is a function $g^* \in \mathrm{Hol}(G)$ such that for every $g \in \mathrm{Hol}(G)$ there exists a sequence $(\sigma_n)_n$ in $\Sigma(G)$ such that $g^* \circ \sigma_n \overset{n\to\infty}{\longrightarrow} g$ locally uniformly in G.

Now let $f \in QR(G)$ be quasiregular with $f = g \circ \phi$. Due to Corollary 3.2.6, there exists a sequence $(\phi_n)_n$ in $Q(G)$ such that $d_{\sup}(\phi_n, \phi) \overset{n\to\infty}{\longrightarrow} 0$, and thus, in particular, ϕ_n converges locally uniformly to ϕ in G. By (4), there exists a sequence $(\sigma_n)_n$ in $\Sigma(G)$ with $g^\star \circ \sigma_n \overset{n\to\infty}{\longrightarrow} g$ locally uniformly in G, where $g^* \in \mathrm{Hol}(G)$ refers to a universal function on G. Using [RS02, Kompositionssatz 3.1.5, p. 88], it follows that

$$g^* \circ \sigma_n \circ \phi_n \overset{n\to\infty}{\longrightarrow} g \circ \phi = f$$

locally uniformly in G as well (after possibly switching to a subsequence of $(\sigma_n)_n$ or $(\phi_n)_n$, respectively). Clearly, it is $\psi_n := \sigma_n \circ \phi_n \in Q(G)$ for each $n \in \mathbb{N}$, therefore the sequence $(g^* \circ \psi_n)_n$ in $QR(G)$ converges to f locally uniformly. Hence, one finally arives at

Corollary 3.2.8.
For every bounded, simply connected domain G in $\mathbb{C}$, there exists $g^* \in \mathrm{Hol}(G)$ such that the set $\left\{ g^* \circ \psi \,\middle|\, \psi \in Q(G) \right\}$ is dense in $QR(G)$ in the topology of locally uniform convergence.

3.3 Baire space property and local compactness

This section considers two interesting aspects concerning the topology of $Q(G)$: The Baire space property and local compactness. Local compactness is interesting to ask for $Q(G)$ inasmuch as the quasiconformal automorphism groups are never compact, independent of the underlying domain G (Proposition 1.3.1(i)). When examining the corresponding situation in $\Sigma(G)$ concerning local compactness and the development of its characterization (see especially [Lau95] and [Lau99]), one might be led to the thought that deriving according results for $Q(G)$ may be a difficult task. However, by studying the topological properties of the subspaces $Q_K(G)$, a general answer for the local compactness of the space $Q(G)$ can be given – among other results to be presented in this section. Asking for the Baire space property, i.e. whether $Q(G)$ can be considered a "*thick*" set from a certain topological point of view (see [Wil70, p. 185]), is interesting due to the incompleteness of $Q(G)$ as shown in the previous chapter.

Similar to the case of separability (see Theorem 3.1.2), one simple necessary condition for $Q(G)$ being locally compact may easily be derived by utilizing the knowledge on the corresponding situation in $\Sigma(G)$:

[1]Without loss of generality, one may assume $\phi \in Q(G)$.

Theorem 3.3.1.
If $Q(G)$ is locally compact, then $\Sigma(G)$ is locally compact as well.

Proof. Assume $Q(G)$ is locally compact. Proposition 1.3.1(ii) gives that $\Sigma(G)$ is closed in $Q(G)$. By a well–known result of General Topology, a closed subspace of a locally compact topological space is itself locally compact (see e.g. [Kel75, p. 146]). □

3.3.1 Empty interior of the subsets $Q_K(G)$ and meagerness of $Q(G)$

In order to draw conclusions on the topological structure of $Q(G)$ concerning the Baire space property, a certain knowledge about the subspaces $Q_K(G)$ is required. Therefore, consider the square

$$R := \left\{ z = x + iy \in \mathbb{C} \,\middle|\, 0 < x, y < 1 \right\},$$

and the diagonal sequence $(f_{m,n_m})_{m\in\mathbb{N}}$ from $Q(R)$ as defined in Example 1.1.11. This sequence has unbounded maximal dilatation, i.e. $K(f_{m,n_m}) \longrightarrow +\infty$ for $m \to +\infty$, even though it converges uniformly on R to id_R. Furthermore, it is $f_{m,n_m} = \mathrm{id}_{\partial R}$ on ∂R for all $m \in \mathbb{N}$ by construction, see [LV73, p. 186]. Let $G \subsetneq \mathbb{C}$ be an arbitrary bounded, simply connected domain in $\mathbb{C}$ with fixed inner point $z_0 \in G$. Via conformal equivalence, the square R corresponds to an (open) square $\widetilde{R} \subsetneq G$ centered at z_0 (with sides parallel to the coordinate axes) that is completely contained in G. The mappings $f_{m,n_m} \in Q(R)$ are consequently transferred to mappings $v_m \in Q(\widetilde{R}), m \in \mathbb{N}$, as well. For $z \in G$, consider the mappings

$$h_m(z) := \begin{cases} v_m(z), & z \in \widetilde{R}, \\ \mathrm{id}_G(z), & z \in G \backslash \widetilde{R} \end{cases}$$

which yields a sequence of quasiconformal automorphisms of G, i.e. $h_m \in Q(G)$ for all $m \in \mathbb{N}$. By construction[2], the sequence h_m converges uniformly to id_G on G and $K(h_m) = K(f_{m,n_m}) \longrightarrow +\infty$ as $m \to +\infty$ (see [BF14, Property P2, p. 31]).

Next, let $g \in Q(G)$ be an arbitrary element of $Q(G)$, then $g \in Q_K(G)$ for some $K \in [1, +\infty) \cap \mathbb{N}$ by the countable representation

$$Q(G) = \bigcup_{K=1}^{\infty} Q_K(G),$$

see (0.6). Due to the $d_{\sup}$–isometry of right multiplication in $Q(G)$ (see Proposition 1.3.6), it follows that

$$d_{\sup}(h_m \circ g, g) = d_{\sup}(h_m, \mathrm{id}_G) < \epsilon \tag{3.3}$$

for every $\epsilon > 0$ and sufficiently large indices $m \geq N = N(\epsilon) \in \mathbb{N}$. The maximal dilatation of the composed mappings $h_m \circ g$ is also unbounded as $m \to +\infty$, as can be seen by considering

$$K(h_m) = K(h_m \circ g \circ g^{-1}) \leq K(h_m \circ g)K(g)$$

and using $K(g^{-1}) = K(g) < +\infty$. This leads to the following result:

Lemma 3.3.2.
Let $\epsilon > 0$ and $g \in Q(G)$. Then the open ϵ–ball $B_\epsilon(g)$ about g contains elements with arbitrarily large maximal dilatations, i.e. $B_\epsilon(g) \notin Q_K(G)$ for each fixed $K \in \mathbb{N}$. In particular, the subsets $Q_K(G)$ have empty interior.

[2] This idea is similar to the construction used in the incompleteness proof of $Q(G)$, see Theorem 2.3.3.

Moreover, since each subset $Q_K(G)$ is closed in $Q(G)$ (see Theorem 2.2.1), the countable representation of the space $Q(G)$ by the subsets $Q_K(G)$ yields

Corollary 3.3.3.
Each set $Q_K(G)$ is nowhere dense in $Q(G)$. In particular, the topological space $Q(G)$ is meager, i.e. of the first category (in itself).

By combining the statement of Corollary 3.3.3 with the fact that a Baire space is always a set of the second category (see e.g. [Sin19, p. 231]), one arrives at the following result:

Theorem 3.3.4.
The topological space $Q(G)$ is never a Baire space.

Remark 3.3.5.

(i) The first statement of Corollary 3.3.3 seems quite surprising at first glance when recalling that an equivalent formulation for $Q_K(G)$ being nowhere dense in $Q(G)$ can be given by considering its complement (see e.g. [Sin19, p. 230]):

$$Q_K(G) \text{ is nowhere dense in } Q(G) \iff Q(G)\backslash Q_K(G) \text{ is dense in } Q(G).$$

Hence, the complement set

$$Q_K(G)^{\mathrm{C}} = \{f \in Q(G) \mid K(f) > K\}$$

is dense and open in $Q(G)$ for every fixed $K \in [1, +\infty)$. By analyzing the derivation of Lemma 3.3.2, this result becomes more clear, since each open ϵ–ball about $g \in Q(G)$ contains infinitely many elements belonging to $Q_K(G)^{\mathrm{C}}$, see (3.3).

(ii) Furthermore, the reasoning in (i) provides:

- *Another way of seeing that $Q(G)$ is no Baire space: As seen above, each subset $Q_K(G)^{\mathrm{C}}$ is open and dense in $Q(G)$. However, for $K \in \mathbb{N}$, the countable intersection*

$$\bigcap_{K=1}^{\infty} Q_K(G)^{\mathrm{C}}$$

  is actually empty, since the maximal dilatation of an element of this intersection would be larger than any finite natural number, thus would be infinite. But if $Q(G)$ were a Baire space, this intersection would necessarily have to be dense in $Q(G)$;
- *Another proof that $Q(G)$ is never discrete (as already shown in Corollary 3.2.6), since the subsets $Q_K(G)^{\mathrm{C}}$ are dense in $Q(G)$.*

3.3.2 Conclusions on local compactness and complete metrizability of $Q(G)$

Due to the far–reaching consequences of the Baire space concept in topology, the statement of Theorem 3.3.4 allows for deriving further, quite interesting properties: On the one hand, the space $Q(G)$ is Hausdorff, since its topology is induced by the d_{sup} metric. A well–known statement on Baire spaces is that a locally compact Hausdorff topological space is a Baire space (see e.g. [Kec95, Theorem (8.4), p. 41]). The contraposition of this statement in combination with the Hausdorff property of $Q(G)$ yields

Theorem 3.3.6.
The topological space $Q(G)$ is never locally compact.

On the other hand, another classical result on Baire spaces is that completely metrizable topological spaces are Baire. Considering the corresponding contraposition gives

Corollary 3.3.7.
The topological space $Q(G)$ is not completely metrizable, i.e. there exists no metric that induces the uniform topology on $Q(G)$ and turns it into a complete metric space. In particular, the space $Q(G)$ is never Polish.

3.4 Path–connectedness

3.4.1 Sufficient condition

This subsection is intended to provide a proof of the following statement (see also [BL23, Theorem 10]):

Theorem 3.4.1.
Let G be a domain having only prime ends of the first kind. Then $Q(G)$ is path–connected.

This statement is the corresponding counterpart for $Q(G)$ to a result of Schmieder[3] concerning the situation in $\Sigma(G)$, see [Sch86] and Proposition 1.2.1(iv). For the proof of Theorem 3.4.1, the following Lemma about paths in $Q(G)$ between quasiconformal automorphisms and their post–compositions with conformal mappings will be helpful (see also [BL23, Lemma 7]):

Lemma 3.4.2.
Let G be a domain having only prime ends of the first kind. Then for every $f \in Q(G)$ and every $\sigma \in \Sigma(G)$, the mappings f and $\sigma \circ f$ can be joined by a path in $Q(G)$.

Proof. By Proposition 1.2.1(iv), $\Sigma(G)$ is path–connected for $\mathcal{P}(G) = \mathcal{P}_1(G)$, therefore id_G and σ can be joined by a path in $\Sigma(G)$, say $\gamma : [0,1] \longrightarrow \Sigma(G)$. Thus the mapping

$$\widetilde{\gamma} : [0,1] \longrightarrow Q(G),\ t \longmapsto \widetilde{\gamma}(t) := \gamma(t) \circ f$$

clearly connects f and $\sigma \circ f$ in $Q(G)$ and is continuous, for let $t_n \longrightarrow t$ in $[0,1]$, then

$$d_{\sup}(\widetilde{\gamma}(t_n), \widetilde{\gamma}(t)) = d_{\sup}(\gamma(t_n) \circ f, \gamma(t) \circ f) = d_{\sup}(\gamma(t_n), \gamma(t)) \longrightarrow 0$$

as $n \longrightarrow \infty$ by the continuity of γ and the isometry property of right multiplication in $Q(G)$ (Proposition 1.3.6). Hence $\widetilde{\gamma}$ is a path in $Q(G)$ joining f and $\sigma \circ f$. □

Proof of Theorem 3.4.1. The proof will be split in two parts: First, the case $G = \mathbb{D}$ will be considered. Afterwards the conjugation mapping $\Phi : Q(\mathbb{D}) \longrightarrow Q(G)$ will provide the proof of the claimed statement.

Let $f, g \in Q(\mathbb{D})$ with complex dilatations $\mu_f, \mu_g \in \mathbb{B}_{L^\infty}(\mathbb{D})$. Without loss of generality, assume that $\mu_f \neq \mu_g$; otherwise, the claim follows from Lemma 3.4.2, since then $g = \sigma \circ f$ for some $\sigma \in \Sigma(\mathbb{D})$ by the Measurable Riemann Mapping Theorem 1.1.2(II). For a fixed $z^* \in \mathbb{D}$, the following normalization according to Definition 1.1.3 will be introduced: By the classical Riemann Mapping Theorem, there exists a uniquely defined conformal automorphism $\sigma_f \in \Sigma(\mathbb{D})$ such that $(\sigma_f \circ f)(0) = 0$ and $(\sigma_f \circ f)(z^*) \in \mathbb{R}^+$, and the complex dilatation remains unchanged in this normalization process, i.e. $\mu_{\sigma_f \circ f} = \mu_f$ almost everywhere in $\mathbb{D}$ (the same normalization is applied to g with $\sigma_g \in \Sigma(\mathbb{D})$). Lemma 3.4.2 provides paths in $Q(\mathbb{D})$ joining f with $\sigma_f \circ f$ and g with $\sigma_g \circ g$, respectively, say $\gamma_f : [0,1] \longrightarrow Q(\mathbb{D})$ and $\gamma_g : [0,1] \longrightarrow Q(\mathbb{D})$. Next, let $\Lambda \subseteq \mathbb{R}$ be an open interval containing $[0,1]$, i.e. $\Lambda = (-a, 1+a)$ for some fixed $a > 0$. In $L^\infty(\mathbb{D})$, consider the straight line

$$\gamma : [0,1] \longrightarrow L^\infty(\mathbb{D}),\ t \longmapsto \gamma(t) := t\mu_g + (1-t)\mu_f$$

which clearly is a path in $\mathbb{B}_{L^\infty}(\mathbb{D})$ joining the complex dilatations of f and g (any open ball in a normed vector space is convex, thus $\gamma(t) \in \mathbb{B}_{L^\infty}(\mathbb{D})$ for every $t \in [0,1]$). Construct a new path Γ

[3] Actually, this result on the path–connectedness for $\Sigma(G)$ in the case $\mathcal{P}(G) = \mathcal{P}_1(G)$ was already implicitly stated in Gaier's work in the proof of [Gai84, Satz 7a, p. 237], even though he did not mention it.

in $\mathbb{B}_{L^\infty}(\mathbb{D})$ as follows:

$$\Gamma : \Lambda \longrightarrow \mathbb{B}_{L^\infty}(\mathbb{D}),\ t \longmapsto \Gamma(t) := \mu_t := \begin{cases} \mu_f, & t \in (-a, 0) \\ \gamma(t), & t \in [0,1] \\ \mu_g, & t \in (1, 1+a) \end{cases}$$

Hence Γ is a path in $\mathbb{B}_{L^\infty}(\mathbb{D})$ joining μ_f and μ_g, consisting of two constant pieces and a non–constant part which is γ. Since both complex dilatations are elements of $\mathbb{B}_{L^\infty}(\mathbb{D})$, there exists a number $k < 1$ such that $\|\mu_t\|_{L^\infty(\mathbb{D})} \leq k$ for every $t \in \Lambda$. Consequently, the family $(\mu_t)_{t\in\Lambda}$ depends continuously on t and has uniformly bounded L^∞–norm. Apply the Measurable Riemann Mapping Theorem 1.1.2 in order to find for each $t \in \Lambda$ the unique normalized $\phi_t \in Q(\mathbb{D})$ whose complex dilatation coincides with μ_t almost everywhere in $\mathbb{D}$. By construction of Γ, it is $\phi_t = \sigma_f \circ f$ for $t \in (-a, 0)$ and $\phi_t = \sigma_g \circ g$ for $t \in (1, 1+a)$. This process induces a mapping

$$H : \Lambda \longrightarrow Q(\mathbb{D}),\ t \longmapsto H(t) := \phi_t$$

and $(\phi_t)_{t\in\Lambda} \subseteq Q(\mathbb{D})$ is a family of K–quasiconformal automorphisms of $\mathbb{D}$ for some finite K due to the uniform boundedness of $\|\mu_t\|_{L^\infty(\mathbb{D})}$ for all $t \in \Lambda$. Now, for each $z \in \mathbb{D}$, consider the evaluation of $H(t) = \phi_t$ at z, i.e. $\phi_t(z) \in \mathbb{D}$. By Proposition 1.1.5, this pointwise evaluation is continuous, i.e. if $t \longrightarrow t_0$ in Λ, then

$$|H(t)(z) - H(t_0)(z)| = |\phi_t(z) - \phi_{t_0}(z)| \longrightarrow 0$$

Since $\mathbb{D}$ is a Jordan domain, it is finitely connected on the boundary, and the limit mapping ϕ_{t_0} is a homeomorphism. Hence the Näkki–Palka Theorem 1.1.10 is applicable, implying that the mapping H is continuous not only with respect to the topology of pointwise convergence, but also with respect to the topology of uniform convergence on $Q(\mathbb{D})$. This yields that H (when restricted to $[0,1]$) is a path in $Q(\mathbb{D})$ joining $H(0) = \sigma_f \circ f$ and $H(1) = \sigma_g \circ g$. The construction carried out and described above is schematically visualized in Figure 3.1.
Now the situation is as follows: The paths γ_f and γ_g join f and g with the corresponding normalized mappings $\sigma_f \circ f$ and $\sigma_g \circ g$ in $Q(\mathbb{D})$, respectively. Furthermore, as seen above, the path H joins $\sigma_f \circ f$ and $\sigma_g \circ g$ in $Q(\mathbb{D})$. Therefore, combining these three paths in an appropriate manner yields the desired result: Reparametrize the preimage domains of γ_f and γ_g to, say, $[-a-1, -a]$ and $[1+a, 2+a]$, respectively, and consider

$$[-a-1, -a] \cup \Lambda \cup [1+a, 2+a] = [-a-1, 2+a] \ni t \longmapsto \begin{cases} \gamma_f(t), & t \in [-a-1, -a] \\ H(t), & t \in \Lambda \\ \gamma_g(3+2a-t), & t \in [1+a, 2+a] \end{cases} \tag{3.4}$$

This is a path in $Q(\mathbb{D})$ joining the mappings f and g (in (3.4), the path γ_g needs to be traversed in opposite direction in order to start at the mapping $\sigma_g \circ g$ and to end at g; thus the term $\gamma_g(3+2a-t)$ is to be used for $t \in [1+a, 2+a]$). Hence $Q(\mathbb{D})$ is path–connected.

For an arbitrary domain G having only prime ends of the first kind, the conjugation mapping $\Phi : Q(\mathbb{D}) \longrightarrow Q(G)$ is continuous by Theorem 2.1.2. Thus $Q(G)$ is path–connected as well. □

Remark 3.4.3.

(i) The statement of Theorem 3.4.1 immediately implies

The space $Q(G)$ is connected if G has only prime ends of the first kind.

which is the quasiconformal version of Gaier's result stated in [Gai84, Satz 7a, p. 237].

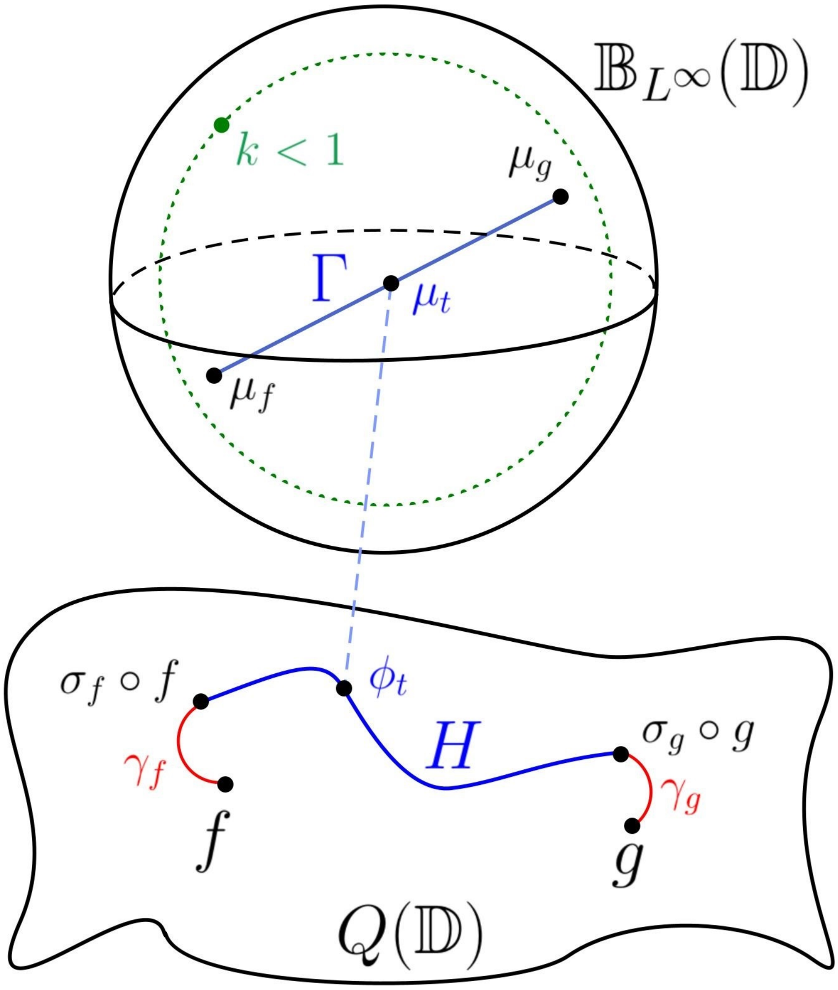

Figure 3.1: Schematic overview of the first part of the proof of Theorem 3.4.1.

(ii) *As for the topological property of path–connectedness of $Q(G)$, Theorem 3.4.1 provides a significant sufficient criterion, which at the same time represents the quasiconformal counterpart to Schmieder's result for the conformal automorphism groups mentioned at the beginning of this section. However, the question for a necessary criterion in order for $Q(G)$ to be path–connected must be left unanswered. One reason why no "easy" necessary criterion in terms of*

$$\text{“}Q(G) \text{ path–connected} \Longrightarrow \Sigma(G) \text{ path–connected”}$$

seems to lie in reach is due to the special nature of path–connectedness, and more generally of the topological notion of connectedness. Both of these properties are not hereditary, i.e. they are in general not passed from a topological space to any of its subspaces, even if these subspaces have additional properties such as being closed in the ambient space (see e.g. [Sin19, p. 53]); this behaviour stands in sharp contrast to the corresponding situation of separable metric space as stated in Proposition 3.1.1.

The statement of Theorem 3.4.1 and the discussion in the previous remark immediately lead to ask the

Question 3.4.4.
Is the requirement of G having only prime ends of the first kind also necessary in order for $Q(G)$ to be path–connected? What can be said about necessary conditions for connectedness of $Q(G)$?

3.4.2 Propagation of path–connectedness from $Q_K(G)$ to $Q(G)$

As stated in Remark 3.4.3(ii), (path–)connectedness is no hereditary topological property, i.e. it is in general neither passed from a topological space to any of its subspaces, nor does the (path–)connectedness of a subspace imply the corresponding property of the ambient superspace. However, the following statement shows that in the case of $Q(G)$, path–connectedness can in fact be propagated from the subspace $Q_K(G)$ to $Q(G)$ under certain circumstances:

Theorem 3.4.5.
Suppose the group operation of $Q(G)$ is continuous. If a subspace $Q_K(G)$ is path–connected for some $K \in (1, +\infty)$, then $Q(G)$ is also path–connected, and all $f, g \in Q(G)$ can be connected by a composition of finitely many paths in $Q_K(G)$. In particular, this is true for $G \in \mathcal{JD}$.

Proof. By the factorization property of quasiconformal mappings (Proposition 1.1.4), the elements $f, g \in Q(G)$ can be written as

$$f = f_1 \circ f_2 \circ \cdots \circ f_{N_1}, \quad g = g_1 \circ g_2 \circ \cdots \circ g_{N_2}$$

with $f_j, g_p \in Q_K(G)$ for all j, p and certain indices $N_1 = N_1(f, K), N_2 = N_2(g, K) \in \mathbb{N}$. Without loss of generality, one may assume $N_1 = N_2$, for if $N_1 < N_2$ (otherwise, switch the roles of f and g), it is

$$f = f_1 \circ f_2 \circ \cdots \circ f_{N_1} \circ \widetilde{f}_{N_1+1} \circ \widetilde{f}_{N_1+2} \circ \cdots \circ \widetilde{f}_{N_2}$$

with $\widetilde{f}_{N_1+q} := \mathrm{id}_G$ for all $q = 1, \ldots, N_2 - N_1$. Hence, it is $N := N_1 = N_2$ and $f_q, g_q \in Q_K(G)$ for all $q = 1, \ldots, N$. Due to the path–connectedness of $Q_K(G)$, there exist paths

$$\gamma_q : [0, 1] \longrightarrow Q_K(G)$$

connecting f_q and g_q in $Q_K(G)$ for all $q = 1, \ldots, N$. Define the mapping

$$\Gamma : [0, 1] \longrightarrow Q(G),\ t \longmapsto \Gamma(t) := \gamma_1(t) \circ \gamma_2(t) \circ \cdots \circ \gamma_N(t),$$

which is continuous by the continuity of the group operation in $Q(G)$. Furthermore, it is

$$\Gamma(0) = \gamma_1(0) \circ \gamma_2(0) \circ \cdots \circ \gamma_N(0) = f_1 \circ f_2 \circ \cdots \circ f_N = f$$

by construction, and likewise $\Gamma(1) = g$. Hence, Γ is a path in $Q(G)$ connecting f and g, and consequently, $Q(G)$ is path–connected. Finally, if $G \in \mathcal{JD}$, then $Q(G)$ is a topological group (Proposition 1.3.3(ii)), therefore the continuity of the group operation is fulfilled. ☐

3.5 Compactness criteria and σ–compactness

Compact subsets of metric (or topological) spaces naturally represent interesting mathematical objects, and the vast amount of results and concepts evolving around compactness impressively show the importance of this notion. In this section, the focus lies on compact subsets of $Q(G)$ and criteria in order to decide whether a non–empty subset $\varnothing \neq M \subsetneq Q(G)$ is compact; the space $Q(G)$ itself is never compact (see Proposition 1.3.1(i)), therefore it suffices to consider proper non–empty subsets.

3.5.1 A necessary compactness criterion: Uniformly bounded dilatation

For a non–empty subset $\varnothing \neq M \subseteq Q(G)$ let

$$K(M) := \sup_{f \in M} K(f) \tag{3.5}$$

denote the **maximal dilatation** of M. Obviously, it is $K(M) \in [1, +\infty]$. For certain compact subsets of $Q(G)$, this number will be finite, as it is shown in

Example 3.5.1.
Let $G = \mathbb{D}$ and $A \subsetneqq [1, +\infty)$ be a compact interval. Consider the family

$$M = \left\{ \mathbb{D} \ni z \longmapsto z|z|^{K-1} \;\middle|\; K \in A \right\} \subsetneqq Q(\mathbb{D})$$

of monomial–like radial stretchings in $\mathbb{D}$ (as introduced in Definition 2.3.1 and (2.3), respectively). From [Bie17, Lemma 2.22, p. 58], it follows that the convergence of a sequence of monomial–like radial stretchings $(z|z|^{K_n-1})_n$ in $Q(\mathbb{D})$ is equivalent to the convergence of the corresponding sequence of exponents $(K_n)_n$ in $\mathbb{R}$. In view of this result, the compactness of the interval A implies the compactness of the family M (and vice versa). Moreover, it is

$$K(M) = \max_{K \in A}\{K\} < +\infty$$

The previous example gives rise to the conjecture that the maximal dilatation of a (non–empty) compact subset of $Q(G)$ is necessarily uniformly bounded from above, at least if certain additional structural properties are given. A clever argument in [MNP98, p. 230], in which the authors present a proof of a similar statement as the mentioned conjecture concerning so–called "quasiconformally homogeneous" subsets of $\mathbb{C}$, uses the following variant of the Baire Category Theorem (see e.g. [RF10, Corollary 4, p. 212]):

Proposition 3.5.2 (Baire Category Theorem)**.**
Let X be a complete metric space and and $(X_n)_{n\in\mathbb{N}}$ a countable collection of closed subsets of X with $X = \bigcup_{n=1}^{\infty} X_n$. Then at least one of the subsets X_n has non–empty interior.

Inspired by the arguments in [MNP98, p. 230], this statement is utilized in order to prove the following result, yielding the affirmative answer for compact subgroups of $Q(G)$ to the conjecture mentioned above for the case $G \in \mathcal{JD}$ (see also [BL23, Theorem 13]):

Theorem 3.5.3.
Let $G \in \mathcal{JD}$ and $\varnothing \neq M \subseteq Q(G)$ be a compact subgroup. Then $K(M) < +\infty$.

Proof. For $n \in \mathbb{N}$, set $M_n = \{f \in M \mid K(f) \leq n\}$, then obviously

$$M = \bigcup_{n=1}^{\infty} M_n$$

First of all, each subset M_n is closed in M by the following reasoning: Let $(f_j)_{j\in\mathbb{N}} \subseteq M_n$ be convergent to $f \in M$. By definition, it is $K(f_j) \leq n$, and due to the general requirement of G being bounded and simply connected, the Hurwitz–type Theorem 1.1.9 is applicable. Hence, the limit mapping f is either constant or n–quasiconformal, but the first case cannot occur due to $f \in M$, thus $f \in M_n$.
Now the non–empty and compact, thus complete metric space M can be expressed as the countable union of closed subsets. The Baire Category Theorem 3.5.2 yields that there is an index $N \in \mathbb{N}$ such that the subset M_N has non–empty interior in the subspace topology of M. Hence there exists $\epsilon > 0$ and a point $\widehat{f} \in M_N$ (the interior point) such that the intersection of the open ϵ–ball $U := B_\epsilon(\widehat{f})$ in $Q(G)$ with M is open in the subspace topology of M and is contained in M_N, i.e.

$$\widehat{f} \in U \cap M \subseteq M_N$$

For further usage, note that U is open in the ambient space $Q(G)$, and it is $K(\widehat{f}) \leq N$. Next, for $g \in M$, consider the mapping

$$L_g : Q(G) \longrightarrow Q(G),\ h \longmapsto L_g(h) := g \circ \widehat{f}^{-1} \circ h$$

which is the left multiplication in $Q(G)$ with the mapping $g \circ \widehat{f}^{-1}$. By construction, it is $L_g(\widehat{f}) = g$. Furthermore, $Q(G)$ is a topological group due to $G \in \mathcal{JD}$ by Proposition 1.3.3(ii), thus the mapping L_g is a homeomorphism of $Q(G)$ onto itself – in particular, L_g is an open mapping. Hence the image of U under L_g is again open in $Q(G)$ and $g \in L_g(U)$. The situation described so far is depicted in Figure 3.2.

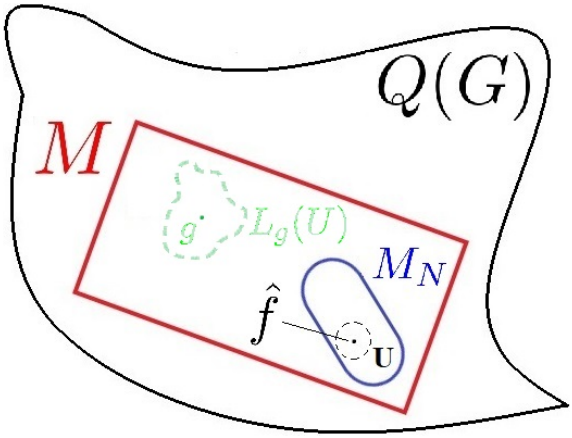

Figure 3.2: The compact set $M \subsetneq Q(G)$ (in red) with the subset M_N (in blue) together with the interior point $\widehat{f}$ and its open neighborhood U. Moreover, the green dashed line denotes the image of U under the left multiplication L_g in the topological group $Q(G)$.

Furthermore, $L_g(U) \cap M$ is open in the subspace topology of M, and in consequence the family

$$(L_g(U) \cap M)_{g \in M}$$

is an open cover (in the subspace topology of M) of the compact space M, which by definition yields a finite subcover: There exist elements $g_1, \ldots, g_m \in M$ such that

$$L_{g_1}(U) \cap M, \ldots, L_{g_m}(U) \cap M$$

cover M, as visualized in Figure 3.3. Hence, every $f \in M$ is contained in some open set $L_{g_j}(U) \cap M$ (in the subspace topology of M) and thus can be written as $f = g_j \circ \widehat{f}^{-1} \circ h$ with $h \in U \cap M \subseteq M_N$. This implies that the maximal dilatation of each of these elements satisfies

$$K(f) \leq K(g_j) \cdot K(\widehat{f}^{-1}) \cdot K(h) \leq \left(\max_{j=1,\ldots,m} K(g_j) \right) \cdot K(\widehat{f}) \cdot N \leq \left(\max_{j=1,\ldots,m} K(g_j) \right) \cdot N^2 < +\infty$$

by construction of the mapping L_g, the property $K(\widehat{f}^{-1}) = K(\widehat{f})$ of quasiconformal mappings (see e.g. [BF14, Property P1, p. 31]) and due to $K(U \cap M) \leq N$ (which holds since $U \cap M \subseteq M_N$). □

Remark 3.5.4.
As Example 1.1.11 shows, the statement of Theorem 3.5.3 cannot be strengthened to arbitrary compact subsets of $Q(G)$: Using the terminology of Example 1.1.11, the set $\{\mathrm{id}_R\} \cup \{f_{m,n_m} \mid m \in \mathbb{N}\}$ yields a compact subset of $Q(R)$, but by construction has unbounded maximal dilatation.

3.5.2 A sufficient compactness criterion: Arzelà–Ascoli for $Q(G)$

In order to formulate a sufficient criterion for compactness in $Q(G)$, it is helpful to recall that $Q(G)$ can canonically be interpreted as a subset of the Banach space $C_b(G)$. By assuming the additional requirement $G \in \mathcal{JD}$, Proposition 1.1.6 yields that each $f \in Q(G)$ extends homeomorphically to $\overline{G}$, thus becoming an element of $C(\overline{G})$ by means of $Q(\overline{G})$ (see Definition 1.1.7). In this situation, the following versions of the Theorem of Arzelà–Ascoli are applicable (see [RF10, Arzelà–Ascoli Theorem and Theorem 3, pp. 208–209]):

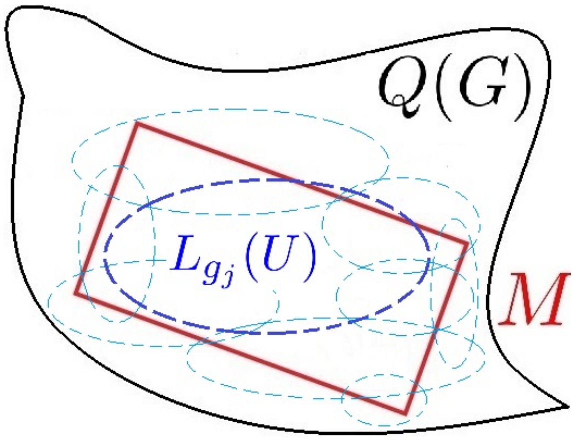

Figure 3.3: The compact set $M \subsetneq Q(G)$ (in red) is covered by the finitely many open subsets $L_{g_j}(U)$ (in blue).

Proposition 3.5.5 (Arzelà–Ascoli)**.**
Let X be a compact metric space.

(i) Let $(f_n)_{n \in \mathbb{N}}$ be a uniformly bounded, equicontinuous sequence of complex–valued[4] *functions on X. Then $(f_n)_n$ has a subsequence that converges uniformly on X to a continuous function f on X.*

(ii) Let $M \subseteq C(X)$. Then M is compact if and only if M is closed in $C(X)$, uniformly bounded and equicontinuous.

In this context, a family of complex–valued mappings $\mathcal{M} \subseteq C(X)$ of a compact metric space (X, d) is called *uniformly equicontinuous* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon$$

holds for all $x, y \in X$ with $d(x, y) < \delta$ and all $f \in \mathcal{M}$ (see [NP73, Section 2.1, p. 428] for the Euclidean case and [RF10, p. 208] for compact metric spaces); occasionally, by irregular nomenclature in the literature, this property is sometimes simply called *equicontinuity*. In view of the situation at hand, it is $X = \overline{G}$ and $M \subseteq Q(\overline{G})$, implying that the uniform boundedness conditions in Proposition 3.5.5 are then automatically fulfilled. Hence, the remaining decisive equicontinuity condition of M is to be studied. Näkki and Palka studied (uniform) equicontinuity of quasiconformal mappings defined on domains in n–dimensional Euclidean spaces $\mathbb{R}^n$ and $\overline{\mathbb{R}}^n$ for $n \geq 2$ in [NP73]. One of their main results is the following statement, which is formulated for this thesis' primarily studied case of quasiconformal mappings defined on domains in $\mathbb{C} \cong \mathbb{R}^2$ (see [NP73, Theorem 3.1 + Remark 3.6(1), pp. 428–431]):

Proposition 3.5.6.
Let M be a family of K–quasiconformal mappings of a domain $\mathbb{C} \neq G \in \mathcal{JD}$ onto a domain G'. Then M is uniformly equicontinuous if and only if each $f \in M$ can be extended to a continuous mapping of $\overline{G}$ onto $\overline{G'}$ and the set $M(z) := \{f(z) \,|\, f \in M\}$ is contained in a compact subset of G' for some point $z \in G$.

Remark 3.5.7.
In Proposition 3.5.6, the requirement "$M(z)$ is contained in a compact subset of G' for some point $z \in G$" can be replaced by several other equivalent statements, see [NP73, Remark 3.6(1), p. 431].

[4]In the cited version of the Arzelà–Ascoli Theorem in [RF10], *real–valued* functions are considered. However, the statement of this theorem remains the same when complex–valued functions are focused, basically due to the completeness of $\mathbb{C}$; see e.g. [Con90, pp. 175–176], where the complex–valued case is included.

Combining these preparatory results and notions, the following sufficient criterion for relative compactness in $Q(G)$ with $G \in \mathcal{JD}$ can now be established:

Theorem 3.5.8.
Let $G \in \mathcal{JD}$ and $M \subseteq Q_K(G)$. Then M is relatively compact (i.e. the closure $\overline{M}$ in $Q(G)$ is compact) if

$$M(z) := \{f(z) \,|\, f \in M\}$$

is contained in a compact subset of G for some point $z \in G$.

Proof. Since $M(z)$ is contained in a compact subset of G for some $z \in G$ and Proposition 1.1.6 is applicable, all requirements of Näkki–Palka's Proposition 3.5.6 are satisfied, implying that M is uniformly equicontinuous. Now let $(f_n)_{n\in\mathbb{N}} \subseteq M$ be a sequence in M, then by Theorem 3.5.5(i), there exists a uniformly convergent subsequence $(f_{n_j})_j$ on $\overline{G}$ with limit mapping $f \in C(\overline{G})$, which is also uniformly convergent when restricted to G (with limit mapping $f|_G$). Due to $M \subseteq Q_K(G)$, the Hurwitz–type Theorem formulated in Proposition 1.1.9 is applicable, yielding that (the restriction of) f is either a K–quasiconformal automorphism of G or a constant with $f \equiv c \in \partial G$. The latter case cannot occur, for each f_{n_j} is bijective, thus $f \in Q_K(G)$ and M is relatively compact. □

The Näkki–Palka Theorem cited in Proposition 3.5.6 may now be used in order to formulate the announced *"Arzelà–Ascoli–type Theorem"* for $Q(G)$ in the case $G \in \mathcal{JD}$:

Corollary 3.5.9 (Arzelà–Ascoli for $Q(G)$)**.**
Let $G \in \mathcal{JD}$ and $M \subseteq Q_K(G)$. Then M is compact if and only if M is closed[5] *in $C(\overline{G})$ and $M(z) = \{f(z) \,|\, f \in M\}$ is contained in a compact subset of G for some point $z \in G$.*

Proof. Suppose M is compact, then (the set of extended mappings of) M is closed in $C(\overline{G})$ and equicontinuous by Proposition 3.5.5(ii). Hence Proposition 3.5.6 yields the claim.
Otherwise, assume that (the set of extended mappings of) M is closed in $C(\overline{G})$ and $M(z)$ is contained in a compact subset of G for some $z \in G$. By Theorem 3.5.8, M is relatively compact, and thus compact due to being closed in $C(\overline{G})$. □

An application of this Arzelà–Ascoli–type Theorem for $Q(G)$ for the special case $G = \mathbb{D}$ will be given in Remark 3.5.11, yielding a classical compactness result for certain subsets of $Q(\mathbb{D})$.

3.5.3 A compactification procedure for $Q_K(G)$

In this subsection, the subspaces

$$Q_K(G) = \big\{f \in Q(G) \,\big|\, K(f) \le K\big\}$$

for $K \ge 1$ are considered once more. Theorem 2.2.1 shows that these subsets are always complete, but never compact. However, a certain compactification procedure actually transforms the sets $Q_K(G)$ into compact subspaces of $Q(G)$ for $G \in \mathcal{JD}$, as will be demonstrated in the following. To this end, the below–mentioned classical result on K–quasiconformal automorphisms of the unit disk $\mathbb{D}$ fixing the origin will prove valuable (see e.g. [Ahl06, Theorem 1, p. 32]):

Proposition 3.5.10.
The set

$$Q_{K,fix}(\mathbb{D}) := \{f \in Q_K(\mathbb{D}) \,|\, f(0) = 0\} \tag{3.6}$$

consisting of all K–quasiconformal automorphisms of $\mathbb{D}$ fixing the origin forms a compact subspace of $Q(\mathbb{D})$.

[5] More precisely, the homeomorphic extensions of the elements of M to $\overline{G}$, assured by Proposition 1.1.6, are referred to in this statement.

Remark 3.5.11.
As an application, the Arzelà–Ascoli Theorem for $Q_K(G)$, formulated in Corollary 3.5.9, contains the statement of Proposition 3.5.10 as a special case for $G = \mathbb{D}$, since in this situation $\{0\}$ is a compact subset of $\mathbb{D}$ and the Hurwitz–type Theorem for sequences of quasiconformal mappings (Proposition 1.1.9) yields the closedness in $C(\overline{\mathbb{D}})$.

In view of Proposition 3.5.10, let

$$Q_{\text{fix}}(\mathbb{D}) := \bigcup_{K\geq 1} Q_{K,\text{fix}}(\mathbb{D}) \tag{3.7}$$

and for $f \in Q(\mathbb{D})$, consider the conformal unit disk automorphism

$$\sigma_f(z) := \frac{z - f(0)}{1 - \overline{f(0)}z} \quad (z \in \mathbb{D})$$

This mapping $\sigma_f \in \Sigma(\mathbb{D})$ is now utilized in order to define the following function on $Q(\mathbb{D})$:

$$\mathfrak{C} : Q(\mathbb{D}) \longrightarrow Q_{\text{fix}}(\mathbb{D}),\ f \longmapsto \mathfrak{C}(f) := \sigma_f \circ f \tag{3.8}$$

By construction, it is $\mathfrak{C}(f)(0) = (\sigma_f \circ f)(0) = 0$ and $K(\mathfrak{C}(f)) = K(f)$ for every $f \in Q(\mathbb{D})$. Further properties of $\mathfrak{C}$ are studied in

Lemma 3.5.12.
The mapping $\mathfrak{C}$ defined in (3.8) is continuous, surjective and idempotent (i.e. $\mathfrak{C}(\mathfrak{C}(f)) = \mathfrak{C}(f)$ for all $f \in Q(\mathbb{D})$). In particular, $\mathfrak{C}$ is not injective.

Proof. In order to show continuity, let $(f_n)_{n\in\mathbb{N}}$ converge in $Q(\mathbb{D})$ to $f \in Q(\mathbb{D})$. The corresponding mappings

$$\sigma_{f_n}(z) = \frac{z - f_n(0)}{1 - \overline{f_n(0)}z}$$

converge pointwise in $\mathbb{D}$ to σ_f as n tends to infinity. Due to $\sigma_{f_n} \in \Sigma(\mathbb{D})$, it is $K(\sigma_{f_n}) = 1$ for all $n \in \mathbb{N}$, thus Proposition 1.1.10 of Näkki–Palka is applicable, implying that the convergence of $(\sigma_{f_n})_n$ to $\sigma_f \in \Sigma(\mathbb{D})$ is not merely pointwise, but in fact uniform on $\mathbb{D}$, i.e. $\sigma_{f_n} \overset{n\to\infty}{\longrightarrow} \sigma_f$ in $Q(\mathbb{D})$. Hence the mapping $f \longmapsto \sigma_f$ is continuous on $Q(\mathbb{D})$. Furthermore, $Q(\mathbb{D})$ is a topological group according to Proposition 1.3.3(ii), yielding that the group multiplication is continuous. Thus the sequence $(\mathfrak{C}(f_n))_n = (\sigma_{f_n} \circ f_n)_n$ converges to the element $\mathfrak{C}(f) = \sigma_f \circ f$, establishing the continuity of the mapping $\mathfrak{C}$.

As for the surjectivity: Let $h \in Q_{\text{fix}}(\mathbb{D})$, then by definition, it is $h(0) = 0$. Choose an element $f \in Q(\mathbb{D})$ with $\mu_f = \mu_h$ a.e. in $\mathbb{D}$ (such an f surely exists, namely $f = \sigma \circ h$ for arbitrary $\sigma \in \Sigma(\mathbb{D})$). The Measurable Riemann Mapping Theorem 1.1.2(II) implies $h = \widetilde{\sigma} \circ f$ for some $\widetilde{\sigma} \in \Sigma(\mathbb{D})$. It follows that

$$\mathfrak{C}(\widetilde{\sigma} \circ f)(z) = (\sigma_{\widetilde{\sigma}\circ f} \circ (\widetilde{\sigma} \circ f))(z) = \frac{(\widetilde{\sigma} \circ f)(z) - (\widetilde{\sigma} \circ f)(0)}{1 - \overline{(\widetilde{\sigma} \circ f)(0)}(\widetilde{\sigma} \circ f)(z)} = \frac{h(z) - h(0)}{1 - \overline{h(0)}h(z)} = h(z)$$

for every $z \in \mathbb{D}$. Hence $\mathfrak{C}(\widetilde{\sigma} \circ f) = h$, showing that $\mathfrak{C}$ is a surjective mapping of $Q(\mathbb{D})$ onto $Q_{\text{fix}}(\mathbb{D})$.

The claimed idempotence property can be seen as follows: Let $f \in Q(\mathbb{D})$, then by construction it is $\mathfrak{C}(f)(0) = 0$ and this yields

$$\mathfrak{C}(\mathfrak{C}(f))(z) = (\sigma_{\mathfrak{C}(f)} \circ \mathfrak{C}(f))(z) = \frac{\mathfrak{C}(f)(z) - \mathfrak{C}(f)(0)}{1 - \overline{\mathfrak{C}(f)(0)}\mathfrak{C}(f)(z)} = \mathfrak{C}(f)(z)$$

for every $z \in \mathbb{D}$, hence $\mathfrak{C}(\mathfrak{C}(f)) = \mathfrak{C}(f)$.
In particular, $\mathfrak{C}$ cannot be injective, for let $g \in Q(\mathbb{D})$ with $g(0) \neq 0$, then the corresponding mapping $\sigma_g \in \Sigma(\mathbb{D})$ is not the identity on $\mathbb{D}$, thus $g \neq \mathfrak{C}(g)$. However, it is $\mathfrak{C}(\mathfrak{C}(g)) = \mathfrak{C}(g)$ by the idempotence of $\mathfrak{C}$. Thus the elements g and $\mathfrak{C}(g) \neq g$ are both mapped to the same image $\mathfrak{C}(g)$ by $\mathfrak{C}$. □

Remark 3.5.13.
In addition to the properties studied above, note that $Q_{fix}(\mathbb{D}) < Q(\mathbb{D})$ (in a group–theoretic sense) and that the mapping $\mathfrak{C}$ satisfies the equation

$$\mathfrak{C}(e^{i\alpha} \cdot f) = e^{i\alpha} \cdot \mathfrak{C}(f)$$

since

$$\mathfrak{C}(e^{i\alpha} \cdot f)(z) = (\sigma_{e^{i\alpha} \cdot f} \circ e^{i\alpha} \cdot f)(z) = \frac{e^{i\alpha} \cdot f(z) - e^{i\alpha} \cdot f(0)}{1 - \overline{e^{i\alpha} \cdot f(0)} \cdot e^{i\alpha} \cdot f(z)} = e^{i\alpha} \cdot \frac{f(z) - f(0)}{1 - \overline{f(0)} f(z)} = e^{i\alpha} \cdot \mathfrak{C}(f)(z)$$

for all $\alpha \in (-\pi, \pi]$, $z \in \mathbb{D}$ and all $f \in Q(\mathbb{D})$ (the dot denoting the usual pointwise–defined multiplication of complex–valued functions).

Now the announced compactness result can be formulated:

Theorem 3.5.14.
For every $K \geq 1$, the set $\mathfrak{C}(Q_K(\mathbb{D})) = Q_{K,fix}(\mathbb{D}) \subseteq Q_{fix}(\mathbb{D})$ is compact.

Proof. It follows from the definition of the mapping $\mathfrak{C}$ that $\mathfrak{C}(Q_K(\mathbb{D})) \subseteq Q_{K,\text{fix}}(\mathbb{D})$. By the surjectivity of $\mathfrak{C}$, for every $f \in Q_{K,\text{fix}}(\mathbb{D})$ there exists a corresponding element $h \in Q_K(\mathbb{D})$ with $\mathfrak{C}(h) = f$, thus $\mathfrak{C}(Q_K(\mathbb{D})) = Q_{K,\text{fix}}(\mathbb{D})$. Proposition 3.5.10 then implies that $Q_{K,\text{fix}}(\mathbb{D})$ is compact. □

Naturally, this result transfers to domains G having only prime ends of the first kind as follows: Let $F : G \longrightarrow \mathbb{D}$ be a conformal mapping with $F(z_0) = 0$ for a fixed $z_0 \in G$. The corresponding conjugation mapping $\Phi : Q(\mathbb{D}) \longrightarrow Q(G)$ is uniformly continuous by Theorem 2.1.4 in this situation, thus the subspace

$$Q_{K,\text{fix}(z_0)}(G) := \Phi(Q_{K,\text{fix}}(\mathbb{D})) \subseteq Q_K(G) \tag{3.9}$$

is compact, and it is $f(z_0) = z_0$ for every $f \in Q_{K,\text{fix}}(G)$ by construction. In turn, this reasoning yields

Corollary 3.5.15.
Let G have only prime ends of the first kind and $K \geq 1$, then the set $Q_{K,fix(z_0)}(G)$ is compact.

3.5.4 σ–compactness of $Q(G)$ and the Lindelöf property

A topological space is called σ**–compact** if it can be written as the countable union of compact subsets. In [Yag99, Lemma 3(i), p. 2730] Yagasaki shows among other results that the following statement is valid:

Proposition 3.5.16.
The space $Q(G)$ is σ–compact in the topology of compact convergence (i.e. in the compact–open topology).

It is the aim of this subsection to show the corresponding result for $G \in \mathcal{JD}$ and the topology primarily used in this thesis for $Q(G)$, i.e. the uniform topology induced by the supremum metric. This will be done by giving two different proofs for the following claim:

Theorem 3.5.17.
The space $Q(G)$ is σ–compact for $G \in \mathcal{JD}$.

First Proof of Theorem 3.5.17. In view of the canonical decomposition

$$Q(G) = \bigcup_{K=1}^{\infty} Q_K(G)$$

of $Q(G)$ with $K \in \mathbb{N}$, it is sufficient to show that each subspace $Q_K(G)$ is σ–compact in order to prove the claim, since a countable union of countable sets is countable. By Proposition 3.5.16, $Q(G)$ is σ–compact in the topology of compact convergence. More precisely, when analyzing the proof in [Yag99], Yagasaki shows that the sets $Q_K(G)$ are σ–compact in the compact–open topology for each $K \in \mathbb{N}$ and then uses the above–mentioned countability reasoning. Thus, it is

$$Q_K(G) = \bigcup_{j=1}^{\infty} M_j$$

with subsets $M_j \subseteq Q_K(G), j \in \mathbb{N}$, being compact in the topology of compact convergence (in other words, each M_j is a closed normal family in $Q_K(G)$). It suffices to show that the subsets M_j are compact in the uniform topology as well. Therefore let $(f_n)_{n\in\mathbb{N}}$ be a sequence in M_j, then by the compactness of M_j in the topology of locally uniform convergence, there exists a subsequence $(f_{n_l})_l$ converging locally uniformly to a limit mapping $f \in M_j$. In particular, f_{n_l} converges pointwise in G to f. Since G is supposed to be a Jordan domain, Proposition 1.1.10 of Näkki–Palka is applicable, implying that f_{n_l} converges uniformly on G to f (recall that $K(f_{n_l}) \leq K$ due to $M_j \subseteq Q_K(G)$). Thus M_j is compact in the uniform topology, and the claim follows. □

As announced above, Theorem 3.5.17 will now be proved using a previously established result, namely the Arzelà–Ascoli–type Theorem for $Q(G)$ given in Corollary 3.5.9.

Second Proof of Theorem 3.5.17. Just as in the first proof given above, it suffices to show that each $Q_K(G)$ is σ–compact for $K \in \mathbb{N}$. To this end, choose three fixed and pairwise distinct points $z_1, z_2, z_3 \in G$. Furthermore, let $\mathcal{B} = \{B_j \mid j \in \mathbb{N}\}$ be a countable basis of the topology of G[6], which surely exists due to the separability of G. For open disks $D_1, D_2, D_3 \subseteq G$ in G with pairwise distinct closures, i.e. $\overline{D_j} \cap \overline{D_{j'}} = \varnothing$ for $j \neq j'$, define

$$\mathcal{F} := \left\{ f \in Q_K(G) \,\middle|\, f(z_j) \in \overline{D_j} \text{ for } j = 1,2,3 \right\}$$

Then $Q_K(G)$ is the countable union of sets of the form $\mathcal{F}$, since: Let $f \in Q_K(G)$ and denote by $w_j := f(z_j)$ the images of the points z_j under f, then the w_j are also pairwise disjoint. In turn, there are open disks B_{w_j} around each w_j with pairwise disjoint closures, implying $f \in \mathcal{F}$ for some $\mathcal{F}$. Since $\mathcal{B}$ is a countable basis of the topology of G, the disks B_{w_j} can be chosen to be contained in $\mathcal{B}$, thus countably many sets $\mathcal{F}$ cover $Q_K(G)$. Hence, it is

$$Q_K(G) = \bigcup_{n=1}^{\infty} \mathcal{F}_n$$

with $\mathcal{F}_n = \left\{ f \in Q_K(G) \mid f(z_j) \in \overline{B_{w_{j,n}}},\ j = 1,2,3 \right\}$. Thus the claim of Theorem 3.5.17 follows if one can show that the sets $\mathcal{F}_n$ are compact. But this statement is implied by the Arzelà–Ascoli–type Theorem 3.5.9 for $Q(G)$, since

a) $\mathcal{F}_n$ is closed in $C(\overline{G})$ by the Hurwitz–type Theorem 1.1.9 for sequences of quasiconformal mappings and due to the fact that the sets $\overline{B_{w_{j,n}}}$ are closed (in G) for $j = 1,2,3$;

[6] A **countable basis** of a topological space $(X, \mathcal{T})$ is a countable sequence $(U_j)_{j\in\mathbb{N}}$ of open subsets $U_j \in \mathcal{T}$ such that for each $x \in X$ and each neighborhood V of x, there exists $j \in \mathbb{N}$ such that $x \in U_j \subseteq V$.

b) $\mathcal{F}_n(z_j) = \{f(z_j) \,|\, f \in \mathcal{F}_n\}$ is contained in the compact set $\overline{B_{w_{j,n}}}$ for $j = 1, 2, 3$.
Hence $\mathcal{F}_n$ is compact, making $Q_K(G)$ a σ–compact space for each $K \in \mathbb{N}$, therefore $Q(G)$ possesses this property as well. □

In the remainder of this subsection, the focus is now switched to the study of a related topological property, namely the Lindelöf property, and its close connection to σ–compactness and separability of $Q(G)$. A topological space is called a **Lindelöf space** if each open cover of the space has a countable subcover, therefore obviously generalizing the notion of a compact space (see [Kel75, p. 50]). The previously proved Theorem 3.5.17 shows that $Q(G)$ is σ–compact for $G \in \mathcal{JD}$. In view of Lindelöf spaces and their connection to σ–compactness and separability, the following two facts are well–known results from point–set topology:

(L1) *Every σ–compact space is a Lindelöf space.*[7].
(L2) *A metric space is a Lindelöf space if and only if it is separable.*[8]

Thus, by combining the two statements (L1) and (L2), a σ–compact metric space is separable. Moreover, separability of the metric space $Q(G)$ occurs if and only if the domain G has only prime ends of the first kind, as shown in Theorems 3.1.2 and 3.1.4. Putting these statements together and applying them to $Q(G)$, one finally arrives at

Corollary 3.5.18.
(i) The space $Q(G)$ is a Lindelöf space if and only if $\mathcal{P}(G) = \mathcal{P}_1(G)$.
(ii) If $G \in \mathcal{JD}$, the space $Q(G)$ is σ–compact. Conversely, if $Q(G)$ is σ–compact, then the domain G has only prime ends of the first kind.

Consequently, the situation in Corollary 3.5.18(ii) immediately rises the

Question 3.5.19.
Is $Q(G)$ a σ–compact space if G has only prime ends of the first kind?

3.6 The space $Q(G)$ for multiply connected domains

The aim of this section is to change the point of view concerning quasiconformal automorphism groups, to be more precise with the underlying domains of these objects, and to extend the investigations of the current chapter to the case of multiply connected domains in $\mathbb{C}$. Therefore, in the following, $G \subsetneq \mathbb{C}$ *will always refer to a bounded domain in $\mathbb{C}$ with finite connectivity and non–degenerate boundary* (i.e. each boundary component of G is supposed to consist of at least two points). Consequently, the definition given in (0.2) for the simply connected case is now canonically transferred to multiply connected domains via

$$Q(G) := \left\{ f : G \longrightarrow G \;\middle|\; f \text{ is quasiconformal mapping of } G \text{ onto } G \right\}$$

and will of course also be called the **quasiconformal automorphism group** of G, its elements $f \in Q(G)$ are called **quasiconformal automorphisms of** G as well. The subset

$$\Sigma(G) := \{ f \in Q(G) \mid f \text{ is conformal} \}$$

is called **conformal automorphism group** of G. As usual, the set $Q(G)$ and its subspaces (via the subspace topology) are always endowed with the supremum metric $d_{\sup}$ if not stated otherwise.

[7] See e.g. [Wil70, 17I.2, p. 126]; This can also be seen directly from the definitions, for let the topological space $X = \bigcup_{n\in\mathbb{N}} A_n$ with compact $A_n \subseteq X$ for each $n \in \mathbb{N}$ and $(U_j)_j$ be any open cover of X, then for every $n \in \mathbb{N}$ it is $A_n \subseteq \bigcup_j U_j$, hence there exists a finite subcover $U_{j_1,n}, \ldots, U_{j_m,n}$ of A_n by compactness. Thus $\bigcup_n \bigcup_{p=1}^m U_{j_p,n}$ is a countable subcover of X, since a countable union of finite sets is countable.

[8] See [Wil70, Theorem 16.11, p. 112] and [Sin19, p. 177].

3.6.1 The space $\Sigma(\mathcal{A})$ for doubly connected domains

Gaier mentions that studying $\Sigma(G)$ for doubly connected domains $G \subseteq \mathbb{C}$ is the only interesting case for quasiconformal automorphism groups besides the simply connected setting (see [Gai84, p. 256]), thus this subsection will be concerned with topological properties of these spaces. Initially, the "regular case" is considered, i.e. domains bounded by Jordan curves. Such a domain is conformally equivalent to an annulus

$$\mathcal{A} := \mathcal{A}_{r,R}(0) := \left\{ z \in \mathbb{C} \,\middle|\, r < |z| < R \right\}$$

for certain constants $0 < r < R < \infty$ with uniquely determined ratio $\frac{R}{r}$ (see e.g. [Dur04, p. 136]), putting the space $\Sigma(\mathcal{A})$ into focus. However, the spaces $\Sigma(\mathcal{A})$ have been studied quite extensively for the topology of *compact convergence*, see for example [Kra04, Section 3.4], [Kra06, Chapter 12] and [RS07, Kapitel 9] as well as [IK99] and [KK05] especially for information on the higher–dimensional situation. Hence, one might expect that some results from the *compact convergence*–setting carry over the the *uniform convergence*–setting. This is actually the case, as will be shown in the following.

Structure and group property of $\Sigma(\mathcal{A})$

Obviously, $\Sigma(\mathcal{A})$ forms a group with respect to composition of mappings and with neutral element being the identity mapping $\mathrm{id}_{\mathcal{A}}$. Moreover, just as for $\Sigma(\mathbb{D})$, the conformal automorphisms of $\mathcal{A}$ are known explicitly (see [Kra04, Example 2, pp. 122–123]):

Lemma 3.6.1. *It is*

$$\Sigma(\mathcal{A}) = U_1 \,\dot{\cup}\, U_2$$

with

$$U_1 := \left\{ \mathcal{A} \ni z \longmapsto e^{i\varphi} z \,\middle|\, \varphi \in [0, 2\pi] \right\} \quad \text{and} \quad U_2 := \left\{ \mathcal{A} \ni z \longmapsto e^{i\varphi} \frac{Rr}{z} \,\middle|\, \varphi \in [0, 2\pi] \right\}$$

Hence, $\Sigma(\mathcal{A})$ is "*just two copies of the circle*" as stated in [Kra06, p. 260], the two sets U_1 and U_2 obviously being disjoint. Moreover, U_1 is itself a group and thus forms a subgroup of $\Sigma(\mathcal{A})$, whereas U_2 is no (sub)group. Besides these elementary facts, from the point of view of group theory, the following result is of importance (see [Mil06, Problem 2–g(3), p. 27]):

Proposition 3.6.2.
The conformal automorphism group $\Sigma(\mathcal{A})$ is isomorphic to the ***orthogonal group***

$$O_2(\mathbb{R}) := \left\{ \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} \,\middle|\, \varphi \in [0, 2\pi] \right\} \dot{\cup} \left\{ \begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ \sin(\varphi) & -\cos(\varphi) \end{pmatrix} \,\middle|\, \varphi \in [0, 2\pi] \right\}$$

(endowed with matrix multiplication) via the mapping

$$H : \Sigma(\mathcal{A}) \longrightarrow O_2(\mathbb{R}),\; f \longmapsto H(f) := \begin{cases} \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}, & f = e^{i\varphi} z \in U_1 \\ \begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ \sin(\varphi) & -\cos(\varphi) \end{pmatrix}, & f = e^{i\varphi} \frac{Rr}{z} \in U_2 \end{cases}$$

In particular, $\Sigma(\mathcal{A})$ is a non–abelian group, even though the subgroup $U_1 < \Sigma(\mathcal{A})$ is abelian. By virtue of the group isomorphism H, the subgroup $U_1 < \Sigma(\mathcal{A})$ corresponds to the **special orthogonal group** $SO_2(\mathbb{R}) < O_2(\mathbb{R})$ of all orthogonal matrices $B \in O_2(\mathbb{R})$ with determinant $\det(B) = +1$, whereas U_2 corresponds to its complement $O_2(\mathbb{R}) \backslash SO_2(\mathbb{R})$ (these matrices having determinant -1). Moreover, the elements of U_2 have the special property of **involution**, i.e. $h^2 = h \circ h = \mathrm{id}_{\mathcal{A}}$ for all $h \in U_2$.

Topological properties of $\Sigma(\mathcal{A})$

The orthogonal group $O_2(\mathbb{R})$ naturally carries the standard topology induced from the Euclidean space $\mathbb{R}^4$ endowed with the canonical Euclidean $\|\cdot\|_2$–norm. In this topology, it is a compact (thus complete and locally compact) space consisting of exactly two (path)–connected components, namely $SO_2(\mathbb{R})$ and its complement in $O_2(\mathbb{R})$; in particular, $O_2(\mathbb{R})$ is not (path)–connected. Moreover, $O_2(\mathbb{R})$ is know to be a Lie group, thus in particular a topological group. The self–evident goal is now to show that the group isomorphism H provided by Proposition 3.6.2 is also a homeomorphism in order to transfer all relevant topological properties of $O_2(\mathbb{R})$ to $\Sigma(\mathcal{A})$. In view of this, the following lemma regarding the distance in the space $\Sigma(\mathcal{A})$ measured in terms of the supremum metric on $\mathcal{A}$ will be useful:

Lemma 3.6.3. *For $e^{i\varphi}z, e^{i\alpha}z \in U_1$, it is*

$$d_{\sup}(e^{i\varphi}z, e^{i\alpha}z) = R \cdot |e^{i\varphi} - e^{i\alpha}|$$

Likewise, for $e^{i\varphi}\frac{Rr}{z}, e^{i\alpha}\frac{Rr}{z} \in U_2$, it is

$$d_{\sup}\left(e^{i\varphi}\frac{Rr}{z}, e^{i\alpha}\frac{Rr}{z}\right) = R \cdot |e^{i\varphi} - e^{i\alpha}|$$

In the "mixed case" $e^{i\varphi}z \in U_1, e^{i\alpha}\frac{Rr}{z} \in U_2$, it is

$$d_{\sup}\left(e^{i\varphi}z, e^{i\alpha}\frac{Rr}{z}\right) = R + r$$

Proof. The first case $e^{i\varphi}z, e^{i\alpha}z \in U_1$, the claim follows from

$$d_{\sup}(e^{i\varphi}z, e^{i\alpha}z) = \sup_{z\in\mathcal{A}} \left|e^{i\varphi}z - e^{i\alpha}z\right| = \left|e^{i\varphi} - e^{i\alpha}\right| \cdot \sup_{z\in\mathcal{A}} |z| = R \cdot \left|e^{i\varphi} - e^{i\alpha}\right|$$

Similarly, the second case $e^{i\varphi}\frac{Rr}{z}, e^{i\alpha}\frac{Rr}{z} \in U_2$ follows also by direct calculation as

$$d_{\sup}\left(e^{i\varphi}\frac{Rr}{z}, e^{i\alpha}\frac{Rr}{z}\right) = \sup_{z\in\mathcal{A}} \left|e^{i\varphi}\frac{Rr}{z} - e^{i\alpha}\frac{Rr}{z}\right| = Rr \cdot \left|e^{i\varphi} - e^{i\alpha}\right| \cdot \sup_{z\in\mathcal{A}} \left|\frac{1}{z}\right| = R \cdot \left|e^{i\varphi} - e^{i\alpha}\right|$$

The third case $e^{i\varphi}z \in U_1, e^{i\alpha}\frac{Rr}{z} \in U_2$ can be seen by the following reasoning:

$$d_{\sup}\left(e^{i\varphi}z, e^{i\alpha}\frac{Rr}{z}\right) = \sup_{z\in\mathcal{A}} \left|e^{i\varphi}z - e^{i\alpha}\frac{Rr}{z}\right| = \sup_{z\in\mathcal{A}} \frac{1}{|z|} \left|z^2 - e^{i(\alpha-\varphi)}Rr\right|$$

Let $\psi = \alpha - \varphi$, then the mapping $\mathcal{A} \ni z \longmapsto \frac{1}{z}(z^2 - e^{i\psi}Rr)$ is holomorphic on the bounded domain $\mathcal{A}$ and continuous on its closure $\overline{\mathcal{A}}$, thus the (strong version of the) maximum modulus principle applies (see e.g. [RS02, Satz 8.5.6, pp. 230–231]), yielding that the maximum distance in question is located on the boundary $\partial\mathcal{A}$. Assume first that the inner boundary component of the annulus $\mathcal{A}$ contains this maximum, then setting $z = re^{it} \in \partial\mathcal{A}$ with $t \in [0, 2\pi]$ concludes in

$$\sup_{z\in\mathcal{A}} \frac{1}{|z|} \left|z^2 - e^{i\psi}Rr\right| = \max_{t\in[0,2\pi]} \frac{1}{r} \left|r^2e^{2it} - e^{i\psi}Rr\right| = \max_{t\in[0,2\pi]} \left|re^{2it} - Re^{i\psi}\right|$$

Due to the compactness of the interval $[0, 2\pi]$ there exists a maximum and it is attained at $t^* = \frac{1}{2}(j\pi + \psi)$ for $j \in \{-1, 1\}$ as a direct calculation shows (here, the value of the index j is to be chosen in such a way that $t^* \in [0, 2\pi]$ is fulfilled, which in turn only depends on ψ). This finally yields

$$\max_{t\in[0,2\pi]} \left|re^{2it} - Re^{i\psi}\right| = \left|re^{2it^*} - Re^{i\psi}\right| = \left|re^{\pm i\pi} - R\right| = R + r$$

If the outer boundary component of $\mathcal{A}$ contains the maximum, analogous reasoning as given above with $z = Re^{it} \in \partial\mathcal{A}$ and $t \in [0, 2\pi]$ applies due to

$$\frac{1}{|z|} \left|z^2 - e^{i\psi}Rr\right| = \frac{1}{R} \left|R^2e^{2it} - e^{i\psi}Rr\right| = \left|Re^{2it} - re^{i\psi}\right| \qquad \square$$

Remark 3.6.4.

(i) *From a geometrical point of view, the third result on the "mixed" situation in Lemma 3.6.3 is not surprising: A mapping $e^{i\varphi}z \in U_1$ simply rotates the annulus $\mathcal{A}$ about the origin, whereas a mapping $e^{i\alpha}Rr/z \in U_2$ also rotates $\mathcal{A}$ but also interchanges the boundary components (see [Kra04, p. 123]). Hence, the maximal distance between these two mappings in the supremum metric – due to the fact that it is located on the boundary by the maximum modulus principle – will be attained at points on the inner (outer) boundary component having maximal distance from the point $Re^{i\psi}$ ($re^{i\psi}$). And this maximal distance is exactly the value $R + r$, as illustrated in Figure 3.4.*

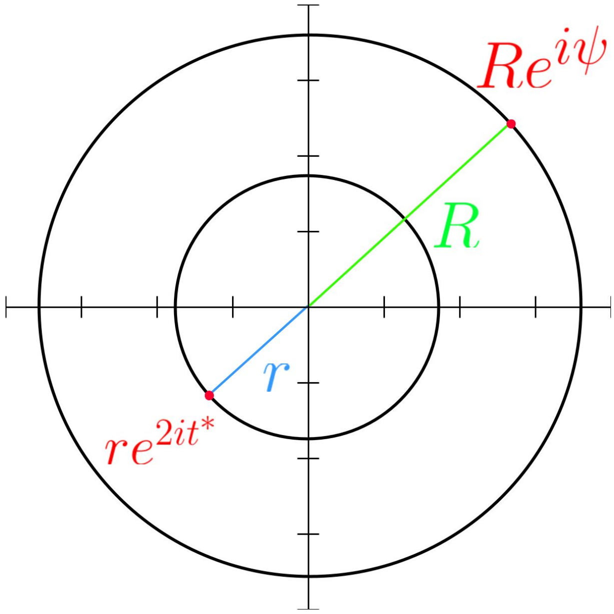

Figure 3.4: Distance in the space $\Sigma(\mathcal{A})$ between a conformal mapping $e^{i\varphi}z \in U_1$ and a conformal reflection $e^{i\alpha}\frac{Rr}{z} \in U_2$ which interchanges the boundary components of the annulus $\mathcal{A}$.

(ii) *In particular, Lemma 3.6.3 states that the two connected components U_1 and U_2 of $\Sigma(\mathcal{A})$ have constant $d_{\sup}$–distance from each other.*

The previous result implies that whenever a sequence of conformal automorphisms of $\mathcal{A}$ converges uniformly to an element of U_m (with $m \in \{1, 2\}$), then, from a certain index on, every element of the sequence already lies in U_m. This observation will be crucial in the proof of

Theorem 3.6.5. *The mapping H defined in Proposition 3.6.2 is a homeomorphism between the metric spaces $(\Sigma(\mathcal{A}), d_{\sup})$ and $(O_2(\mathbb{R}), \|\cdot\|_2)$.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ in $\Sigma(\mathcal{A})$ be convergent with limit $f \in \Sigma(\mathcal{A})$. Following the previous discussion, two cases have to be considered:

(a) $f_n = e^{i\varphi_n}z \in U_1$ for sufficiently large n: Then $f = e^{i\varphi}z \in U_1$ since U_1 is closed in $\Sigma(\mathcal{A})$. The convergence of f_n to f implies $|e^{i\varphi_n} - e^{i\varphi}| \longrightarrow 0$, hence it is $\cos(\varphi_n) \longrightarrow \cos(\varphi)$ and $\sin(\varphi_n) \longrightarrow \sin(\varphi)$ for $n \longrightarrow \infty$. By Proposition 3.6.2, it is $H(f_n), H(f) \in SO_2(\mathbb{R})$, therefore

$$\|H(f_n) - H(f)\|_2 = \left\| \begin{pmatrix} \cos(\varphi_n) - \cos(\varphi) & -\sin(\varphi_n) + \sin(\varphi) \\ \sin(\varphi_n) - \sin(\varphi) & \cos(\varphi_n) - \cos(\varphi) \end{pmatrix} \right\|_2 \overset{n\to\infty}{\longrightarrow} 0$$

showing the continuity of H in the first case.

(b) $f_n = e^{i\varphi_n}\frac{Rr}{z} \in U_2$ for sufficiently large n: Then $f = e^{i\varphi}\frac{Rr}{z} \in U_2$ since U_2 is closed in $\Sigma(\mathcal{A})$. The convergence of f_n to f implies $|e^{i\varphi_n} - e^{i\varphi}| \longrightarrow 0$, hence it is $\cos(\varphi_n) \longrightarrow \cos(\varphi)$ and $\sin(\varphi_n) \longrightarrow \sin(\varphi)$ for $n \longrightarrow \infty$. By Proposition 3.6.2, it is $H(f_n), H(f) \in O_2(\mathbb{R}) \backslash SO_2(\mathbb{R})$, therefore

$$\|H(f_n) - H(f)\|_2 = \left\| \begin{pmatrix} \cos(\varphi_n) - \cos(\varphi) & \sin(\varphi_n) - \sin(\varphi) \\ \sin(\varphi_n) - \sin(\varphi) & -\cos(\varphi_n) + \cos(\varphi) \end{pmatrix} \right\|_2 \overset{n\to\infty}{\longrightarrow} 0$$

showing the continuity of H in the second case.

Hence, the mapping H is continuous. In the same manner and using the same reasoning as above, the continuity of the inverse mapping H^{-1} can be seen, establishing the homeomorphism property of H. □

Since the mapping H is both, a group isomorphism and a homeomorphism, all group–theoretical and topological properties of $O_2(\mathbb{R})$ are transported to $\Sigma(\mathcal{A})$, an important result that is summarized in

Corollary 3.6.6 (Properties of $\Sigma(\mathcal{A})$).
The metric space $(\Sigma(\mathcal{A}), d_{\sup})$ is compact and consists of exactly two (path)–connected components, U_1 and U_2; in particular, the space is complete, locally compact, separable and disconnected, but not totally disconnected and thus not discrete. The connected component U_1 is a closed and therefore compact subspace. Moreover, $\Sigma(\mathcal{A})$ is a topological group, i.e. composition and inversion in $\Sigma(\mathcal{A})$ are continuous mappings, and U_1 is the identity component (i.e. the connected component of the identity $\mathrm{id}_{\mathcal{A}}$); in particular, $U_1 \triangleleft \Sigma(\mathcal{A})$.

Corollary 3.6.6 reveals substantial differences between the simply connected and the doubly connected situation, as can be seen by comparison with Proposition 1.2.1. Moreover, due to the previously mentioned fact that the orthogonal group $O_2(\mathbb{R})$ is in fact even a Lie group, the question may be raised whether this special and rich mathematical structure is also transferred to $\Sigma(\mathcal{A})$ via the mapping H, i.e. whether composition and inversion are actually *analytic* mappings in $\Sigma(\mathcal{A})$. A classical result of H. Cartan is that $\Sigma(G)$ is in fact *always* a Lie group when endowed with the topology of locally uniform convergence for any bounded domain, independent of the domain's connectivity (see [RS07, p. 202]); this circumstance might lead to the conjecture that $\Sigma(\mathcal{A})$ is a Lie group in the topology of uniform convergence as well, especially in view of Näkki–Palka's Proposition 1.1.10 on convergent sequences of K–quasiconformal automorphisms for uniformly bounded maximal dilatation K.

Another aspect to be considered is the behaviour of inversion as well as right and left multiplication in $\Sigma(\mathcal{A})$ with respect to the notion of distance provided by the supremum metric. In the simply connected setting, Gaier showed that right multiplication is always an isometry of $\Sigma(G)$ onto itself (for every bounded, simply connected domain G), but left multiplication is not isometric in general (see [Gai84, p. 234]). These circumstances change completely for annulus domains with non–degenerated boundary components, as shown in

Theorem 3.6.7 (Group operations in $\Sigma(\mathcal{A})$ are isometric).
Inversion, left and right multiplication in $\Sigma(\mathcal{A})$ are isometric self–mappings. In particular, it is $d_{\sup}(f, \mathrm{id}_{\mathcal{A}}) = d_{\sup}(f^{-1}, \mathrm{id}_{\mathcal{A}})$ for all $f \in \Sigma(\mathcal{A})$.

Proof. The claim that right multiplication in $\Sigma(\mathcal{A})$ is isometric is proved in the very same manner as in the case of simply connected domains (see Proposition 1.3.6). For the left multiplication $L_g(f) := g \circ f$, the two cases $g \in U_1$ and $g \in U_2$ need to be treated:

(a) $\underline{g = e^{i\varphi}z \in U_1}$: Let $f, h \in \Sigma(\mathcal{A})$, then

$$d_{\sup}(L_g(f), L_g(h)) = d_{\sup}(g \circ f, g \circ h) = \sup_{z \in \mathcal{A}} \left| e^{i\varphi} f(z) - e^{i\varphi} h(z) \right| = d_{\sup}(f, h)$$

hence left multiplication with elements of U_1 is isometric.

(b) $\underline{g = e^{i\varphi}\frac{Rr}{z} \in U_2}$: Three subcases will be considered:

(b.1) Let $f = e^{i\alpha}z, h = e^{i\psi}z \in U_1$, then

$$\begin{aligned} d_{\sup}(L_g(f), L_g(h)) = d_{\sup}(g \circ f, g \circ h) &= \sup_{z \in \mathcal{A}} \left| e^{i\varphi}\frac{Rr}{e^{i\alpha}z} - e^{i\varphi}\frac{Rr}{e^{i\psi}z} \right| = Rr \left| e^{-i\alpha} - e^{-i\psi} \right| \cdot \sup_{z \in \mathcal{A}} \left| \frac{1}{z} \right| \\ &= R \left| e^{i\alpha} - e^{i\psi} \right| = d_{\sup}(f, h) \end{aligned}$$

hence left multiplication with elements of U_2 is isometric on U_1.

(b.2) Let $f = e^{i\alpha}\frac{Rr}{z}, h = e^{i\psi}\frac{Rr}{z} \in U_2$, then

$$\begin{aligned} d_{\sup}(L_g(f), L_g(h)) = d_{\sup}(g \circ f, g \circ h) &= \sup_{z \in \mathcal{A}} \left| e^{i\varphi}\frac{Rr}{e^{i\alpha}Rr/z} - e^{i\varphi}\frac{Rr}{e^{i\psi}Rr/z} \right| = \sup_{z \in \mathcal{A}} \left| e^{-i\alpha}z - e^{-i\psi}z \right| \\ &= R \left| e^{-i\alpha} - e^{-i\psi} \right| = R \left| e^{i\alpha} - e^{i\psi} \right| = d_{\sup}(f, h) \end{aligned}$$

hence left multiplication with elements of U_2 is isometric on U_2.

(b.3) Let $f = e^{i\alpha}z \in U_1, h = e^{i\psi}\frac{Rr}{z} \in U_2$, then

$$\begin{aligned} d_{\sup}(L_g(f), L_g(h)) = d_{\sup}(g \circ f, g \circ h) &= \sup_{z \in \mathcal{A}} \left| e^{i\varphi}\frac{Rr}{e^{i\alpha}z} - e^{i\varphi}\frac{Rr}{e^{i\psi}Rr/z} \right| = \sup_{z \in \mathcal{A}} \left| e^{-i\alpha}\frac{Rr}{z} - e^{-i\psi}z \right| \\ &= \sup_{z \in \mathcal{A}} \left| e^{i\alpha}e^{i\psi} \right| \cdot \left| e^{-i\alpha}\frac{Rr}{z} - e^{-i\psi}z \right| = \sup_{z \in \mathcal{A}} \left| e^{i\psi}\frac{Rr}{z} - e^{i\alpha}z \right| = d_{\sup}(f, h) \end{aligned}$$

hence left multiplication with elements of U_2 is isometric in the "mixed case".

All in all, left multiplication is $d_{\sup}$–isometric on all of $\Sigma(\mathcal{A})$. Now, the isometry of the inversion in $\Sigma(\mathcal{A})$ will be shown. Let $f = e^{i\varphi}z, g = e^{i\alpha}z \in U_1$, then

$$d_{\sup}(f^{-1}, g^{-1}) = \sup_{z \in \mathcal{A}} \left| e^{-i\varphi}z - e^{-i\alpha}z \right| = R \left| e^{-i\varphi} - e^{-i\alpha} \right| = R \left| e^{i\varphi} - e^{i\alpha} \right| = d_{\sup}(f, g)$$

For $f, g \in U_2$, one obtains $d_{\sup}(f^{-1}, g^{-1}) = d_{\sup}(f, g)$ immediately due to the fact that the elements of U_2 satisfy the involution property $f^{-1} = f$. In the "mixed case" $f = e^{i\varphi}z \in U_1, g = e^{i\alpha}\frac{Rr}{z} \in U_2$, Lemma 3.6.3 yields $d_{\sup}(f^{-1}, g) = R + r = d_{\sup}(f, g)$, hence

$$d_{\sup}(f^{-1}, g^{-1}) = d_{\sup}(f^{-1}, g) = R + r = d_{\sup}(f, g) \qquad \square$$

3.6.2 Doubly connected domains: Topological properties of $Q(\mathcal{A})$

General radial stretchings in the doubly–connected case

Now the quasiconformal automorphism group of the annulus $\mathcal{A} = \mathcal{A}_{r,R}(0)$ will be considered. Naturally, the set $Q(\mathcal{A})$ forms a group with the group operation being the composition of mappings and with neutral element $\mathrm{id}_{\mathcal{A}}$, analogously to the situation for simply connected domains in $\mathbb{C}$. Furthermore, it is $\Sigma(\mathcal{A}) \leq Q(\mathcal{A})$, but beside this, the set $Q(\mathcal{A})$ contains proper quasiconformal automorphisms $f \in Q(\mathcal{A}) \backslash \Sigma(\mathcal{A})$, an example for such a mapping is presented in [Bie17, pp. 18–20], see also (3.10). Moreover, an important and interesting subgroup in the simply connected setting are general radial stretchings. In the unit disk case $G = \mathbb{D}$, these mappings are defined in

Definition 2.3.1. The crucial point now is that in this process, the interval $[0,1]$ can obviously and easily be replaced by $[r,R]$, yielding a mapping

$$f_\rho(z) := \rho(r)e^{i\varphi}$$

for $z = re^{i\varphi} \in \mathcal{A}$ which clearly is a bijective, continuous mapping of the annulus $\mathcal{A}$ onto itself. This naturally leads to

Definition 3.6.8.
Let $\rho : [r,R] \longrightarrow [r,R]$ be continuous, bijective and strictly increasing. Then the mapping

$$f_\rho : \mathcal{A} \longrightarrow \mathbb{C},\ z = re^{i\varphi} \longmapsto f_\rho(z) := \rho(r)e^{i\varphi}$$

is called ***(general) radial stretching*** *of $\mathcal{A}$.*

In turn, if the radial dilation mapping ρ is chosen appropriately in Definition 3.6.8, then f_ρ represents a quasiconformal automorphism of $\mathcal{A}$, i.e. $f_\rho \in Q(\mathcal{A})$. Thus, in the same manner as in the simply connected case (see Lemma 2.3.2), one arrives at

Lemma 3.6.9.
For each mapping $\rho \in C([r,R])$ as in Definition 3.6.8 such that ρ is a piecewise C^1–mapping on $[r,R]$, the corresponding general radial stretching f_ρ is a quasiconformal automorphism of $\mathcal{A}$.

Incompleteness and conjugation mapping for $Q(\mathcal{A})$

In the simply connected case, general radial stretchings were used in order to show the incompleteness of the metric space $Q(G)$, see Theorem 2.3.3. The very same construction may be used in the annulus case as well, leading to

Theorem 3.6.10.
The metric space $Q(\mathcal{A})$ is incomplete. In particular, this space is not compact.

Proof. The interval $[0,1]$, corresponding to the "unit disk situation" in the proof of Theorem 2.3.3, is homeomorphic to the interval $[r,R]$ in the present annulus case via the affine–linear mapping

$$w(x) = (R-r)x + r$$

which is obviously a C^1–mapping and thus uniformly continuous. Hence, the construction utilized in the mentioned incompleteness proof for the simply connected case transfers to $Q(\mathcal{A})$, yielding that the sequence $(w \circ \rho_n \circ w^{-1})_{n\in\mathbb{N}}$ of piecewise C^1–mappings of the interval $[r,R]$ converges uniformly to a non–injective limit mapping on $[r,R]$, the mappings ρ_n being defined by (2.4). The corresponding general radial stretchings

$$f_n(z) = (w \circ \rho_n \circ w^{-1})(r)e^{i\varphi}$$

for $z = re^{i\varphi} \in \mathcal{A}$ are quasiconformal automorphisms of $\mathcal{A}$ according to Lemma 3.6.9, uniformly converging to a non–injective limit mapping. □

So far, only the "standard" domain in the doubly–connected case was treated, i.e. the annulus $\mathcal{A} = \mathcal{A}_{r,R}(0)$. Thus the question for the transmission of the group–theoretic and topological properties to more general domains of this type arises naturally. This question is answered by utilizing the well–known conjugation mapping between the automorphism groups $Q(\mathcal{A})$ and $Q(G)$ for a "well–behaved" doubly–connected domain G in

Lemma 3.6.11. *Let $G \subsetneq \mathbb{C}$ be a doubly–connected domain whose boundary components consist of Jordan curves and $F : \mathcal{A} \longrightarrow G$ be a conformal mapping. Then the induced conjugation mapping*

$$\Phi : Q(\mathcal{A}) \longrightarrow Q(G),\, g \longmapsto \Phi(g) := F \circ g \circ F^{-1}$$

is a homeomorphism and a group isomorphism. In particular, the conformal automorphism (sub)groups $\Sigma(\mathcal{A})$ and $\Sigma(G)$ are homeomorphic.

Proof. Let $\epsilon > 0$ and $(g_n)_n$ be convergent in $Q(\mathcal{A})$ to $g \in Q(\mathcal{A})$. Then for $w \in G$:

$$|\Phi(g_n)(w) - \Phi(g)(w)| = |F(g_n(z)) - F(g(z))| \le \omega_F(|g_n(z) - g(z)|) \le \omega_F(d_{\sup}(g_n, g))$$

where ω_F denotes the modulus of continuity of F and $z = F^{-1}(w) \in \mathcal{A}$. Since F is uniformly continuous on $\mathcal{A}$, the last term becomes $< \epsilon$ for sufficiently large n. Applying the supremum over $w \in G$ to both sides of the resulting inequality shows that Φ is continuous. The same arguments apply to Φ^{-1} due to the uniform continuity of F^{-1}, concluding in the desired result. The claim on the conformal automorphism (sub)groups follows from the facts that Φ is a group isomorphism (which, in turn, can be seen in the very same manner as in the simply connected case, see Proposition 1.3.2(i)) and that composition of conformal maps is conformal. □

The following statement is a classical result on the conformal automorphism group of domains in $\mathbb{C}$ having arbitrary, but finite connectivity and with respect to the topology of locally uniform convergence (see [Kra06, Theorem 12.2.3, p. 263]):

> *Let $\Omega \subsetneq \mathbb{C}$ be a bounded domain with C^1 boundary (i.e., the boundary consists of finitely many simple, closed, continuously differentiable curves). If Ω has non–compact automorphism group* (w.r.t. locally uniform convergence), *then Ω is conformally equivalent to the unit disk $\mathbb{D}$.*

Using the previously established observations, this result can be transferred to the situation of uniform convergence as follows:

Theorem 3.6.12.
Let $G \subsetneq \mathbb{C}$ be a bounded domain in $\mathbb{C}$ whose boundary consists of finitely many Jordan curves. If the space $\Sigma(\Omega)$ is non–compact in the uniform topology, then G is conformally equivalent to $\mathbb{D}$.

Proof. If G would be doubly connected, there would exist a conformal mapping $F : G \longrightarrow \mathcal{A}_{r,R}$ for certain radii $0 < r < R < \infty$ which extends homeomorphically to the boundary. Lemma 3.6.11 implies that the induced conjugation mapping $\Phi : \Sigma(G) \longrightarrow \Sigma(\mathcal{A})$ is a homeomorphism, contradicting the compactness of $\Sigma(\mathcal{A})$ (see Corollary 3.6.6). If G would be n–connected for some $n \in \mathbb{N}, n \ge 3$, then the classical result of Koebe on the cardinality of conformal automorphism groups (see e.g. [Gai84, p. 256] and [Kra06, p. 278]) yields that $\Sigma(G)$ is a finite group, hence the space $\Sigma(G)$ would surely be compact in the topology induced by $d_{\sup}$, which is a contradiction. □

A concrete example for an automorphism of $\mathcal{A}$

In this subsection, let $R > 1$ be a fixed constant, $\mathcal{A} := \mathcal{A}_{1,R}$ and define

$$f : \mathcal{A} \longrightarrow \mathbb{C},\ z \longmapsto f(z) := z|z|^{\frac{2\pi i}{\ln(R)}} \tag{3.10}$$

The mapping f is called a **Full Dehn twist** and was already investigated in [Bie17, pp. 18–20]. The mapping f can be written as $f(z) = ze^{2\pi i \frac{\ln(|z|)}{\ln(R)}}$ and in a certain sense reminds of a general radial stretching via the function $\rho(t) = t^{1+\frac{2\pi i}{\ln(R)}}$ (however, f is *not* a radial stretching of $\mathcal{A}$ as defined in Definition 3.6.8, for ρ does not map the interval $[1, R]$ onto itself).

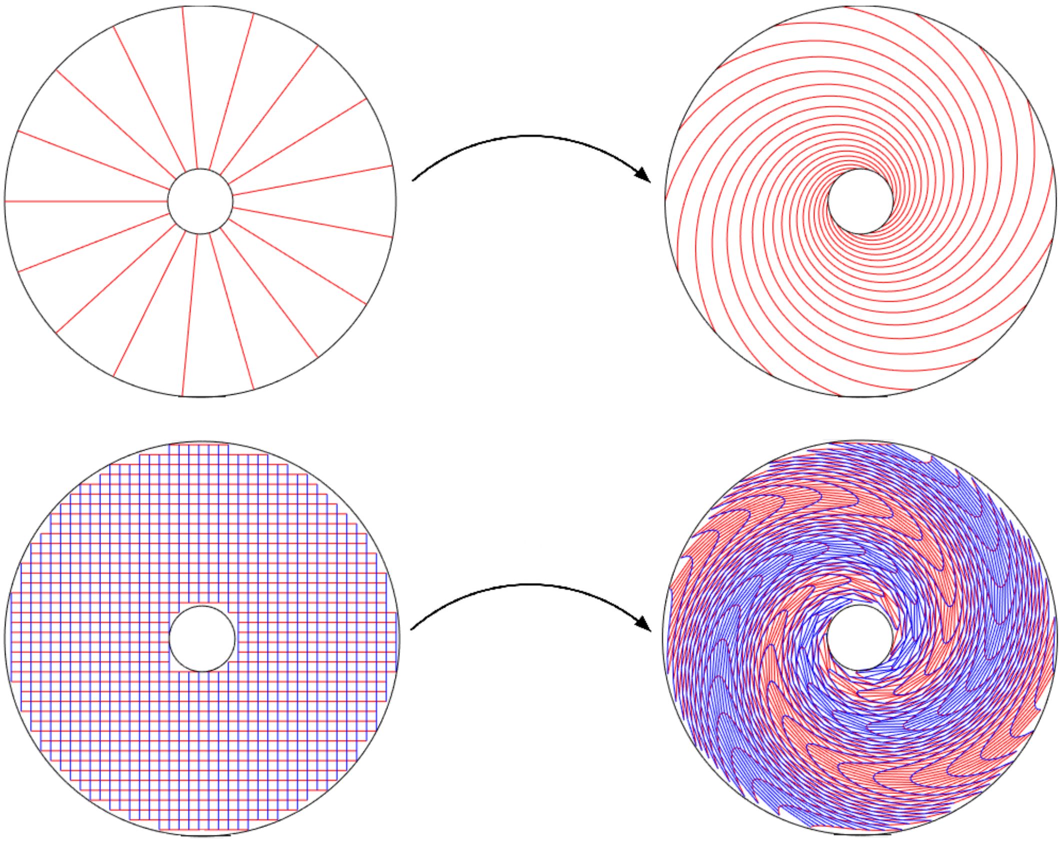

Figure 3.5: The image of the annulus $A_{1,6}$ under the corresponding Dehn twist f. The first figure shows that the radial rays are mapped onto spirals winding around the inner boundary circle, whereas the Euclidean grid is mapped onto a "spiral–like" grid as visualized by the second figure.

It turns out that f maps the annulus $\mathcal{A}$ homeomorphically onto itself, coincides with the identity on $\partial\mathcal{A}$ and is in fact a quasiconformal automorphism of $\mathcal{A}$, i.e. $f \in Q(\mathcal{A})$ (see [GL00, p. 204]). Thus, one may consider the iterated mapping

$$f^{[n]} := f \circ f \circ \cdots \circ f \ \ (n \text{ times})$$

for every $n \in \mathbb{N}_0$ with $f^{[0]} := \mathrm{id}_{\mathcal{A}}$, thereby defining a sequence $(f^{[n]})_{n\in\mathbb{N}_0}$ in $Q(\mathcal{A})$. A simple induction argument shows

Lemma 3.6.13.
For every $n \in \mathbb{N}_0$ and $z \in \mathcal{A}$, it is

$$f^{[n]}(z) = z|z|^{n\frac{2\pi i}{\ln(R)}} = z\left[e^{2\pi i\frac{\ln(|z|)}{\ln(R)}}\right]^n \tag{3.11}$$

Proof. The case $n = 0$ is obviously true. Assume that (3.11) is true for some $n \in \mathbb{N}_0$ and all $z \in \mathcal{A}$, and consider

$$f^{[n+1]}(z) = f^{[n]}(f(z)) = z|z|^{\frac{2\pi i}{\ln(R)}} \cdot \left|z|z|^{\frac{2\pi i}{\ln(R)}}\right|^{n\frac{2\pi i}{\ln(R)}} = z|z|^{\frac{2\pi i}{\ln(R)}} \cdot \left(|z|\left||z|^{\frac{2\pi i}{\ln(R)}}\right|\right)^{n\frac{2\pi i}{\ln(R)}}$$

The second absolute value in the bracket evaluates to

$$\left||z|^{\frac{2\pi i}{\ln(R)}}\right| = \left|e^{\ln(|z|)\frac{2\pi i}{\ln(R)}}\right| = \left|e^{i\cdot\frac{2\pi\ln(|z|)}{\ln(R)}}\right| = 1$$

thus one arrives at

$$f^{[n+1]}(z) = z|z|^{\frac{2\pi i}{\ln(R)}} \cdot |z|^{n\frac{2\pi i}{\ln(R)}} = z|z|^{(n+1)\frac{2\pi i}{\ln(R)}}$$ □

Let $\alpha \in (0,1) \cap (\mathbb{R}\backslash\mathbb{Q})$ be an irrational number, then $t_\alpha := R^\alpha \in (1,R) \subseteq \mathcal{A}$ and thus

$$f^{[n]}(t_\alpha) = t_\alpha e^{2\pi i n\alpha}$$

by the previous Lemma 3.6.13. Hence, the resulting sequence of points $(f^{[n]}(t_\alpha))_{n\in\mathbb{N}_0} \subseteq \mathcal{A}$ clearly has no convergent subsequence, yielding that the sequence of quasiconformal automorphisms $(f^{[n]})_n$ cannot possess a subsequence convergent in $Q(\mathcal{A})$ to a quasiconformal limit mapping. This shows again that the space $Q(\mathcal{A})$ is non–compact, a result that was already stated in Theorem 3.6.10. Furthermore, since every bounded, doubly–connected domain in $\mathbb{C}$ having Jordan curves as its boundary components is conformally equivalent to an annulus $\mathcal{A}$, this insight in connection with Lemma 3.6.11 implies

Corollary 3.6.14.
Let $G \subsetneq \mathbb{C}$ be a doubly–connected domain whose boundary components consist of Jordan curves. Then the space $Q(G)$ is non–compact.

3.6.3 Topological characteristics of $Q(G_N)$ for finitely connected domains

In this subsection, G_N will denote a bounded, N–connected domain in $\mathbb{C}$ for $N > 1$ with finitely many boundary components, all of which are supposed to be non–degenerated.

If $N \geq 3$, the classical result of Koebe states that every domain G_N admits only *finitely many* conformal automorphisms, i.e. the group $\Sigma(G_N)$ is finite (this was already stated in the proof of Theorem 3.6.12). Consequently, the space $\Sigma(G_N)$ is compact and discrete, therefore a lot of information about the topological structure of these spaces is known. In contrast to this situation, it is shown in the following that the topology of $Q(G_N)$ differs from $\Sigma(G_N)$ in a versatile manner. In fact, by embedding the "standard domain" $\mathbb{D}$ into G_N via conformal equivalence, many of the topological properties of $Q(G)$ for bounded, simply connected domains G carry over to $Q(G_N)$.

Let $z_0 \in G_N$ be a fixed inner point and $\epsilon > 0$ such that the open ball $U := B_\epsilon(z_0)$ is completely contained in G_N. Then by conformal equivalence[9], the unit disk corresponds to U. Consequently, every $g \in Q(\mathbb{D})$ corresponds to an element $\widetilde{g} \in Q(U)$, and the spaces $Q(\mathbb{D})$ and $Q(U)$ are homeomorphic. In addition, if the continuous extension of g to $\overline{\mathbb{D}}$ satisfies $g \equiv \mathrm{id}_{\partial\mathbb{D}}$ on $\partial\mathbb{D}$, a quasiconformal automorphism h of G_N can be constructed by

$$h(z) := \begin{cases} \widetilde{g}(z), & z \in U \\ \mathrm{id}_{G_N}(z), & z \in G_N\backslash U \end{cases} \tag{3.12}$$

for $z \in G_N$. By utilizing this construction appropriately, the following information can be retrieved for $Q(G_N)$:

- **Incompleteness:** The metric space $(Q(G_N), d_{\sup})$ is incomplete, since the corresponding space $Q(U)$ is incomplete. Using the convergent sequence with non–injective limit mapping from the proof of Theorem 2.3.3 shows the existence of an analogously defined sequence in $Q(G_N)$ (via the construction (3.12)) that converges uniformly on G_N to a non–injective limit mapping. In particular, this shows that the space $Q(G_N)$ is non–compact.

- **Uncountability:** The set $Q(G_N)$ is always uncountable. This can be seen by taking into account the set of monomial–like radial stretchings $f_K(z) = z|z|^{K-1}$ for $K \in \mathbb{R}^+$ (see (2.3)) and transferring them via (3.12) to $Q(G_N)$.

[9] Of course, this conformal equivalence can explicitly be specified by simply translating and scaling $\mathbb{D}$.

- **$Q(G_N)$ is no Baire space and conclusions:** The subsets $Q_K(G_N)$ as defined in (0.5) can of course be analogously considered for the domains G_N. Based on this, the following conclusions can be drawn:
    - (i) The reasoning prior to Lemma 3.3.2, where it is derived that the subsets $Q_K(G)$ have empty interior for bounded, simply connected domains G, can be transferred to the subsets $Q_K(G_N)$, since the used arguments are independent of the connectivity of the underlying domains (in particular, the d_{sup}–isometry of right multiplication does neither depend on the particular domain, nor on the fact that the mappings are quasiconformal). Hence, the subsets $Q_K(G_N)$ have empty interior.
    - (ii) Furthermore, the subsets $Q_K(G_N)$ are also closed in $Q(G_N)$ by means of the Hurwitz–type Theorem (Proposition 1.1.9) for every $K \in [1, +\infty)$, thus $Q_K(G_N)$ is nowhere dense in $Q(G_N)$. An immediate consequence of this is that the space $Q(G_N)$ is meager, which in turn implies that $Q(G_N)$ is no Baire space.
    - (iii) Finally, the statements of Theorem 3.3.6 and Corollary 3.3.7 are also valid for the domains G_N: The topological space $Q(G_N)$ is not locally compact and not completely metrizable.

Furthermore, regarding dense subsets of $Q(G_N)$, the statements of Theorem 3.2.5 and Remark 3.3.5(i) remain valid for multiply connected domains, which can be seen as follows:

- Concerning Theorem 3.2.5: The crucial part in the proof of this theorem is Kiikka's result on the approximation of quasiconformal mappings by C^∞–diffeomorphisms which are also quasiconformal, see Proposition 3.2.3. However, the connectivity of the underlying domain plays no role in Kiikka's theorem, thus this statement is also valid for G_N (see also Remark 3.2.7(i)). This implies that the quasiconformal C^∞–diffeomorphisms of G_N are dense in $Q(G_N)$.

- In view of Remark 3.3.5(i) regarding the denseness of the complement of $Q_K(G)$ in $Q(G)$: Since the analogous reasoning was applied above for $Q(G_N)$, this result also holds in the multiply connected case, i.e. the subset

$$Q(G_N)\backslash Q_K(G_N) = \{ f \in Q(G_N) \mid K(f) > K \}$$

  is dense and open in the space $Q(G_N)$ for every $K \in [1, +\infty)$. In particular, the space $Q(G_N)$ is not discrete, since id_{G_N} can be approximated arbitrarily well with respect to d_{sup} due to (3.3).

A summary of the previously established results for the space $Q(G_N)$ is given in

Theorem 3.6.15.
Let G_N be a bounded, N–connected domain in $\mathbb{C}$ with $N > 1$ and finitely many, non–degenerated boundary components. Then the quasiconformal automorphism group $Q(G_N)$ of G_N is uncountable. Furthermore, the topological space $Q(G_N)$ has the following properties:
(i) $Q(G_N)$ is neither compact nor locally compact.
(ii) $Q(G_N)$ is incomplete and not completely metrizable.
(iii) $Q(G_N)$ is not discrete and contains the following dense subsets:

- *The quasiconformal C^∞–diffeomorphisms;*
- *The subsets $Q(G_N)\backslash Q_K(G_N)$ for every fixed $K \in [1, +\infty)$.*

Several differences between confomal and quasiconformal automorphism groups of simply and multiply connected (bounded) domains are described in the following tabulated comparison:

Property	*Connectivity*	$Q(G)$	$\Sigma(G)$
Compactness	$N = 1$	Not compact	Not compact
	$N = 2$	Not compact	Depending on ∂G
	$N \geq 3$	Not compact	Compact
Local compactness	$N = 1$	Not locally compact	Depending on ∂G
	$N = 2$	Not locally compact	Depending on ∂G
	$N \geq 3$	Not locally compact	Locally compact
Completeness	$N = 1$	Incomplete	Complete
	$N \geq 2$	Incomplete	Complete
Discreteness	$N = 1$	Not discrete	Depending on ∂G
	$N = 2$	Not discrete	Depending on ∂G
	$N \geq 3$	Not discrete	Discrete
Cardinality	$N \leq 2$	Uncountable	Uncountable
	$N \geq 3$	Uncountable	Finite

Table 3.1: Similarities and differences between $Q(G)$ and $\Sigma(G)$ for bounded, simply and multiply connected domains $G \subsetneq \mathbb{C}$ with finite connectivity and non–degenerated boundary components.

A concrete multiply connected domain and a corresponding quasiconformal automorphism are presented in

Example 3.6.16.
Let the domain Ω_N be defined as follows: Consider the unit disk $\mathbb{D}$ from which $N-1$ mutually disjoint, compact (proper) subsets of the closed annulus $\overline{\mathcal{A}_{1/3,2/3}}$ are removed, i.e.

$$\Omega_N := \mathbb{D} \setminus \dot{\bigcup_{j=1}^{N-1}} C_j \tag{3.13}$$

with

$$C_j := \left\{ z \in \mathbb{D} \;\middle|\; \frac{1}{3} \leq |z| \leq \frac{2}{3},\ \operatorname{Arg}(z) \in [\theta_{j-1}, \theta_j] \right\}$$

and the angle intervals $[\theta_{j-1}, \theta_j]$ are appropriately chosen subintervals of the canonical angle interval $(-\pi, \pi]$ for $j = 1, \ldots, N-1$. An example of such a domain Ω_N with $N = 3$ is shown in Figure 3.6.

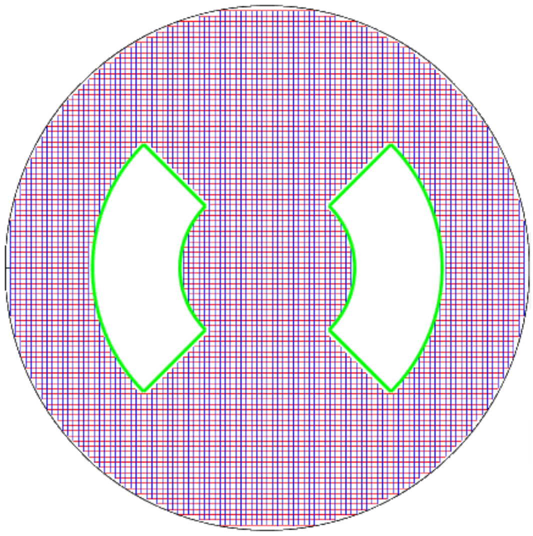

Figure 3.6: A triply–connected domain Ω_3 equipped with a Euclidean grid, outer boundary curve $\partial\mathbb{D}$ and inner boundary components ∂C_1 and ∂C_2 (in green) corresponding to the angle intervals $[-\frac{\pi}{4}, \frac{\pi}{4}]$ and $[\frac{3\pi}{4}, \pi] \cup (-\pi, -\frac{3\pi}{4})$. The inner boundary components are clearly symmetric with respect to both, the real and imaginary axis.

Now, an appropriately chosen subfamily of general radial stretchings introduced in Definition 3.6.8 will be considered, thereby defining a family of quasiconformal automorphisms of the domain Ω_N as follows: The radial dilation mapping $\rho : [0,1] \longrightarrow [0,1]$ is supposed to be a bijective, piecewise C^1–mapping having fixed points at $r = 0, r = \frac{1}{3}, r = \frac{2}{3}$ and $r = 1$; such a mapping surely exists, probably the most simple example is given by $\rho = \mathrm{id}_{[0,1]}$, but of course there are many further functions of this particular kind. Next, for $z = re^{i\varphi} \in \Omega_N$ define

$$f(z) = \rho(r)e^{i\varphi}$$

which can be regarded as a "general radial stretching" of the domain Ω_N. By construction, f is a continuous injective mapping of Ω_N onto itself (hence a homeomorphism by [LV73, Lemma 1.1, p. 6]), and – basically due to Lemma 2.3.2 which is concerned with the simply connected case in $\mathbb{D}$ and the special construction of the radial dilation mapping ρ – the mapping f turns out to be a quasiconfomal automorphism of Ω_N. An example for a radial dilation mapping fulfilling the demanded requirements described above is given by the piecewise defined function

$$\rho^*(r) := \begin{cases} 9r^3, & r \in \left[0, \frac{1}{3}\right) \\ \frac{1}{3}\left(\frac{\ln(3r)}{\ln(2)} + 1\right), & r \in \left[\frac{1}{3}, \frac{2}{3}\right) \\ \frac{1}{3}\left(\frac{1}{15}((3r-1)^4 + 14) + 1\right), & r \in \left[\frac{2}{3}, 1\right] \end{cases} \tag{3.14}$$

which is clearly a piecewise C^1–mapping on the interval $[0,1]$. The image of the domain Ω_3 shown in Figure 3.6 under this particular mapping $f^*(z) := \rho^*(r)e^{i\varphi}$ (with $z = re^{i\varphi} \in G_3$) is depicted in Figure 3.7.

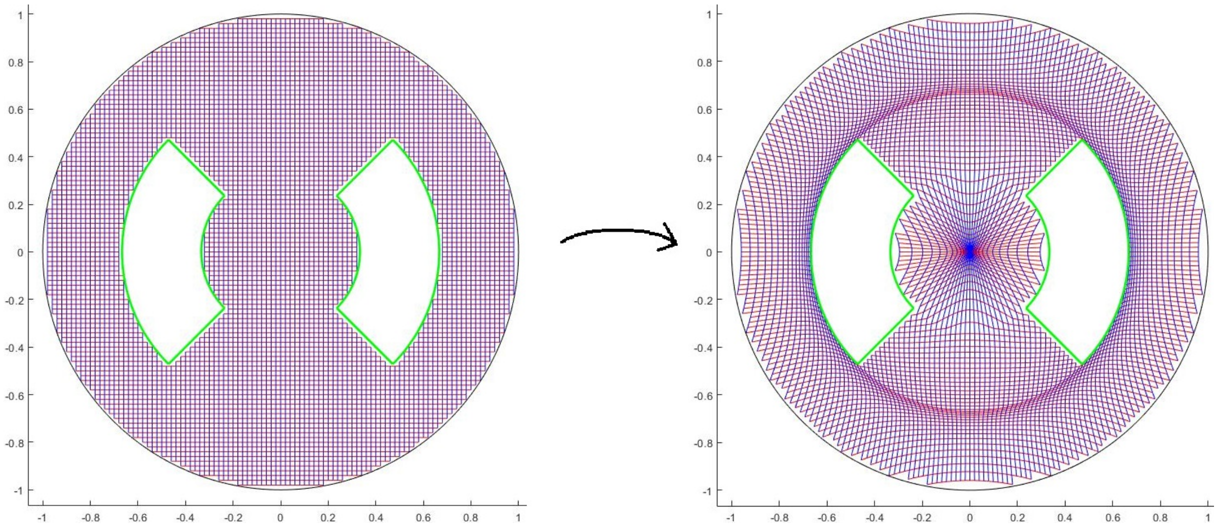

Figure 3.7: The image of the triply–connected domain Ω_3 and the belonging Euclidean grid (see Figure 3.6) under the general radial stretching f^* using the radial dilation mapping ρ^* in (3.14).

The mapping properties of the radial dilation mapping ρ^* are clearly visible in the corresponding graph of the "radial stretching" f^*: In the inner disk $|z| \leq \frac{1}{3}$, the Euclidean grid is pulled towards the origin due to the fact that the corresponding subfunction of ρ^* satisfies $9r^3 \leq r$ for $r \in [0, \frac{1}{3})$. On the annulus $\frac{1}{3} < |z| < \frac{2}{3}$, the graph of the subfunction $\frac{1}{3}\left(\frac{\ln(3r)}{\ln(2)} + 1\right)$ of ρ^* is very similar to the identity $\mathrm{id}_{[\frac{1}{3},\frac{2}{3}]}$, thus the Euclidean grid experiences virtually no deformation. In the last case, i.e. the outer annulus $\frac{2}{3} \leq |z| \leq 1$, the grid is again pulled towards the origin by the mapping behaviour of the third subfunction of ρ^*.

Remark 3.6.17.
Obviously, the definition of the domains Ω_N in (3.13) only depends on the angle intervals $[\theta_{j-1}, \theta_j]$ and is therefore reasonable for any finite connectivity N with appropriately chosen angles. Moreover, the choice of the radii $\frac{1}{3}$ and $\frac{2}{3}$ is also of minor importance and could be changed to other values.

Chapter 4

Quasiconformal automorphisms of $\mathbb{D}$: Subspaces, constructions, applications

This final chapter of the thesis at hand treats specialized topics intertwined with the quasiconformal automorphism group of the unit disk in several ways, ranging from a special subset of $Q(\mathbb{D})$ up to concrete examples of mappings of this kind and an outlook on an application based on a recent research paper.
Section 4.1 introduces the class of *harmonic quasiconformal automorphisms* of $\mathbb{D}$. This particular class of mappings has drawn much attention in the past, as demonstrated by the vast number of publications in connection with these objects. After introducing the set $HQ(\mathbb{D})$ in Definition 4.1.1 and deriving some basic facts, topological properties of this subset are investigated. Furthermore, it is shown in Theorem 4.1.21 that $HQ(\mathbb{D})$ forms a incomplete subspace of quasiconformal unit disk automorphisms. This is achieved by means of a famous mapping in Real Analysis, the *Cantor function*, which is often used as a counterexample for certain allegedly valid statements. As a major tool in studying the harmonic quasiconformal unit disk automorphisms, the Theorem of Radó–Kneser–Choquet on the representation of harmonic mappings is utilized.
Section 4.2 presents a novel and rather unexpected construction method for examples of quasiconformal automorphisms of $\mathbb{D}$. Again, monomial–like radial stretchings will be of particular importance as, on the one hand, they are studied in terms of additional algebraic structure, namely *commutative semirings*. On the other hand, the mentioned construction principle related to classical *Cesàro summation* in Real Analysis is applied to these mappings in order to generate interesting examples of quasiconformal automorphisms of $\mathbb{D}$.
Finally, Section 4.3 presents a possible construction of a *Quasiconformal Cryptosystem*, giving an application–oriented ending of this thesis.

4.1 Harmonic quasiconformal automorphisms of $\mathbb{D}$

The aim of this section is to investigate certain properties of interest of a special class of mappings which have drawn a huge amount of attention in recent years in the respective research area: The harmonic quasiconformal mappings. Initiated by Martio in 1968 (see [Kal08, p. 238] and [Pav02, p. 366]), this particular class of homeomorphisms attracted a huge amount of interest in the recent past, see [BM10, Introduction], [Kal08], [KN99], [PS99], [Pav02], [Pav14, Section 10.3] and the references therein, to name only a few. Especially Kalaj and Pavlović both worked intensively in this area and achieved numerous results, among others several characterization statements for harmonic quasiconformal automorphisms of the unit disk (see [Kal08, Theorem A, p. 239] and Proposition 4.1.17 below). Due to the Theorem of Radó–Kneser–Choquet (Proposition 4.1.2), the following discussion will focus on the case $G = \mathbb{D}$, i.e. the class of harmonic quasiconformal automorphisms of the unit disk.

4.1.1 Definition and basic properties

Definition 4.1.1.

$$HQ(\mathbb{D}) := \left\{ f \in Q(\mathbb{D}) \,\middle|\, f \text{ is harmonic} \right\}$$

That is, the elements of $HQ(\mathbb{D})$ are the harmonic quasiconformal automorphisms of $\mathbb{D}$. Here and henceforth, a complex–valued mapping $f = u + iv$ defined on a domain is called *harmonic* if both, its real and imaginary parts, are real–valued harmonic mappings, which in turn are defined via the *Laplace equation*

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

in which the differential polynomial $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the *Laplace operator*. An immediate conclusion to be drawn is $\Sigma(\mathbb{D}) \subseteq HQ(\mathbb{D})$; in particular, it is $\mathrm{id}_{\mathbb{D}} \in HQ(\mathbb{D})$ and therefore $HQ(\mathbb{D}) \neq \varnothing$. An important fact about harmonic mappings in $\mathbb{C}$ and their representation is given by the following result due to Radó, Kneser and Choquet ([Dur04, pp. 33–34], [DS87, pp. 154–156] and [Pav14, Theorem 1.1, p. 5]):

Proposition 4.1.2 (Radó–Kneser–Choquet, extended version)**.**
*Let $G \subsetneq \mathbb{C}$ be a convex Jordan domain and $\gamma : \partial\mathbb{D} \longrightarrow \partial G$ be a **weak homeomorphism**, i.e. a continuous mapping of $\partial\mathbb{D}$ onto ∂G such that the preimage $\gamma^{-1}(\xi)$ of each $\xi \in \partial G$ is either a point or a closed subarc of $\partial\mathbb{D}$. Then the **harmonic extension***

$$\mathfrak{P}[\gamma](z) := \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(t - \phi) + r^2} \gamma(e^{it}) \,\mathrm{d}t, \; z = re^{i\phi} \in \mathbb{D}, \tag{4.1}$$

defines an injective harmonic mapping of $\mathbb{D}$ onto G; moreover, $\mathfrak{P}[\gamma]$ is unique. Conversely, if $G \subsetneq \mathbb{C}$ is a strictly[1] convex Jordan domain and $f : \mathbb{D} \longrightarrow G$ is an injective harmonic mapping, then f has a continuous extension to $\overline{\mathbb{D}}$ which defines a weak homeomorphism of $\partial\mathbb{D}$ onto ∂G. Moreover, if $f \in C(\overline{\mathbb{D}})$ is harmonic in $\mathbb{D}$, then $f|_{\mathbb{D}}$ can be written in the form (4.1).

Let $\mathcal{H}^*(\partial\mathbb{D}, \partial G)$ denote the set of all weak homeomorphisms of $\partial\mathbb{D}$ onto ∂G and in the special case $G = \mathbb{D}$ define $\mathcal{H}^*(\partial\mathbb{D}) := \mathcal{H}^*(\partial\mathbb{D}, \partial\mathbb{D})$. Consequently, let $\mathcal{H}^+(\partial\mathbb{D}, \partial G)$ and $\mathcal{H}^+(\partial\mathbb{D})$ denote the corresponding subsets of all orientation–preserving homeomorphisms, respectively.

[1] A set $S \subseteq \mathbb{C}$ is called *strictly convex* if every point on the line segment connecting $x, y \in \overline{S}$ other than the endpoints is contained in the interior of S ([Dur04, p. 34]). For example, a circle is strictly convex (and in particular convex), while a rectangle is not strictly convex (yet convex).

Remark 4.1.3.
The harmonic extension $\mathfrak{P}[\gamma]$ defined by (4.1) is also called the ***Poisson transformation*** *of $\gamma \in \mathcal{H}^*(\partial\mathbb{D}, \partial G)$, and the corresponding integral kernel*

$$\frac{1 - r^2}{1 - 2r\cos(t) + r^2}$$

is called the ***Poisson kernel****, see [Dur04, p. 12] and [Pav14, pp. 5–6] for further information. Moreover, the Poisson transformation is intimately related to the* ***Dirichlet problem****, whose solution is given explicitly by (4.1); see [ABR01, pp. 12–15] and [Dur04, Section 1.4].*

From the Radó–Kneser–Choquet Theorem 4.1.2, one obtains the following (see also [KN99, pp. 337–338])

Corollary 4.1.4.

$$HQ(\mathbb{D}) = Q(\mathbb{D}) \bigcap \left\{\mathfrak{P}[\gamma] \,\middle|\, \gamma \in \mathcal{H}^+(\partial\mathbb{D})\right\}$$

Corollary 4.1.4 also makes sense when recalling that every quasiconformal automorphism of a Jordan domain admits a homeomorphic boundary extension by Proposition 1.1.6. In particular, the induced boundary mapping is injective, hence an element of $\mathcal{H}^+(\partial\mathbb{D})$. A concrete harmonic automorphism of the unit disk is visualized in

Example 4.1.5.
For $x \in [0,1]$, consider the piecewise–defined function

$$\phi(x) = \begin{cases} 2x, & x \in [0, \frac{1}{3}] \\ \frac{2}{3}, & x \in [\frac{1}{3}, \frac{3}{4}] \\ \frac{4}{3}x - \frac{1}{3}, & x \in [\frac{3}{4}, 1] \end{cases}$$

which is easily seen to map the interval $[0,1]$ continuously, but not injectively onto itself while keeping the endpoints $x = 0$ and $x = 1$ fixed. Transferring ϕ to the interval $[0, 2\pi]$ by conjugating it via the mapping $x \longmapsto t = 2\pi x$ yields a function $\widetilde{\varphi} \in C([0, 2\pi])$ with the very same properties. Consequently, the mapping

$$\gamma(e^{it}) = e^{i\widetilde{\varphi}(t)} \tag{4.2}$$

for $e^{it} \in \partial\mathbb{D}$ defines a weak homeomorphism of $\partial\mathbb{D}$ onto itself, i.e. $\gamma \in \mathcal{H}^*(\partial\mathbb{D})$. The corresponding harmonic extension provided by Proposition 4.1.2 therefore yields a harmonic homeomorphism $\mathfrak{P}[\gamma]$ of $\mathbb{D}$ onto itself. Figure 4.1 shows the (approximated) mapping behaviour of this harmonic extension, visualized by concentric circles around the origin, radial rays and an Euclidean grid. However, the mapping $\mathfrak{P}[\gamma]$ is not quasiconformal due to the fact that its boundary function – which equals γ by construction – is not injective, but this would be a necessary requirement by Proposition 1.1.6.

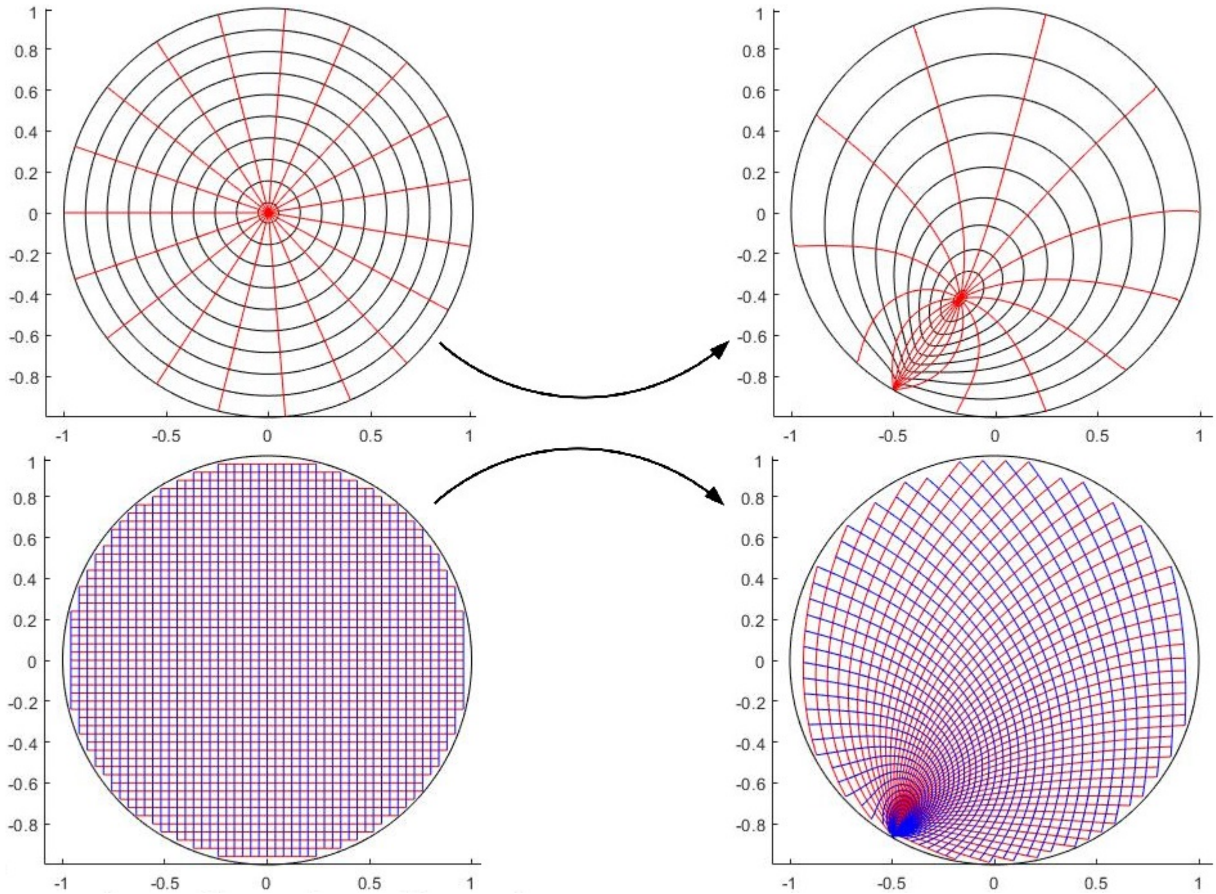

Figure 4.1: Preimage (left) and image (right) of concentric circles and radial rays (top) as well as of an Euclidean grid (bottom) in $\mathbb{D}$ under the harmonic extension of γ defined by (4.2).

Remark 4.1.6.
In particular, the harmonic extension $\mathfrak{P}[\gamma]$ discussed in Example 4.1.5, with γ given by (4.2), provides a concrete example of a sense–preserving homeomorphism of the unit disk that is not quasiconformal, i.e.

$$\mathfrak{P}[\gamma] \in \mathcal{H}^+(\mathbb{D}) \backslash Q(\mathbb{D})$$

Another example of such a mapping will be presented in Proposition 4.1.23.

A basic fact in the theory of harmonic mappings is that the composition of two such mappings is not necessarily harmonic again (see [Dur04, p. 2]). In the same manner, the inverse mapping of an injective harmonic mapping is also not harmonic in general, except for special situations, as stated in (see [Dur04, Theorem, pp. 145–148][2])

Proposition 4.1.7 (Choquet–Deny)**.**
Suppose f is an orientation–preserving injective harmonic mapping defined on a simply connected domain $\Omega \subseteq \mathbb{C}$, and suppose that f is neither analytic nor affine. Then the inverse mapping f^{-1} is harmonic if and only if f has the form

$$f(z) = \alpha\left(\beta z + 2i\,\mathrm{Arg}(\gamma - e^{-\beta z})\right) + \delta \tag{4.3}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ are constants with $\alpha\beta\gamma \neq 0$ and $\left|e^{-\beta z}\right| < |\gamma|$ for all $z \in \Omega$.

[2]In the cited statement of Choquet–Deny's Theorem in [Dur04], there is a small error in the formula of the harmonic mapping f: In the argument function, Duren writes $\gamma - e^{\beta z}$; however, as becomes clear from the book's proof of this Theorem as well, it should actually be $\gamma - e^{-\beta z}$, i.e. the minus sign in the exponent is missing.

This result and the previously stated basic facts immediately imply (see also [Pav14, Problem 10.1, p. 311])

Theorem 4.1.8.
$HQ(\mathbb{D})$ is no semigroup with respect to composition of mappings. In particular, $HQ(\mathbb{D})$ is no subgroup of $Q(\mathbb{D})$.

Furthermore, the Choquet–Deny Theorem 4.1.7 yields that $HQ(\mathbb{D})$ is not closed under inversion. In view of these circumstances, this raises the

Question 4.1.9.
Can a mapping of the form (4.3) be an automorphism of the unit disk $\mathbb{D}$ if the parameter values are chosen appropriately? If yes, can such a mapping be quasiconformal?

4.1.2 Topological properties of $HQ(\mathbb{D})$

This subsection is intended to study certain important topological properties of $HQ(\mathbb{D})$. To this end, certain convergence results for uniformly convergent sequences of harmonic mappings will prove valuable, to be stated in

Proposition 4.1.10.
Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of harmonic mappings on a domain $G \subseteq \mathbb{C}$.

*(i) If $(f_n)_n$ converges locally uniformly on G to some function f, then f is harmonic[3] (**Weierstraß–type Theorem**, see [ABR01, Theorem 1.23, p. 16]).*

*(ii) If all f_n are injective and the sequence $(f_n)_n$ converges locally uniformly on G to f, then f is either injective, a constant mapping, or $f(G)$ lies on a straight line (**Hurwitz–type Theorem**, see [BH94, Theorem 1.5]).*

The first result concerning certain topological aspects of $HQ(\mathbb{D})$ is given in

Theorem 4.1.11.

(i) The space $HQ(\mathbb{D})$ is separable and non–compact.

(ii) The set $HQ(\mathbb{D})$ is closed in the space $Q(\mathbb{D})$.

Proof.

(i) As for the separability of $HQ(\mathbb{D})$, it suffices to observe that the ambient metric space $Q(\mathbb{D})$ is separable by [BL23, Theorem 6, p. 5]. The claimed separability of $HQ(\mathbb{D})$ is then implied by the fact that subspaces of separable metric spaces are also separable. In order to see that $HQ(\mathbb{D})$ is a non–compact space, suppose the contrary, i.e. $HQ(\mathbb{D})$ is compact in the uniform topology. Due to the completeness of $\Sigma(\mathbb{D})$ (see [Gai84, Satz 1, p. 229]), the space $\Sigma(\mathbb{D})$ is closed in the ambient space $HQ(\mathbb{D})$. However, this yields that $\Sigma(\mathbb{D})$ would also be compact as a closed subspace of the compact space $HQ(\mathbb{D})$, contradicting the non–compactness of $\Sigma(\mathbb{D})$ (see [Gai84, Satz 1, p. 229]).

(ii) If $(f_n)_n \subseteq HQ(\mathbb{D})$ converges uniformly on $\mathbb{D}$ to $f \in Q(\mathbb{D})$, then f is harmonic by Proposition 4.1.10(i), hence $f \in HQ(\mathbb{D})$. Therefore, $HQ(\mathbb{D})$ is closed in $Q(\mathbb{D})$. □

An elementary persistence property in the interplay between harmonic and holomorphic mappings is that the post–composition of a holomorphic function with a harmonic one remains harmonic (see [Dur04, p. 2]). This basic fact is utilized in order to prove

[3] Actually, the cited version of the Weierstraß–type Theorem from [ABR01] is concerned only with the real and imaginary parts of the harmonic functions f_n, but this situation transfers immediately to the general case considered in the current section, since by definition, f_n is harmonic if and only if $\mathrm{Re}(f_n)$ and $\mathrm{Im}(f_n)$ satisfy the Laplace equation.

Theorem 4.1.12.
*The space $HQ(\mathbb{D})$ is **dense–in–itself**, i.e. it does not contain any isolated points.*

Proof. The space $Q(\mathbb{D})$ is a topological group (see Proposition 1.3.3(ii)) and not discrete (Corollary 3.2.6); in particular, $\Sigma(\mathbb{D})$ is not discrete (as already noticed in [Gai84, p. 230]). Hence, let $h \in HQ(\mathbb{D})$ be arbitrary and choose a sequence $(f_n)_{n\in\mathbb{N}}$ in $\Sigma(\mathbb{D})\backslash\{\mathrm{id}_{\mathbb{D}}\}$ converging to $\mathrm{id}_{\mathbb{D}}$. Then, for each $n \in \mathbb{N}$, the mapping $g_n := h \circ f_n$ is harmonic (by the mentioned persistence property) and quasiconformal (since $Q(\mathbb{D})$ is a group), thus $(g_n)_n$ is a sequence in $HQ(\mathbb{D})$. The continuity of left multiplication in the topological group $Q(\mathbb{D})$ yields $d_{\sup}(g_n, h) = d_{\sup}(h \circ f_n, h) \overset{n\to\infty}{\longrightarrow} 0$ due to $d_{\sup}(f_n, \mathrm{id}_{\mathbb{D}}) \overset{n\to\infty}{\longrightarrow} 0$. □

Combining Theorem 4.1.11 and Theorem 4.1.12 yields

Corollary 4.1.13.
*The space $HQ(\mathbb{D})$ is **perfect**, i.e. it is closed in $Q(\mathbb{D})$ and contains no isolated points.*

The last property to be studied in this subsection is the *contractibility* of $HQ(\mathbb{D})$, i.e. whether the identity map on $HQ(\mathbb{D})$ is homotopic to some constant map $c(f) = f_0$ from $HQ(\mathbb{D})$ to an element $f_0 \in HQ(\mathbb{D})$ (see [Wil70, Definition 32.6, p. 224]). In this context, the following integral operator will be of importance (see [Pav02, p. 367] and [Pav14, p. 305]):

Definition 4.1.14 (Hilbert transformation)**.**
For periodic $\varphi \in L^1([0, 2\pi])$ and $x \in \mathbb{R}$, the expression

$$\mathfrak{H}(\varphi)(x) := -\frac{1}{\pi} \lim_{\epsilon\to 0^+} \int_{\epsilon}^{\pi} \frac{\varphi(x+t) - \varphi(x-t)}{2\tan(t/2)}\, \mathrm{d}t \tag{4.4}$$

*is called the **(periodic) Hilbert transformation** of φ.*

Remark 4.1.15.

(i) In Fourier theory and trigonometric series, the Hilbert transformation plays a prominent role. However, the definition of the operator $\mathfrak{H}$ is not completely consistent in the vast literature about this topic. For example, a different formulation is given by

$$\mathfrak{H}(\varphi)(x) = -\frac{1}{\pi} \lim_{\epsilon\to 0^+} \int_{\epsilon}^{\pi} \frac{\varphi(x+t) - \varphi(x-t)}{t}\, \mathrm{d}t$$

which is – at least for existence questions – equivalent to (4.4) due to $2\tan(t/2) - t = 0$ for $t \longrightarrow 0$ (see [Pav14, p. 306] and [Zyg02, Vol. I, p. 52]).

(ii) The notion of "the Hilbert transformation" is also present in further mathematical areas, for example in the theory of quasiconformal mappings in $\mathbb{C}$ (see [Leh87, p. 25] and [LV73, pp. 156–160]) and Teichmüller spaces (see [GL00, pp. 319–320]). However, the circumstance that the definitions are in parts considerably different from each other is also present in these contexts.

Due to the presence of the tangent function in the integrand's denominator in (4.4), the question for existence of $\mathfrak{H}$ raises, partially answered in (see [Pav02, p. 367] and [Zyg02, Vol. I, p. 52])

Lemma 4.1.16.
For periodic $\varphi \in L^1([0, 2\pi])$, the Hilbert transformation $\mathfrak{H}(\varphi)(x)$ exists for almost every $x \in \mathbb{R}$. Furthermore, $\mathfrak{H}(\varphi)(x)$ exists if $\varphi'(x)$ exists and is finite at $x \in \mathbb{R}$.

Now the announced connection between the Hilbert transformation $\mathfrak{H}$ and $HQ(\mathbb{D})$ will be clarified. By the Radó–Kneser–Choquet Theorem 4.1.2, every mapping $\gamma = e^{i\varphi} \in \mathcal{H}^+(\partial\mathbb{D})$ defines a harmonic automorphism of $\mathbb{D}$ by means of the Poisson transformation $\mathfrak{P}[e^{i\varphi}]$ (this statement remains true even for $\gamma \in \mathcal{H}^*(\partial\mathbb{D})$, see also [KN99, (1.3), p. 338]). The question for whether this harmonic extension is quasiconformal has been answered in a characterizing manner by Pavlović in [Pav02], and is stated in (see [Pav14, Theorem 10.18, p. 305])

Proposition 4.1.17.
Let $f : \mathbb{D} \longrightarrow \mathbb{D}$ be an orientation–preserving harmonic homeomorphism of the unit disk onto itself. Then the following conditions are equivalent:
(i) f is quasiconformal, i.e. $f \in HQ(\mathbb{D})$;
(ii) $f = \mathfrak{P}[e^{i\varphi}]$, where the function φ has the following properties:
(a) $\varphi(t+2\pi) - \varphi(t) = 2\pi$ for all $t \in \mathbb{R}$;
(b) φ is strictly increasing and bi–Lipschitz;
(c) the Hilbert transformation of φ' is an element of $L^\infty(\mathbb{R})$.

A mapping $g : X \longrightarrow Y$ between metric spaces (X, d_X) and (Y, d_Y) is called *bi–Lipschitz* if there exists a constant $L \in [1, +\infty)$ such that

$$\frac{1}{L} d_X(x_1, x_2) \leq d_Y(g(x_1), g(x_2)) \leq L d_X(x_1, x_2)$$

for all $x_1, x_2 \in X$, thus sharpening the classical notion of a Lipschitz–continuous mapping. Now, in view of Corollary 4.1.4 and Proposition 4.1.17, the following characterization for the elements of the space $HQ(\mathbb{D})$ is valid: A harmonic (orientation–preserving) homeomorphism $\mathfrak{P}[e^{i\varphi}]$ of $\mathbb{D}$ onto itself is quasiconformal if and only if the corresponding mapping φ is an element of

$$\mathcal{H}^+_{qc} := \left\{ \varphi \in C([0,2\pi]) \;\middle|\; \varphi \text{ is strictly increasing and bi–Lipschitz, } \varphi(2\pi) - \varphi(0) = 2\pi,\ \mathfrak{H}(\widetilde{\varphi}') \in L^\infty(\mathbb{R}) \right\}$$

Here, $\widetilde{\varphi}$ denotes the canonical extension of $\varphi \in \mathcal{H}^+_{qc}$ to all of $\mathbb{R}$ via

$$\widetilde{\varphi}(t + 2k\pi) := \varphi(t) + 2k\pi$$

for all $t \in [0, 2\pi]$ and every $k \in \mathbb{Z}$. By the requirement of strict increasing monotonicity, every mapping $\varphi \in \mathcal{H}^+_{qc}$ is differentiable almost everywhere in (the interior of) $[0, 2\pi]$. Consequently, each extended mapping $\widetilde{\varphi} \in C(\mathbb{R})$ is differentiable almost everywhere in $\mathbb{R}$ with $\widetilde{\varphi}'$ being 2π–periodic by construction. Furthermore, the assumption that φ is bi–Lipschitz yields $\varphi' \in L^1([0, 2\pi])$ (see [RF10, Theorem 10, p. 124]). Therefore, the condition $\mathfrak{H}(\widetilde{\varphi}') \in L^\infty(\mathbb{R})$ is reasonable. In view of the topic of the current subsection and the mentioned path–connectedness of $HQ(\mathbb{D})$, the first important observation to be made is

Lemma 4.1.18.
The subset $\mathcal{H}^+_{qc} \subsetneq C([0, 2\pi])$ is convex. In particular, $\mathcal{H}^+_{qc}$ is contractible in the Banach space $C([0, 2\pi])$.

Proof. Let $\varphi_1, \varphi_2 \in \mathcal{H}^+_{qc}$, $\lambda \in [0, 1]$ and consider the mapping $\lambda\varphi_1 + (1-\lambda)\varphi_2$.
(i) **<u>Monotonicity:</u>** For $t, t' \in [0, 2\pi]$ with $t < t'$, it is

$$\lambda\varphi_1(t) + (1-\lambda)\varphi_2(t) < \lambda\varphi_1(t') + (1-\lambda)\varphi_2(t')$$

due to $\lambda, (1-\lambda) \geq 0$, hence $\lambda\varphi_1 + (1-\lambda)\varphi_2$ is strictly increasing.

(ii) **Bi–Lipschitz property:** Let $t, t' \in [0, 2\pi]$ and $L := \max\{L_1, L_2\}$ with L_j denoting the bi–Lipschitz constant of φ_j, $j = 1, 2$. Then on the one hand, by means of the triangle inequality, it is

$$\begin{aligned}\left|\lambda\varphi_1(t) + (1-\lambda)\varphi_2(t) - \lambda\varphi_1(t') - (1-\lambda)\varphi_2(t')\right| &\le \lambda\left|\varphi_1(t) - \varphi_1(t')\right| + (1-\lambda)\left|\varphi_2(t) - \varphi_2(t')\right| \\ &\le \lambda L|t-t'| + (1-\lambda)L|t-t'| = L|t-t'|\end{aligned}$$

Hence $\lambda\varphi_1 + (1-\lambda)\varphi_2$ is Lipschitz–continuous with Lipschitz constant L. Without loss of generality, assume $t > t'$, then on the other hand, it is (recall that $\lambda\varphi_1 + (1-\lambda)\varphi_2$ is strictly increasing by (i))

$$\begin{aligned}\lambda\varphi_1(t) + (1-\lambda)\varphi_2(t) - \lambda\varphi_1(t') - (1-\lambda)\varphi_2(t') &= \lambda(\varphi_1(t) - \varphi_1(t')) + (1-\lambda)(\varphi_2(t) - \varphi_2(t')) \\ &\ge \lambda\frac{1}{L}(t-t') + (1-\lambda)\frac{1}{L}(t-t') = \frac{1}{L}(t-t')\end{aligned}$$

Finally, switching the roles of t and t' shows that $\lambda\varphi_1 + (1-\lambda)\varphi_2$ is bi–Lipschitz continuous on $[0, 2\pi]$ with bi–Lipschitz constant L.

(iii) **Image interval has length 2π:** It is

$$\begin{aligned}&\lambda\varphi_1(2\pi) + (1-\lambda)\varphi_2(2\pi) - (\lambda\varphi_1(0) + (1-\lambda)\varphi_2(0)) \\ &= \lambda(\varphi_1(2\pi) - \varphi_1(0)) + (1-\lambda)(\varphi_2(2\pi) - \varphi_2(0)) \\ &= \lambda 2\pi + (1-\lambda)2\pi = 2\pi\end{aligned}$$

(iv) **Hilbert transformation:** First of all, $\lambda\widetilde{\varphi_1} + (1-\lambda)\widetilde{\varphi_2}$ is differentiable almost everywhere in $\mathbb{R}$ by (i) with

$$(\lambda\widetilde{\varphi_1} + (1-\lambda)\widetilde{\varphi_2})' = \lambda\widetilde{\varphi_1}' + (1-\lambda)\widetilde{\varphi_2}'$$

The function $\lambda\varphi_1' + (1-\lambda)\varphi_2'$ is contained in $L^1([0, 2\pi])$ as the linear combination of such elements. Following Definition 4.1.14, the Hilbert transformation of $\lambda\widetilde{\varphi_1}' + (1-\lambda)\widetilde{\varphi_1}'$ is given by

$$\begin{aligned}\mathfrak{H}(\lambda\widetilde{\varphi_1}' + (1-\lambda)\widetilde{\varphi_2}')(x) &= -\frac{1}{\pi}\int_{0^+}^{\pi} \frac{\lambda\widetilde{\varphi_1}'(x+t) + (1-\lambda)\widetilde{\varphi_2}'(x+t) - \lambda\widetilde{\varphi_1}'(x-t) - (1-\lambda)\widetilde{\varphi_2}'(x-t)}{2\tan(t/2)}\,\mathrm{d}t \\ &= -\frac{1}{\pi}\int_{0^+}^{\pi} \frac{\lambda(\widetilde{\varphi_1}'(x+t) - \widetilde{\varphi_1}'(x-t)) + (1-\lambda)(\widetilde{\varphi_2}'(x+t) - \widetilde{\varphi_2}'(x-t))}{2\tan(t/2)}\,\mathrm{d}t\end{aligned}$$

Since $\varphi_1, \varphi_2 \in \mathcal{H}_{qc}^+$, it is $\mathfrak{H}(\widetilde{\varphi_1}'), \mathfrak{H}(\widetilde{\varphi_2}') \in L^\infty(\mathbb{R})$ by definition of the set $\mathcal{H}_{qc}^+$, thus using the linearity of (improper) integrals the previous equation can be rewritten as

$$\begin{aligned}\mathfrak{H}(\lambda\widetilde{\varphi_1}' + (1-\lambda)\widetilde{\varphi_2}')(x) &= -\frac{\lambda}{\pi}\int_{0^+}^{\pi} \frac{\widetilde{\varphi_1}'(x+t) - \widetilde{\varphi_2}'(x-t)}{2\tan(t/2)}\,\mathrm{d}t - \frac{1-\lambda}{\pi}\int_{0^+}^{\pi} \frac{\widetilde{\varphi_2}'(x+t) - \widetilde{\varphi_2}'(x-t)}{2\tan(t/2)}\,\mathrm{d}t \\ &= \lambda\mathfrak{H}(\widetilde{\varphi_1}')(x) + (1-\lambda)\mathfrak{H}(\widetilde{\varphi_2}')(x)\end{aligned}$$

Since $L^\infty(\mathbb{R})$ is a $\mathbb{R}$–vector space, the previous equation yields $\mathfrak{H}(\lambda\widetilde{\varphi_1}' + (1-\lambda)\widetilde{\varphi_2}') \in L^\infty(\mathbb{R})$. All in all, the mapping $\lambda\varphi_1 + (1-\lambda)\varphi_2$ is contained in $\mathcal{H}_{qc}^+$ for every λ, hence $\mathcal{H}_{qc}^+$ is convex. Thus, as a subset of the normed vector space $C([0, 2\pi])$, $\mathcal{H}_{qc}^+$ is also contractible. □

Continuing the investigation, the set $\mathcal{H}_{qc}^+$ now gives rise to consider the mapping

$$\Lambda : \mathcal{H}_{qc}^+ \longrightarrow HQ(\mathbb{D}),\ \varphi \longmapsto \left(\mathbb{D} \ni z \longmapsto \Lambda(\varphi)(z) := \mathfrak{P}[e^{i\varphi}](z)\right) \tag{4.5}$$

By Corollary 4.1.4 and Pavlović's Proposition 4.1.17, the mapping Λ is surjective. Endowing the involved sets in (4.5) with the respective metric structures concludes in the pleasant

Theorem 4.1.19.
The mapping $\Lambda : (\mathcal{H}^+_{qc}, d_{\sup}) \longrightarrow (HQ(\mathbb{D}), d_{\sup})$ *as defined in (4.5) is continuous and surjective.*

Proof. The fact that Λ is surjective was already mentioned above. Hence, let $(\varphi_n)_{n\in\mathbb{N}}$ converge in $\mathcal{H}^+_{qc}$ to $\varphi \in \mathcal{H}^+_{qc}$. The characterization of elements in $HQ(\mathbb{D})$ stated in Proposition 4.1.17 implies that $(\Lambda(\varphi_n))_{n\in\mathbb{N}}$ is a sequence in $HQ(\mathbb{D})$ and $\Lambda(\varphi) \in HQ(\mathbb{D})$. In particular, $\Lambda(\varphi_n)$ and $\Lambda(\varphi)$ are harmonic quasiconformal automorphisms of $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$ and coincide with $e^{i\varphi_n}$ and $e^{i\varphi}$ on $\partial\mathbb{D}$, respectively (see also [Dur04, p. 12]). Therefore, since $\Lambda(\varphi_n) - \Lambda(\varphi)$ is harmonic as well, the maximum principle for harmonic mappings applies (see [ABR01, pp. 7–9]), concluding in

$$\begin{aligned}\sup_{z\in\mathbb{D}} |\Lambda(\varphi_n)(z) - \Lambda(\varphi)(z)| &= \sup_{z\in\partial\mathbb{D}} |\Lambda(\varphi_n)(z) - \Lambda(\varphi)(z)| \\ &= \sup_{t\in[0,2\pi]} \left| e^{i\varphi_n(t)} - e^{i\varphi(t)} \right| \\ &\le \sup_{t\in[0,2\pi]} |\varphi_n(t) - \varphi(t)| = d_{\sup}(\varphi_n, \varphi)\end{aligned}$$

In the estimate, the basic inequality $|e^{ix} - e^{iy}| \le |x - y|$ for $x, y \in \mathbb{R}$ was used. The last expression tends to zero for $n \to \infty$, proving the continuity of Λ. □

Finally, combining the statements of Lemma 4.1.18 and Theorem 4.1.19 yields the announced

Theorem 4.1.20.
The space $HQ(\mathbb{D})$ *is contractible. In particular, the space* $HQ(\mathbb{D})$ *is path–connected.*

4.1.3 Incompleteness of $HQ(\mathbb{D})$: Statement, auxiliary results and proof

This subsection is concerned with the proof of the following statement:

Theorem 4.1.21.
The space $HQ(\mathbb{D})$ *is incomplete.*

In order to prove this claim, some helpful results are collected in the following. The principal idea of the proof of Theorem 4.1.21 is to construct a sequence of homeomorphic mappings of the interval $[0,1]$ onto itself converging uniformly to the **Cantor function** $\mathcal{C} : [0,1] \longrightarrow [0,1]$; for basic information on this function, see [DMRV06] and [RF10, Section 2.7, pp. 49–53]. A result of Božin and Mateljević shows that, via the Poisson transformation, an appropriately modified variant of the mapping $\mathcal{C}$ induces a harmonic homeomorphism of the unit disk $\mathbb{D}$ onto itself which is *not* quasiconformal (see Proposition 4.1.23). However, this harmonic homeomorphism will be seen to arise as the uniform limit of harmonic quasiconformal automorphisms of $\mathbb{D}$, thus implying that $HQ(\mathbb{D})$ cannot be complete.

First of all, an approximation procedure for the Cantor function $\mathcal{C}$ in terms of a certain recursively defined sequence is stated (see [DMRV06, Proposition 4.2, p. 9]):

Lemma 4.1.22.
Let $B([0,1])$ *denote the Banach space of bounded real–valued functions on* $[0,1]$. *The Cantor function* $\mathcal{C}$ *is the unique element of* $B([0,1])$ *for which*

$$\mathcal{C}(x) = \begin{cases} \frac{1}{2}\mathcal{C}(3x), & 0 \le x \le \frac{1}{3} \\ \frac{1}{2}, & \frac{1}{3} < x < \frac{2}{3} \\ \frac{1}{2} + \frac{1}{2}\mathcal{C}(3x-2), & \frac{2}{3} \le x \le 1 \end{cases}$$

Moreover, for arbitrary $\psi_0 \in B([0,1])$, the sequence $(\psi_n)_{n\in\mathbb{N}_0}$ defined by

$$\psi_{n+1}(x) := \begin{cases} \frac{1}{2}\psi_n(3x), & 0 \le x \le \frac{1}{3} \\ \frac{1}{2}, & \frac{1}{3} < x < \frac{2}{3} \\ \frac{1}{2} + \frac{1}{2}\psi_n(3x-2), & \frac{2}{3} \le x \le 1 \end{cases} \tag{4.6}$$

for $n \in \mathbb{N}_0$ converges uniformly on $[0,1]$ to $\mathcal{C}$.

The principal idea of the approximation procedure and the mappings ψ_n is that the initial mapping ψ_0 is "copied" and gets "duplicated in a scaled fashion", being added to the graph of ψ_n more and more times as the index increases. From this, it is clear that all continuity and differentiability questions regarding ψ_n depend solely on the behaviour of the initial mapping ψ_0 (and eventually existing derivatives) at the boundary points $x = 0$ and $x = 1$ of the starting interval. Furthermore, in Lemma 4.1.22, the stated approximation part and the related uniqueness of $\mathcal{C}$ is based on Banach's Contraction Principle (see [RF10, p. 216]). The following proposition contains the mentioned result of Božin/Mateljević concerning a harmonic homeomorphism of $\mathbb{D}$ which fails to be quasiconformal (see [BM10, Example 3.2, pp. 29–30]):

Proposition 4.1.23.
For $t \in [0,2\pi]$, define $\varphi_{\mathcal{C}}(t) := \pi(\mathcal{C}(\frac{t}{2\pi}) + \frac{t}{2\pi})$ and $\gamma_{\mathcal{C}}(t) := e^{i\varphi_{\mathcal{C}}(t)}$. Then the function $h_{\mathcal{C}} := \mathfrak{P}[\gamma_{\mathcal{C}}]$ is a harmonic homeomorphism of $\mathbb{D}$ onto itself that is not quasiconformal.

Remark 4.1.24. *The Cantor function $\mathcal{C}$, named for the German mathematician Georg Cantor who introduced it in 1883, is an often utilized counterexample to many situations in real (one–dimensional) analysis, especially concerning delicate subtleties of integration and continuity questions. The main obstacle in Proposition 4.1.23 that denies $h_{\mathcal{C}}$ to be quasiconformal is thereby the fact that the Cantor function is continuous and increasing (thus $\varphi_{\mathcal{C}}$ is strictly increasing by construction), but not absolutely continuous (see [DMRV06, Proposition 2.1, p. 3] and [RF10, Example, p. 120]), as pointed out in [BM10, p. 29].*

Now all preparations are made in order to prove the claim of Theorem 4.1.21:

Proof of Theorem 4.1.21. Consider the polynomial function

$$\psi_0 : [0,1] \longrightarrow \mathbb{R},\ x \longmapsto \psi_0(x) := 6x^5 - 15x^4 + 10x^3$$

whose first and second derivatives satisfy

$$\psi_0'(0) = \psi_0'(1) = 0 = \psi_0''(0) = \psi_0''(1) \tag{4.7}$$

Furthermore, ψ_0 is strictly increasing on $(0,1)$ and leaves the boundary points fixed – in other words, ψ_0 maps $[0,1]$ homeomorphically onto itself. Lemma 4.1.22 implies that the corresponding sequence $(\psi_n)_{n\in\mathbb{N}_0}$ defined via (4.6) converges uniformly on $[0,1]$ to the Cantor function $\mathcal{C}$, and by construction, it is $\psi_n \in C^2([0,1])$ for every $n \in \mathbb{N}_0$ due to (4.7). Transferring the ψ_n to the interval $[0,2\pi]$ via

$$\varphi_n(t) := \pi\left(\psi_n\left(\frac{t}{2\pi}\right) + \frac{t}{2\pi}\right),\ t \in [0,2\pi], \tag{4.8}$$

yields a sequence $(\varphi_n)_n$ of C^2–homeomorphism of $[0,2\pi]$ onto itself. Accordingly, this sequence $(\varphi_n)_n$ clearly converges uniformly on $[0,2\pi]$ to the mapping $\varphi_{\mathcal{C}}$ defined in Proposition 4.1.23. As a next step, the mappings φ_n and $\varphi_{\mathcal{C}}$ are extended to all of $\mathbb{R}$ by setting

$$\varphi_n(t + 2k\pi) := \varphi_n(t) + 2k\pi \tag{4.9}$$

for $k \in \mathbb{Z}$ and $t \in [0, 2\pi]$, yielding a sequence $(\varphi_n)_{n\in\mathbb{N}_0} \subseteq C^2(\mathbb{R})$; likewise, the mappings ψ_n and $\mathcal{C}$ are extended in the same manner (the extended mappings are denoted by the same letter). In particular, the φ_n are differentiable with $\varphi_n'(t + 2k\pi) = \varphi_n'(t)$ for all $t \in \mathbb{R}$ by construction, i.e. the φ_n' (and thus the φ_n'' as well) are continuous 2π–periodic mappings. Lifting these mappings to the unit circle by

$$\gamma_n(e^{it}) := e^{i\varphi_n(t)}$$

for $t \in [0, 2\pi]$ and each $n \in \mathbb{N}$ yields orientation–preserving homeomorphisms of $\partial\mathbb{D}$ onto itself, hence the harmonic extensions $\mathfrak{P}[\gamma_n]$ by means of the Radó–Kneser–Choquet Theorem 4.1.2 are (orientation–preserving) harmonic homeomorphisms of $\mathbb{D}$ onto itself. Moreover, by Pavlović's characterization result stated in Proposition 4.1.17, the mappings $\mathfrak{P}[\gamma_n]$ in fact define quasiconformal automorphisms of $\mathbb{D}$, which can be seen as follows:

It is $\varphi_n \in C^2(\mathbb{R})$ strictly increasing with $\varphi_n(t + 2\pi) = \varphi_n(t) + 2\pi$ for all $t \in \mathbb{R}$ by construction, see (4.8) and (4.9). Furthermore, as C^2–homeomorphisms, each mapping φ_n is Lipschitz–continuous, and the corresponding inverse mappings φ_n^{-1} are also C^2 by construction due to (4.7), thus also Lipschitz–continuous. In consequence, the mappings φ_n are bi–Lipschitz. Hence, in view of Proposition 4.1.17(ii), the Hilbert transformation condition (c) needs to be verified. Therefore, let $x \in \mathbb{R}$, then it is

$$\left|\varphi_n'(x+t) - \varphi_n'(x-t)\right| \le L_n \cdot |x + t - (x - t)| = 2L_n|t|$$

since $\varphi_n \in C^2(\mathbb{R})$, thus φ_n' is Lipschitz–continuous on $\mathbb{R}$ with Lipschitz constant $L_n \in \mathbb{R}^+$. This yields

$$\left|\int_{0^+}^{\pi} \frac{\varphi_n'(x+t) - \varphi_n'(x-t)}{t}\mathrm{d}t\right| \le \int_{0^+}^{\pi} \frac{|\varphi_n'(x+t) - \varphi_n'(x-t)|}{t}\mathrm{d}t \le \int_{0^+}^{\pi} \frac{2L_n t}{t}\mathrm{d}t = 2\pi L_n < +\infty$$

and now Remark 4.1.15(i) implies that $\mathfrak{H}(\varphi_n')$ is (essentially) bounded for $\varphi_n, n \in \mathbb{N}_0$ (note that the conclusion could also have been drawn from Lemma 4.1.16 since φ_n' and φ_n'' are periodic and continuous on $\mathbb{R}$). Thus Proposition 4.1.17 shows that the mappings $\mathfrak{P}[\gamma_n]$ are quasiconformal automorphisms of $\mathbb{D}$.

Finally, it will be shown that the mappings $\mathfrak{P}[\gamma_n]$ converge uniformly on $\mathbb{D}$ to the non–quasiconformal mapping $h_{\mathcal{C}}$ in question (from Proposition 4.1.23), which is essentially based on the same idea as the proof of Theorem 4.1.19: Applying the maximum principle for harmonic functions to $\mathfrak{P}[\gamma_n] - h_{\mathcal{C}}$ yields

$$\sup_{z\in\mathbb{D}} |\mathfrak{P}[\gamma_n](z) - h_{\mathcal{C}}(z)| = \max_{z\in\partial\mathbb{D}} |\mathfrak{P}[\gamma_n](z) - \mathfrak{P}[\gamma_{\mathcal{C}}](z)| = \max_{t\in[0,2\pi]} \left|e^{i\varphi_n(t)} - e^{i\varphi_{\mathcal{C}}(t)}\right| \le \max_{t\in[0,2\pi]} |\varphi_n(t) - \varphi_{\mathcal{C}}(t)|$$

Since φ_n converges uniformly on $[0, 2\pi]$ to φ (this, in turn, follows from the fact that ψ_n converges uniformly on $[0, 1]$ to the Cantor function $\mathcal{C}$), the claim follows: The sequence $(\mathfrak{P}[\gamma_n])_{n\in\mathbb{N}_0}$ in $HQ(\mathbb{D})$ converges uniformly to $h_{\mathcal{C}} \notin HQ(\mathbb{D})$, showing that the space $HQ(\mathbb{D})$ is incomplete. ☐

In particular, by considering the homeomorphically extended mappings on $\overline{\mathbb{D}}$, the proof of Theorem 4.1.21 immediately implies the following statement, which is used in Theorem 2.5.5:

Corollary 4.1.25.
The subspace $Q(\overline{\mathbb{D}})$ is not closed in the homeomorphism group $\mathcal{H}(\overline{\mathbb{D}})$.

Proof. The homeomorphic extensions to $\overline{\mathbb{D}}$ of the mappings $\mathfrak{P}[\gamma_n], n \in \mathbb{N}$, in $HQ(\mathbb{D}) \subsetneq Q(\mathbb{D})$ converge uniformly on $\overline{\mathbb{D}}$ to (the extension of) $h_{\mathcal{C}} \in \mathcal{H}(\overline{\mathbb{D}})\backslash Q(\overline{\mathbb{D}})$, and the claim follows. ☐

Furthermore, as an addition to the results in Theorem 4.1.11, the proof of Theorem 4.1.21 implies as a direct consequence:

Corollary 4.1.26.
The set $HQ(\mathbb{D})$ is not closed in the space $(\mathrm{Har}^+(\mathbb{D}), d_{\sup})$, where $\mathrm{Har}^+(\mathbb{D})$ denotes the class of all orientation–preserving harmonic homeomorphisms of $\mathbb{D}$ onto itself.

4.2 Radial stretchings revisited: Semirings and a Cesàro–type construction

In the progress of the thesis at hand hitherto, a couple of examples of quasiconformal automorphisms have already been presented on several occasions:

(i) For the case $G = \mathbb{D}$: The class of general radial stretchings given by $f_\rho(z) = \rho(|z|)e^{i\,\mathrm{Arg}(z)}$ for $z \in \mathbb{D}$ with appropriate $\rho \in C([0,1])$, see Definition 2.3.1, containing the monomial–like radial stretchings $f_K(z) = z|z|^{K-1}$ for $K \in \mathbb{R}^+$ as a special case. More concrete examples of quasiconformal unit disk automorphisms are given by the principal encryption mappings in (4.25) and the actual encryption mappings (4.26) utilized in Subsection 16.

(ii) Furthermore, in the case of the unit disk, a sequence of harmonic quasiconformal automorphisms of $\mathbb{D}$ is utilized in the proof of Theorem 4.1.21.

(iii) In the multiply connected setting treated in Chapter 3.6: The *Full Dehn twist* $f(z) = z|z|^{\frac{2\pi i}{\ln(R)}}$ as defined in (3.10).

The class of monomial–like radial stretchings, as defined by (2.3), are mappings of the form

$$f_K(z) = z|z|^{K-1}$$

for $z \in \mathbb{D}$ and $K \in \mathbb{R}^+ = (0, +\infty)$, thereby constituting a useful set of quasiconformal automorphisms of the unit disk (Lemma 2.3.2). This subsection is concerned with further investigation of these mappings regarding algebraic structure and a construction related to classical *Cesàro summation* from real analysis. To this end, denote by $\mathcal{R}(\mathbb{D})$ the set of all monomial–like radial stretchings of $\mathbb{D}$ and consider the mapping

$$T : \mathbb{R}^+ \longrightarrow \mathcal{R}(\mathbb{D}),\ K \longmapsto T(K) := \left(\mathbb{D} \ni z \longmapsto z|z|^{K-1}\right)$$

Obviously, T constitutes a bijective mapping. Furthermore, the mapping T defines a group isomorphism between the abelian group $(\mathbb{R}^+, \cdot)$ and the group[4] $(\mathcal{R}(\mathbb{D}), \circ)$, i.e.

$$T(K \cdot L) = z|z|^{K\cdot L-1} = z|z|^{K-1} \circ z|z|^{L-1} = T(K) \circ T(L)$$

for all $K, L \in \mathbb{R}^+$ (see [Bie17, (2.12), p. 43]). In particular, $(\mathcal{R}(\mathbb{D}), \circ)$ forms a commutative semigroup (even a monoid), a fact that will be utilized in the remainder of this section. Furthermore, examination of the basic fact that $\mathbb{R}^+$ also carries the algebraic structure of a commutative semigroup with respect to addition of real numbers (i.e. addition is an associative binary operation on $\mathbb{R}^+$) gives rise to introduce the mapping

$$\oplus : \mathcal{R}(\mathbb{D}) \times \mathcal{R}(\mathbb{D}) \longrightarrow \mathcal{R}(\mathbb{D}),\ (z|z|^{K-1}, z|z|^{L-1}) \longmapsto z|z|^{K-1} \oplus z|z|^{L-1} := z|z|^{K+L-1}$$

which might be called *addition* in $\mathcal{R}(\mathbb{D})$. Consequently, by construction, the mapping T defined above also behaves in a pleasant way with respect to this operation $\oplus$ on $\mathcal{R}(\mathbb{D})$:

$$T(K + L) = z|z|^{K+L-1} = z|z|^{K-1} \oplus z|z|^{L-1} = T(K) \oplus T(L)$$

[4]The fact that $\mathcal{R}(\mathbb{D})$ forms a group is easily verified, see e.g. [Bie17, Theorem 2.11, p. 43].

In other words, T defines a homomorphism between the commutative semigroup $(\mathbb{R}^+, +)$ and the tuple $(\mathcal{R}(\mathbb{D}), \oplus)$, hence – since the set–theoretic mapping properties of T remain unchanged – a *semigroup isomorphism.* In particular, $(\mathcal{R}(\mathbb{D}), \oplus)$ is itself a commutative semigroup. The self–evident consequence now is to introduce the triple $(\mathcal{R}(\mathbb{D}), \oplus, \circ)$ and to confirm the following

Lemma 4.2.1.
The triple $(\mathcal{R}(\mathbb{D}), \oplus, \circ)$ forms a commutative semiring[5]*, i.e. both $(\mathcal{R}(\mathbb{D}), \oplus)$ and $(\mathcal{R}(\mathbb{D}), \circ)$ are commutative semigroups, and the binary operations $\oplus$ and $\circ$ satisfy the distributive law*

- $z|z|^{K-1} \circ \left(z|z|^{L-1} \oplus z|z|^{L'-1}\right) = z|z|^{K-1} \circ z|z|^{L-1} \oplus z|z|^{K-1} \circ z|z|^{L'-1}$ *and*
- $\left(z|z|^{L-1} \oplus z|z|^{L'-1}\right) \circ z|z|^{K-1} = z|z|^{L-1} \circ z|z|^{K-1} \oplus z|z|^{L'-1} \circ z|z|^{K-1}$

Proof. The facts that $(\mathcal{R}(\mathbb{D}), \oplus)$ and $(\mathcal{R}(\mathbb{D}), \circ)$ form commutative semigroups were already established previously. In order to verify the distributive laws, the homomorphic mapping properties of T are utilized:

$$\begin{aligned} z|z|^{K-1} \circ \left(z|z|^{L-1} \oplus z|z|^{L'-1}\right) &= T(K) \circ \left(T(L) \oplus T(L')\right) = T(K) \circ T(L + L') = T(K \cdot (L + L')) \\ &= T(K \cdot L + K \cdot L') = T(K \cdot L) \oplus T(K \cdot L') \\ &= T(K) \circ T(L) \oplus T(K) \circ T(L') \\ &= z|z|^{K-1} \circ z|z|^{L-1} \oplus z|z|^{K-1} \circ z|z|^{L'-1} \end{aligned}$$

This proves the first claim. The second equation follows in the very same manner. ☐

Finally, the announced Cesàro–type construction will be briefly presented: For a given (not necessarily convergent) sequence $\left(z|z|^{K_j-1}\right)_{j\in\mathbb{N}}$ of monomial–like radial stretchings in $\mathcal{R}(\mathbb{D})$ and $n \in \mathbb{N}$, define the **Cesàro–type mapping**

$$s_n : \mathbb{D} \longrightarrow \mathbb{C},\ z \in \mathbb{D} \longmapsto s_n(z) := \frac{1}{n} \sum_{j=1}^{n} z|z|^{K_j-1} \tag{4.10}$$

which, by means of $z = re^{i\varphi} \in \mathbb{D}$, can be rewritten as

$$s_n(z) = \frac{e^{i\varphi}}{n} \sum_{j=1}^{n} r^{K_j}$$

Due to $z|z|^{K_j-1} \in Q(\mathbb{D})$ for all $j \in \mathbb{N}$, it follows immediately that $s_n \in C(\mathbb{D})$ with homeomorphic extension to $\partial\mathbb{D}$ and the extended mapping satisfies $s_n\big|_{\partial\mathbb{D}} \equiv \mathrm{id}_{\partial\mathbb{D}}$. In particular, s_n maps the unit disk continuously onto itself and keeps both, the boundary and the origin, pointwise fixed. Furthermore, by considering the real–valued mapping

$$g(r) := \frac{1}{n} \sum_{j=1}^{n} r^{K_j}$$

for $r \in [0, 1]$, one sees that g is a continuous mapping of the interval $[0, 1]$ onto itself with $g(0) = 0$ and $g(1) = 1$. On the open interval $(0, 1)$, the corresponding derivative

$$\frac{\mathrm{d}}{\mathrm{d}r} g(r) = \frac{1}{n} \sum_{j=1}^{n} K_j r^{K_j-1}$$

[5]The notion of a semiring is not consistently used in the mathematical literature, thus the definition used in this thesis is stated in Lemma 4.2.1.

is positive due to $K_j \in \mathbb{R}^+$, thus g is strictly increasing, and in particular injective. Consequently, the mapping s_n is continuous and injective on the open plane set $\mathbb{D}$, hence a homeomorphism by a classical result (see [LV73, Lemma 1.1, p. 6]). Next, since every summand $z|z|^{K_j-1}$ in (4.10) is an ACL mapping (due to $z|z|^{K_j-1} \in Q(\mathbb{D})$ by Definition 1.1.1), so is the finite sum s_n as linear combination of such mappings (see [RF10, p. 120], using the basic fact that finite unions of measurable sets of measure zero remain having measure zero). In the same spirit, one may consider the Wirtinger derivatives of s_n, which compute as (see [AIM08, p. 28])

$$(s_n)_z(z) = \frac{1}{2n}\sum_{j=1}^{n}(K_j+1)|z|^{K_j-1}$$

$$(s_n)_{\overline{z}}(z) = \frac{z}{\overline{z}}\frac{1}{2n}\sum_{j=1}^{n}(K_j-1)|z|^{K_j-1}$$

Therefore, by utilizing the elementary strict inequality $|x-1| < x+1$ for all $x \in \mathbb{R}^+$, the following estimate can be deduced:

$$|(s_n)_{\overline{z}}(z)| \leq \frac{1}{2n}\sum_{j=1}^{n}|K_j-1|\,|z|^{K_j-1} < \frac{1}{2n}\sum_{j=1}^{n}(K_j+1)\,|z|^{K_j-1} = |(s_n)_z(z)|$$

Consequently, this concludes in the fact that the Wirtinger derivatives of the mapping s_n satisfy

$$|(s_n)_{\overline{z}}| \leq k\,|(s_n)_z|$$

for some constant $k < 1$. Finally, in view of Definition 1.1.1, these arguments yield the

Lemma 4.2.2.
The mappings defined via (4.10) are quasiconformal automorphisms of $\mathbb{D}$, i.e. $s_n \in Q(\mathbb{D})$ for every $n \in \mathbb{N}$.

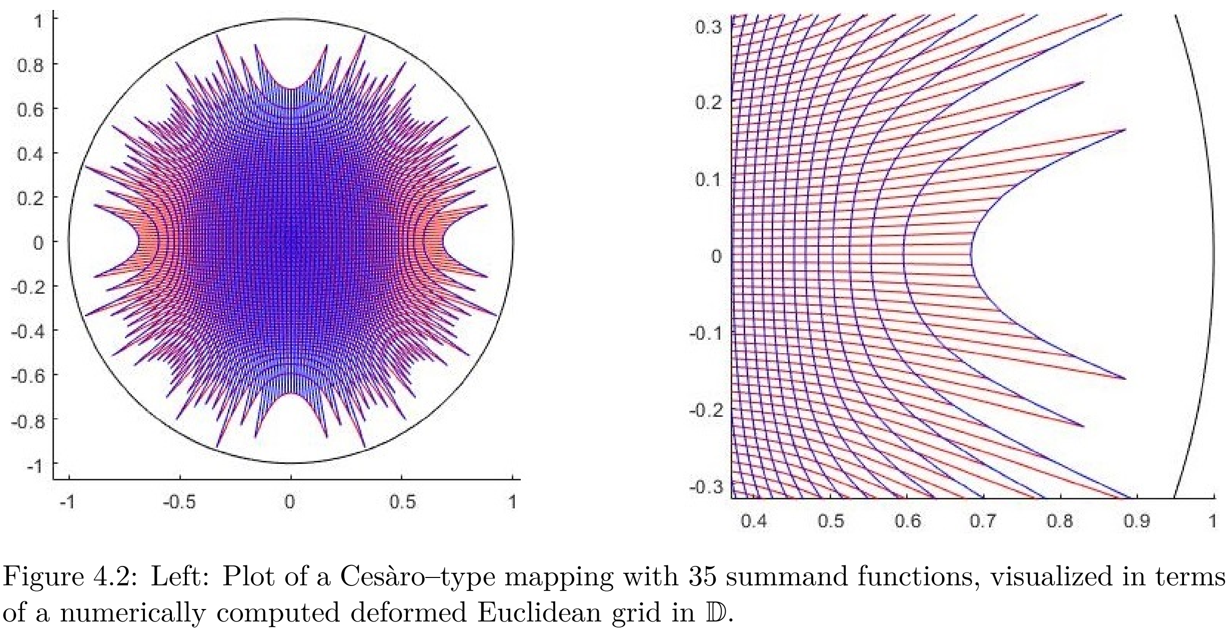

Figure 4.2: Left: Plot of a Cesàro–type mapping with 35 summand functions, visualized in terms of a numerically computed deformed Euclidean grid in $\mathbb{D}$.
Right: Magnified region in $\mathbb{D}$ of the left–hand side plot, highlighting the special mapping properties.

The geometric behaviour of the mappings s_n in (4.10) – which is rather unusual due to the naturally present "non– additivity" of quasiconformal mappings, see Remark 4.2.3(i) – is visualized in Figure 4.2 by showing a (numerically approximated) plot of the following mapping of this particular type: For $j \in \mathbb{N}$, denote by $p_j \in \mathbb{N}$ the j–th prime number (thus, for example, $p_1 = 2, p_2 = 3, p_3 = 5$, etc.) and let

$$K_j = p_j^{(-1)^j} + 1 = \begin{cases} \frac{1}{p_j} + 1, & j = 2m+1 \text{ odd} \\ p_j + 1, & j = 2m \qquad \text{even} \end{cases} \tag{4.11}$$

Then the corresponding Cesàro–type mappings are given by

$$s_n(z) = \frac{1}{n}\sum_{j=1}^{n} z|z|^{p_j^{(-1)^j}} = \frac{z}{n}\left(|z|^{\frac{1}{2}} + |z|^3 + |z|^{\frac{1}{5}} + |z|^7 + \cdots + |z|^{p_n^{(-1)^n}}\right)$$

for $z \in \mathbb{D}$ and $n \in \mathbb{N}$. The concrete mapping shown in the left–hand side plot of Figure 4.2 consists of $n = 35$ summands, i.e. the first 35 values of the numbers K_j according to (4.11) are used. In a certain sense, the geometric behaviour of the depicted mapping s_{35} is controlled by two "opposed forces":

- On the one hand, the summands having the exponents $\frac{1}{p_{2m-1}} + 1$ are close to the identity mapping $\mathrm{id}_{\mathbb{D}}$ (in either sense of the word "close"), since the values of the reciprocal prime number tend to zero rapidly, thus leaving the Euclidean grid more or less unaltered;
- On the other hand, the summands belonging to the exponents $p_{2m} + 1$ with even index values show the effect of "pulling" the grid towards the origin, as can also be seen in Figure 2.2.

This concludes in the interesting pictures shown in Figure 4.2, in which both mapping effects are visible: In the center of the unit disk, close to the origin, the Euclidean grid is virtually unchanged, whereas in the outer regions near $\partial\mathbb{D}$, the grid shows strong signs of distortion. This effect becomes especially apparent in the right–hand side plot of Figure 4.2, depicting a magnified region inside $\mathbb{D}$ of the Euclidean grid's image under the described mapping s_{35}.

Remark 4.2.3.
As already mentioned previously, the particular construction of the Cesàro–type mappings s_n in (4.10) is remarkable inasmuch as automorphic mappings, thus in particular (quasi)conformal mappings, are not preserved by linear combinations in general (under the fundamental assumption that operations similar to "addition" and "scalar multiplication" can be reasonably defined). This can easily be seen by considering a simple counterexample, e.g.

$$\mathbb{D} \ni z \longmapsto \frac{1}{2}\left(-z + \frac{z - \frac{1}{2}}{1 - \frac{1}{2}z}\right) = \frac{1 - z^2}{2(z-2)}$$

which is not even an injective[6] mapping, even though each summand inside the bracket is a conformal unit disk automorphism.

Since the Cesàro–type mappings s_n define a sequence in $\mathbb{D}$, the question for the convergence in $Q(\mathbb{D})$ arises naturally. In particular, it is interesting to ask whether the sequence $(s_n)_n$ can be convergent, even though the "basic functions" $z|z|^{K_j - 1}$ are divergent. The following Example 4.2.4 is concerned with these questions, demonstrating two concrete prototypical situations:

[6] For example, the specified mapping assumes the value $w = -\frac{1}{4}$ twice in $\mathbb{D}$, namely for $z = 0$ and $z = \frac{1}{2}$, as a simple calculation shows.

Example 4.2.4.

(i) Let $K_j = 1 + \frac{1}{j}$ for $j \in \mathbb{N}$, thus the basic functions are given by

$$z|z|^{K_j - 1} = z|z|^{\frac{1}{j}}$$

for $z \in \mathbb{D}$. The corresponding sequence of monomial–like radial stretchings $(\mathbb{D} \ni z \mapsto z|z|^{\frac{1}{j}})_j$ clearly consists of 2–quasiconformal mappings and converges uniformly on $\mathbb{D}$ to the identity mapping $\mathrm{id}_{\mathbb{D}}$, as the following reasoning shows: Let $z = re^{i\varphi} \in \mathbb{D}$, then

$$\left| z|z|^{\frac{1}{j}} - z \right| = r\left(1 - r^{\frac{1}{j}}\right)$$

The supremum of this expression, i.e. the maximal distance in $\mathbb{D}$, occurs for $r_j = (\frac{j}{j+1})^j$, as can be seen by a simple extreme value calculation, yielding

$$d_{\sup}(z|z|^{\frac{1}{j}}, \mathrm{id}_{\mathbb{D}}) = r_j(1 - r_j^{\frac{1}{j}}) = r_j \frac{1}{j+1} \longrightarrow 0$$

due to the well–known[7] limit $r_j \longrightarrow e^{-1}$ for $j \to +\infty$. In equivalence with the classical result of Cauchy concerning the limits of arithmetic means of convergent sequences of (real or complex) numbers (see [Heu09, Satz 27.1, p. 177] and also [Zyg02, Vol. I, p. 75]), the corresponding Cesàro–type sequence

$$s_n(z) = \frac{z}{n} \sum_{j=1}^{n} |z|^{\frac{1}{j}}$$

also converges in $Q(\mathbb{D})$ to the identity mapping. This claim can be proved in the classical manner (see e.g. [Heu09, pp. 176–177]):
Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ sufficiently large such that $d_{\sup}(z|z|^{\frac{1}{j}}, \mathrm{id}_{\mathbb{D}}) < \frac{\epsilon}{2}$ for $j \geq N$. Moreover, due to $\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{N} d_{\sup}(z|z|^{\frac{1}{j}}, \mathrm{id}_{\mathbb{D}}) = 0$ (since N is independent of n), there exists $M \in \mathbb{N}$ with $\frac{1}{n} \sum_{j=1}^{N} d_{\sup}(z|z|^{\frac{1}{j}}, \mathrm{id}_{\mathbb{D}}) < \frac{\epsilon}{2}$ for each $n \geq M$. Hence, for $n \geq \max(N, M)$, consider

$$\begin{aligned} |s_n(z) - z| &= \left| \frac{1}{n} \sum_{j=1}^{n} z|z|^{\frac{1}{j}} - z \right| = \frac{1}{n} \left| \sum_{j=1}^{n} (z|z|^{\frac{1}{j}} - z) \right| = \frac{1}{n} \left| \sum_{j=1}^{N} (z|z|^{\frac{1}{j}} - z) + \sum_{j=N+1}^{n} (z|z|^{\frac{1}{j}} - z) \right| \\ &\leq \frac{1}{n} \sum_{j=1}^{N} \left| z|z|^{\frac{1}{j}} - z \right| + \frac{1}{n} \sum_{j=N+1}^{n} \left| z|z|^{\frac{1}{j}} - z \right| \leq \frac{1}{n} \sum_{j=1}^{N} d_{\sup}(z|z|^{\frac{1}{j}}, \mathrm{id}_{\mathbb{D}}) + \frac{1}{n} \sum_{j=N+1}^{n} d_{\sup}(z|z|^{\frac{1}{j}}, \mathrm{id}_{\mathbb{D}}) \end{aligned}$$

The left–hand sum is smaller than $\frac{\epsilon}{2}$ by construction, and each summand of the right–hand sum is smaller than $\frac{\epsilon}{2}$ by the choice of N. Thus, continuing the inequality chain above, these estimates yield

$$|s_n(z) - z| < \frac{\epsilon}{2} + \frac{(n-N)\epsilon}{2n} \leq \epsilon$$

Switching to the supremum over all $z \in \mathbb{D}$ finally concludes in $d_{\sup}(s_n, \mathrm{id}_{\mathbb{D}}) \leq \epsilon$, hence the Cesàro–type mappings s_n converge to the identity in $Q(\mathbb{D})$.

(ii) For this second example, let

$$K_j = (-1)^j + 2 = \begin{cases} 1, & j = 2t + 1 \ \textit{odd} \\ 3, & j = 2t \quad\ \textit{even} \end{cases}$$

[7] For example, this limit can be seen immediately by writing $r_j^{-1} = (1 + \frac{1}{j})^j \overset{j\to+\infty}{\longrightarrow} e$; see [Heu09, Beispiel 9, p. 149].

for $j \in \mathbb{N}$, which is clearly a divergent sequence of real numbers. Consequently, the corresponding sequence of monomial–like radial stetchings of $\mathbb{D}$ given by $(z|z|^{((-1)^j+1)})_j$ cannot converge in $Q(\mathbb{D})$ either. However, it will be shown in the following that the respective Cesàro–type mappings

$$s_n(z) = \frac{1}{n}\sum_{j=1}^{n} z|z|^{K_j-1} = \frac{z|z|}{n}\sum_{j=1}^{n}|z|^{(-1)^j} = \frac{z|z|}{n}\left(|z|^{-1}+|z|+|z|^{-1}+\cdots+|z|^{(-1)^j}\right)$$

in fact converge to the limit mapping

$$f(z) := \frac{1}{2}z(1+|z|^2)$$

which is also a quasiconformal unit disk automorphism. To this end, distinguish two cases:

- *For even $n = 2m \in \mathbb{N}$, it is*

$$s_n(z) = \frac{z|z|}{n}\left(m|z|^{-1}+m|z|\right) = \frac{z}{2}\left(1+|z|^2\right) = f(z)$$

  since the terms $|z|^{-1}$ and $|z|$ both occur m times in the finite sum, respectively;

- *On the contrary, for odd $n = 2m+1 \in \mathbb{N}$, one finds*

$$s_n(z) = \frac{z|z|}{n}\left((m+1)|z|^{-1}+m|z|\right) = \frac{z}{n}\left(1+m(1+|z|^2)\right) = \frac{z}{n}+\frac{n-1}{n}\frac{z}{2}(1+|z|^2) = \frac{z}{n}+\frac{n-1}{n}f(z)$$

  due to the fact that the term $|z|^{-1}$ occurs $m+1$ times in the finite sum.

From this, it is only a small step towards the announced result. For even index values n, it is already $s_n \equiv f$. And for odd index values n, it is

$$d_{\sup}(s_n, f) = \sup_{z\in\mathbb{D}}\left|\frac{z}{n}+\frac{n-1}{n}f(z)-f(z)\right| = \frac{1}{n}\sup_{z\in\mathbb{D}}|z-f(z)| = \frac{1}{2n}\sup_{z\in\mathbb{D}}\left|z(1-|z|^2)\right|$$

Since the function $z(1-|z|^2)$ is obviously continuous on the compact set $\overline{\mathbb{D}}$, this supremum is finite[8], concluding in the desired result: s_n converges in $Q(\mathbb{D})$ to the limit mapping f, which is clearly (and also necessarily) continuous. Moreover, f is certainly an ACL mapping due to $f = u + iv$ with $u(x,y) = \frac{1}{2}(x^3+xy^2+x)$ and $v(x,y) = \frac{1}{2}(y^3+x^2y+y)$. The Wirtinger derivatives of f are given by

$$f_z(z) = \frac{1}{2}(1+2|z|^2)$$
$$f_{\overline{z}}(z) = \frac{1}{2}z^2$$

which yields $|f_{\overline{z}}| = \frac{1}{2}|z|^2 < \frac{1}{2}+|z|^2 = |f_z|$. Hence, the mapping f fulfills the requirements of quasiconformality on $\mathbb{D}$ of the Analytic Definition 1.1.1. Finally, by writing

$$f(re^{i\varphi}) = \frac{1}{2}r(1+r^2)e^{i\varphi}$$

one sees that f maps the unit disk bijectively onto itself, yielding $f \in Q(\mathbb{D})$.

[8]This supremum can of course be explicitly computed by means of a simple extreme value calculation and setting $z = re^{i\varphi} \in \mathbb{D}$: It is $d_{\sup}(s_n, f) = \frac{2}{3\sqrt{3}}$, attained for $r = \frac{1}{\sqrt{3}}$.

4.3 A view towards applications: Quasiconformal Cryptosystems

This section is concerned with obtaining a glimpse of application for the theory of $Q(G)$, to be more precise, towards a theory of *cryptosystems* based on quasiconformal automorphisms. First of all, the primordial idea in order to focus this topic will be amplified: The motivation for considering *Quasiconformal Cryptosystems* was laid by the article "Biholomorphic Cryptosystems" by N.J. Daras, recently published in the scientific compendium *Advancements in Complex Analysis – From Theory to Practice*, see [Dar20]. In this paper, Daras presents "*... a physical adaption of classical cryptological discrete structures within the environment of complex variables*" ([Dar20, p. 51]). More precisely, he proposes two aspects of classical cryptological research in an complex–analytical framework: On the one hand, Daras presents an encoding–/decoding scheme based on biholomorphic (conformal) mappings, i.e. an algorithm for translating characters used in human language into objects that can be handled by the specific mappings investigated in Complex Analysis. More explicitly, he proposes a method to embed a human–readable set of characters "*into an initial simpl[y] connected domain of the complex plane* $\mathbb{C}$*, which is then transformed successively to other connected domains of* $\mathbb{C}$" ([Dar20, pp. 51–52]). Obviously, such a procedure is demanded in a quasiconformal setting as well, therefore a certain encoding–/decoding scheme adapted to this section's setting is presented in Subsection 4.3.1. On the other hand, Daras describes a cryptosystem based on biholomorphic mappings, i.e. algorithms to transform plaintext characters (which were properly encoded before) into corresponding ciphertext characters by means of biholomorphic mappings: This is realized with the aid of mappings from domains in $\mathbb{C}^n$ into $\mathbb{C}^n$ with $n \geq 2$, hence in the multi–dimensional situation, justifying the usage of the word "biholomorphic" rather than "conformal"[9]. Here, the number n refers to the length of the plaintext message to be encrypted, i.e. Daras' encryption scheme is designed as a so–called *block cipher*, i.e. the plaintext is divided into blocks of constant length and each block is encrypted individually using the corresponding encryption key (see [MvOV01, p. 224]). The quasiconformal cryptosystem to be introduced in Subsection 4.3.2 will be based on the very same concept, altered by the fact that both, the plaintext and ciphertext space, are modeled as a subset of the complex plain rather than being contained in some higher–dimensional unitary space $\mathbb{C}^n$.

This very special and promising situation – complex–analytic topics applied to application–oriented real–world problems – paved the way for the idea to study this field of interdisciplinary mathematics and to transfer Daras' proposed framework to the more general situation of *Quasiconformal Cryptosystems*, which are based on quasiconformal automorphisms of domains in $\mathbb{C}$. In view of this situation, the first two subsections of the current section primarily focus on the definition of quasiconformal cryptosystems, therefore the following general assumptions are made and will be used henceforth:

(1) All messages, plaintext and ciphertext, are assumed to have a fixed length, say n, for some natural number $n \in \mathbb{N}$ with $n > 1$. If an encoded plaintext P is to be encrypted, the single characters of P are divided into blocks of length n and each of those blocks is encrypted one after another; if necessary, some appropriate padding[10] scheme is used in order to arrive at a message with overall length being an integer multiple of the block length n.
(2) The encryption scheme to be proposed in the following is based on quasiconformal automorphisms of a certain (bounded) domain $G \subseteq \mathbb{C}$. Due to this circumstance and for the sake of simplicity, one may take $G = \mathbb{D}$ without loss of generality due to the (Measurable) Riemann Mapping Theorem (see Proposition 1.1.2 and e.g. [RS07, p. 173]).

[9] See e.g. [Kra04, p. 162] for further information on the difference between the notions of *conformal* and *biholomorphic* mappings, especially in the case of several complex variables.

[10] In this context, *padding* means to add certain prescribed data to a plaintext in order to arrive at a integer multiple of the block length used in the corresponding block cipher.

4.3.1 Preliminaries and basics of symmetric cryptography

Symmetric cryptosystems

In order to be able to establish an algorithm that is capable of defining a process for encrypting and decrypting certain information, the following definition of such a *cryptosystem* is required (see [KL20, pp. 4–5] and [MvOV01, pp. 11–12]):

Definition 4.3.1.
*A **(symmetric) private–key cryptosystem** is a tuple $(\mathcal{M}, \mathcal{C}, \mathcal{K}, E, D)$ consisting of*

*(i) The **plaintext space** $\mathcal{M}$ consisting of all possible messages, called **plaintext messages** or simply **plaintexts**;*
*(ii) The **ciphertext space** $\mathcal{C}$ consisting of all possible ciphertexts, called **ciphertext messages** or **ciphertexts**;*
*(iii) The **key space** $\mathcal{K}$ consisting of all possible **keys**;*
*(iv) An **encryption function***

$$E : \mathcal{M} \times \mathcal{K} \longrightarrow \mathcal{C},\ (m,k) \longmapsto E(m,k) := E_k(m)$$

such that $E(\cdot, k)$ is injective for each fixed key $k \in \mathcal{K}$;
*(v) A **decryption function***

$$D : \mathcal{C} \times \mathcal{K} \longrightarrow \mathcal{M},\ (c,k) \longmapsto D(c,k) := D_k(c)$$

such that $D(\cdot, k)$ is injective for each fixed key $k \in \mathcal{K}$,

*and with the additional property that the following **correctness condition** is fulfilled:*

$$\forall k \in \mathcal{K}\ \exists! k' \in \mathcal{K}\ \forall m \in \mathcal{M} :\ D_{k'}(E_k(m)) = m \tag{4.12}$$

In almost every cryptographic algorithm, it is $k' = k$ in the correctness condition (4.12), see [MvOV01, p. 12 / p. 15], i.e. encryption and decryption process are executed using the same key element. The historical development of symmetric cryptosystems is a interesting and ample science in its own right, including important characters such as Caesar and Alan Turing; see [KL20, Section 1.3] and [MvOV01, Section 1.14] as well as the sources mentioned therein for more information. Furthermore, due to the limited dimensions available in real–world applications imposed by a physically limited universe (e.g. computational power of CPUs, main memory [RAM] and disk storage space [HDD], transmission rate, computation time), the sets involved in Definition 4.3.1 – the plaintext space $\mathcal{M}$, the ciphertext space $\mathcal{C}$ and the key space $\mathcal{K}$ – are all *finite* sets. In sharp contrast to this intuitively plausible constraint, the cryptosystem to be proposed in the following is not bounded to this finiteness condition. In fact, all involved sets will have uncountably many elements. In view of the classical paradigm

"A necessary, but usually not sufficient, condition for an encryption scheme to be secure is that the key space be large enough to preclude exhaustive search"

in cryptography (see [MvOV01, Fact 1.40, p. 21]), the uncountability of the key space guarantees that the cryptosystem to be proposed satisfies the cited paradigm.

Remark 4.3.2.

*(i) In the corresponding literature, there is no completely unified definition of a symmetric cryptosystem. For example, some sources demand an additional item given by a so–called **key–generation function (algorithm)*** KeyGen *that serves as the source for retrieving a key from the key space $\mathcal{K}$ (see [Buc16, Definition 3.1, p. 73] and [KL20, p. 4]). Moreover, some authors define a cryptosystem to be based on probabilistic algorithms, e.g. the encryption function is sometimes supposed to be probabilistic, i.e. depending on stochastic principles and phenomena ([Buc16, p. 73]).*

(ii) In contrast to symmetric cryptosystems which are based on a single key in order to encrypt and decrypt information, there exists another extremely important and successful encryption scheme, so–called ***public–key systems****. Public–key cryptography was invented in the 1970s and has ever since been applied in a vast manner to basically every aspect of telecommunication infrastructure and beyond; see e.g. [Buc16, pp. 165–166] for more details.*

(iii) The usage of the symbol k for a key element $k \in \mathcal{K}$ in this section is not related to the previous usage of this letter in this thesis so far, connected to the maximal dilatation of quasiconformal mappings (see Definition 1.1.1(ii)). Since there won't be any risk of misunderstanding in the remainder of the current section, this naming convention for key elements – widely used in literature about cryptography – will be used tacitly.

Encoding a character set into $\mathbb{D}$

In elementary terms, the process of encoding is simply the translation of some human–readable (finite) *alphabet* (a character set) into another format that can be handled appropriately by some algorithm. A standard example is given by the *ASCII*[11] *code*, which is a widely used encoding scheme in electronic communication systems (see [Buc16, Beispiel 3.8, p. 83]). In the situation at hand, this means that the standard alphabet

$$\mathcal{A} := \{\text{a, b, c}, \dots, \text{x, y, z, A, B, C}, \dots, \text{X, Y, Z}, 0, 1, 2, \dots, 7, 8, 9\} \tag{4.13}$$

is to be embedded into the open unit disk $\mathbb{D}$ in a reasonable manner. Obviously, this task is solvable in numerous different ways, and further characters may be added to the alphabet $\mathcal{A}$, e.g. special symbols like @ or &. However, due to the fact that the concrete encoding scheme to be used is not of crucial importance for the topic to follow – and for the sake of simplicity –, the character set $\mathcal{A}$ given by (4.13) will be used throughout this section. The proposed encoding scheme to be presented in the current subsection is one of those mentioned possibilities: First, the set $\mathcal{A}$ is divided into three distinct subsets via

$$\begin{aligned} \mathcal{A}_1 &:= \{\text{a, b, c}, \dots, \text{x, y, z}\} \\ \mathcal{A}_2 &:= \{\text{A, B, C}, \dots, \text{X, Y, Z}\} \\ \mathcal{A}_3 &:= \{0, 1, 2, \dots, 7, 8, 9\} \end{aligned}$$

Thus, clearly, it is $\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2 \uplus \mathcal{A}_3$. Then, for each $j = 1, 2, 3$, every element $m \in \mathcal{A}_j$ is assigned the uniquely defined natural number $\mathfrak{o} = \mathfrak{o}(m) \in \mathbb{N}$ at which the character m occurs in the corresponding subalphabet $\mathcal{A}_j$ when adapting the canonical ordering of letters and digits as shown above; in order to clarify this described assignment, consider the following

Example 4.3.3.

1. *In the subalphabet $\mathcal{A}_1$, choose $m = c$, then it is $\mathfrak{o}(c) = 3$ since the letter c occurs at the third position in $\mathcal{A}_1$. Likewise, it is $\mathfrak{o}(z) = 26$.*
2. *In the subalphabet $\mathcal{A}_2$, choose $m = X$, then it is $\mathfrak{o}(X) = 24$ since the letter X occurs at the corresponding twenty–fourth position in $\mathcal{A}_2$.*
3. *In the subalphabet $\mathcal{A}_3$, choose $m = 5$, then it is $\mathfrak{o}(5) = 6$ since the digit 5 occurs at the sixth position in $\mathcal{A}_3$.*

Consequently, these assignments give rise to the mappings

$$\eta_j : \mathcal{A}_j \longrightarrow \mathbb{N},\ m \longmapsto \eta_j(m) := \mathfrak{o}(m)$$

for $j = 1, 2, 3$. Now the following encoding mapping can be defined:

[11] **A**merican **S**tandard **C**ode for **I**nformation **I**nterchange

Definition 4.3.4.
Fix three numbers $0 < r_1 < r_2 < r_3 < 1$. *The mapping*

$$Enc : \mathcal{A} \longrightarrow \mathbb{D},\ m \longmapsto Enc(m) := \begin{cases} r_1 \exp\left(2\pi i \frac{\eta_1(m)}{|\mathcal{A}_1|}\right), & m \in \mathcal{A}_1 \\ r_2 \exp\left(2\pi i \frac{\eta_2(m)}{|\mathcal{A}_2|}\right), & m \in \mathcal{A}_2 \\ r_3 \exp\left(2\pi i \frac{\eta_3(m)}{|\mathcal{A}_3|}\right), & m \in \mathcal{A}_3 \end{cases} \tag{4.14}$$

is called the ***canonical encoding mapping*** *of* $\mathbb{D}$.

Clearly, the canonical encoding mapping proposed above embeds the standard alphabet $\mathcal{A}$ as defined by (4.13) into the unit disk $\mathbb{D}$. Figure 4.3 depicts a concrete example of a canonical encoding mapping of $\mathbb{D}$ with the characters used in Example 4.3.3.
As already stated before, it is obvious that there are numerous further possibilities for defining such an encoding mapping, as can be seen, among others, in the free choice of the radii r_j. Consequently, other encoding mappings are possible and probably reasonable. For example, one may post–compose the mapping Enc with one (or several) injective self–maps of $\mathbb{D}$ in order to arrive at different encoded character sets. However, in the remainder of this section, the canonical encoding mapping (4.14) will be utilized in order to define a cryptosystem based on quasiconformal automorphisms of $\mathbb{D}$.

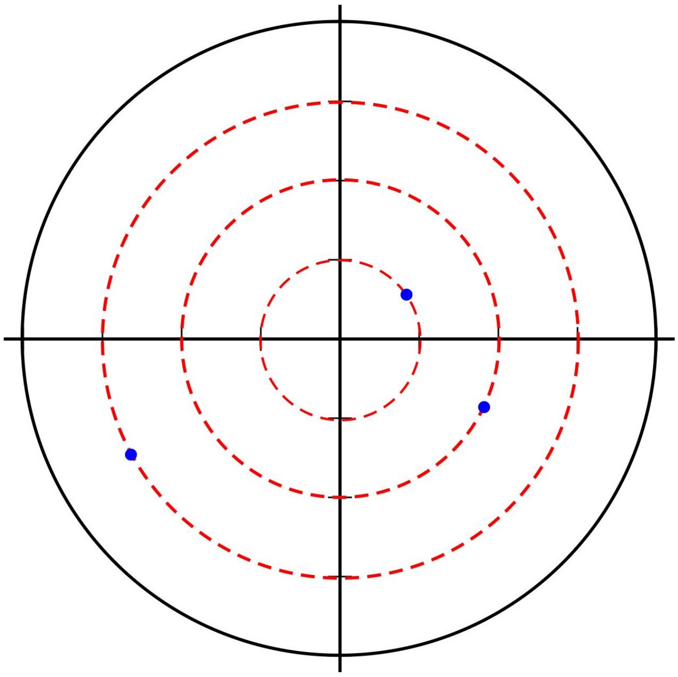

Figure 4.3: The unit disk (with boundary $\partial\mathbb{D}$ in black) with three inscribed smaller circles (dashed red lines), each with radius $r_j = \frac{j}{4}$ for $j = 1, 2, 3$. The blue dots are the (approximated values of the) characters $c \in \mathcal{A}_1$, $X \in \mathcal{A}_2$ and $5 \in \mathcal{A}_3$ as outlined in Example 4.3.3, located on their corresponding inner circles.

4.3.2 Symmetric quasiconformal cryptosystems modeled on $Q(\mathbb{D})$

In this subsection, the central idea for defining a symmetric encryption scheme using quasiconformal automorphisms of $\mathbb{D}$ is presented and described precisely step by step. Since the cryptosystem is supposed to operate on message blocks of block length n, each character of which is an element of the unit disk, the definition of the plaintext $\mathcal{M}$ is immediately obvious: It is

$$\mathcal{M} = \mathbb{D}^n$$

The ciphertext space $\mathcal{C}$, however, needs to have a slightly different form, namely by excluding the origin from the base set $\mathbb{D}$ and instead including the point $z = 1$:

$$\mathcal{C} = ((\mathbb{D} \cup \{1\}) \backslash \{0\})^n$$

This special construction is due to a necessary distinction of cases which will become clear in the following (see (4.22)).

The key space $\mathcal{K}$ and the key–generation process

In the following, the cryptographic configuration of the key space $\mathcal{K}$ is explained. Choose and fix non–trivial, pairwise distinct mappings $f_1, \ldots, f_n \in Q(\mathbb{D}) \backslash \{\mathrm{id}_{\mathbb{D}}\}$, which represent the *basic encryption functions*; here, as stated at the beginning of the current section, the number n denotes the block length of the messages to be en–/decrypted. Without loss of generality, one may assume that $f_j^{-1} \neq f_{j'}$ for $j, j' \in \{1, \ldots, n\}$, i.e. a quasiconformal automorphism of $\mathbb{D}$ and its inverse element in $Q(\mathbb{D})$ do not occur simultaneously in the chosen listing. For the next step, the following data sets are to be determined:

- An n–tuple $t := (t_1, \ldots, t_n) \in (\mathbb{Z} \backslash \{0\})^n$ consisting of non–zero integers, which will serve as exponents of the basic encryption functions f_j;
- Three non–trivial permutations of the set $\{1, \ldots, n\}$, i.e. non–identity elements of the *symmetric group* S_n (see [KM17, Beispiel 2.1, p. 20]), as follows:
  - A permutation σ_f which will operate on the encryption functions;
  - A permutation σ_p operating on the plaintext p, permuting the order of the plaintext characters;
  - A permutation σ_c operating on intermediate ciphertexts c, permuting the order of previously encrypted characters.

These elements constitute the **encryption key**

$$k = \big((t_1, \ldots, t_n), (\sigma_f, \sigma_p, \sigma_c)\big) \in (\mathbb{Z} \backslash \{0\})^n \times (S_n)^3 \tag{4.15}$$

such that the **key space** $\mathcal{K}$ is given by

$$\mathcal{K} = (\mathbb{Z} \backslash \{0\})^n \times (S_n)^3 \tag{4.16}$$

Remark 4.3.5 (Key–generation process)**.**
In the previously described construction of the key space $\mathcal{K}$, there was no reference on how to explicitly "choose" the involved parameters, for example the exponent tuple $t \in (\mathbb{Z} \backslash \{0\})^n$. In cryptography, this parameter selection process is a special question, interesting and important in its own right. In most situations, in particular in real–world applications, the selection of key para-meter values is given by the discrete uniform probability distribution on the corresponding (finite) sets (see [Buc16, p. 74] and [KL20, p. 25]). In the current situation, however, this scenario is not applicable due to the fact that parts of the key space $\mathcal{K}$ defined by (4.16) contain infinitely many elements (the Cartesian product $(\mathbb{Z} \backslash \{0\})^n$). A possible way to bypass this principal obstacle could be the following workaround for approximately choosing a uniformly distributed key from $\mathcal{K}$:
Instead of using the whole Cartesian product $(\mathbb{Z} \backslash \{0\})^n$, use a sufficiently large natural number N, use the discrete uniform distribution on $(\{-N, \ldots, N\} \backslash \{0\})^n$ and apply the n–fold product measure. On the finite set $(S_n)^3$, the product measure of the discrete uniform distribution can be used without any altering.

This yields an approximation procedure for generating a uniformly distributed key from the key space $\mathcal{K}$. Nevertheless, as already mentioned, key generation and key management are important topics in cryptography in their own right. Therefore, the key–generation process will not be pursued any further in this section.

The encryption mapping and the encryption algorithm

As the next important construction step, the central part of the quasiconformal cryptosystem will be introduced, namely the symmetric encryption mapping E and its mode of operation. Let

$$k = \big((t_1, \dots, t_n), (\sigma_f, \sigma_p, \sigma_c)\big) \in \mathcal{K}$$

be a given encryption key as presented in (4.15). As a first preparatory step, consider for the index values $j = 1, \dots, n$ the mappings

$$\widehat{f_j} := \big(f_{\sigma_f(j)}\big)^{t_j} = \underbrace{f_{\sigma_f(j)} \circ f_{\sigma_f(j)} \circ \dots \circ f_{\sigma_f(j)}}_{t_j \text{ times}} \tag{4.17}$$

These mappings $\widehat{f_j} \in Q(\mathbb{D})$ will serve as the *actual encryption functions*. In (4.17), the usage of the permutation $\sigma_f \in S_n$ has the effect of interchanging the position of the original mappings f_j in the encryption process, whereas the exponents t_j alter the (previously interchanged) mappings within the group $Q(\mathbb{D})$ (here and in the remainder of this section, the group operation of $Q(\mathbb{D})$, i.e. composition of mappings, is written multiplicatively as in (4.17)).

Remark 4.3.6.
The construction of the mappings $\widehat{f_j}$ in (4.17) shows that the order of applying the permutation σ_f and the exponentiation using the integers t_j is not interchangeable, i.e. it is crucial to keep the arrangement of these operations fixed in order to arrive at a well–defined cryptographic algorithm. In a formula–like notation, this statement reads as

$$\big(f_{\sigma_f(j)}\big)^{t_j} \neq \big(f^{t_j}\big)_{\sigma_f(j)} \tag{4.18}$$

Of course, the right–hand side of (4.18) could also be used for the proposed encryption scheme; in the first instance, it is of no deeper meaning that the encryption mappings $\widehat{f_j}$ are defined via (4.17), even though – from a cryptological point of view – there might be "good" reasons to change the definition of the encryption mappings $\widehat{f_j}$ accordingly.

In order to describe the encryption process defined by the encryption mapping E, let $P = P_1 \cdots P_V$ be a given plaintext message consisting of $V \in \mathbb{N}$ message blocks $P_v = p_{v,1} \cdots p_{v,n} \in \mathcal{M}$ for all $v = 1, \dots, V$, each[12] of which has block length n. Consider the first message block P_1 and permute the order of the corresponding plaintext characters using the plaintext permutation σ_p via

$$\widehat{p}_{1,j} := p_{1,\sigma_p(j)}$$

for $j = 1, \dots, n$, yielding an intermediate plaintext $\widehat{P}_1 := \widehat{p}_{1,1} \cdots \widehat{p}_{1,n}$.
Now the central encryption part will be explained. The intermediate plaintext $\widehat{P}$ is now encrypted characterwise in the following recursive way: First of all, set $\widehat{c}_{1,0} := 1$ for reasons that will become clear in an instant. Next, choose an **initialization vector**

$$IV_1 := (w_{1,1}, \dots, w_{1,n}) \in (\mathbb{D} \backslash \{0\})^n \tag{4.19}$$

[12] Without loss of generality, one may assume that each block has block length n by applying an appropriate padding scheme, as already stated in the beginning of this section.

using an appropriate generation scheme. Then, for every index value $j = 1, \ldots, n$ successively, the value $\widehat{p}_{1,j}$ is multiplied[13] with the corresponding element $w_{1,j}$ of IV_1 and the $(j-1)$–th intermediate ciphertext character:

$$\chi_{1,j} := w_{1,j} \cdot \widehat{c}_{1,j-1} \cdot \widehat{p}_{1,j} \in \mathbb{D} \tag{4.20}$$

Afterwards, the resulting number $\chi_{1,j}$ is encrypted using the j–th encryption mapping $\widehat{f}_j$, i.e. the value

$$\widehat{f}_j(\chi_{1,j}) \in \mathbb{D} \tag{4.21}$$

is computed. In order to establish a well–defined encryption/decryption scheme, the following distinction is necessary: In case $\widehat{f}_j(\chi_{1,j}) = 0$, define $\widehat{c}_{1,j} := 1$ (which serves as a "wildcard value", to be explained below), otherwise set $\widehat{c}_{1,j} := \widehat{f}_j(\chi_{1,j})$. This procedure yields the corresponding intermediate ciphertext character $\widehat{c}_{1,j}$, which can be summarized as

$$\widehat{c}_{1,j} := \begin{cases} 1, & \text{if } \ \widehat{f}_j(\chi_{1,j}) = 0 \\ \widehat{f}_j(\chi_{1,j}), & \text{otherwise} \end{cases} \tag{4.22}$$

The distinction of the two cases in (4.22) is necessary since in the decryption process – which is basically "just the inversion" of the encryption mapping –, one needs to divide by $\widehat{c}_{1,j}$, and of course this cannot be done if one of the corresponding ciphertext characters is zero. The fact that the "wildcard value" 1 is used in (4.22) is of minor importance; actually, any other value $z_0 \in \mathbb{C} \backslash \mathbb{D}$ could be used equally well, since each mapping $\widehat{f}_j$ is a quasiconformal automorphism of the unit disk. The described encryption process so far is well–defined due to the following elementary

Lemma 4.3.7.
The set $\mathbb{D}\backslash\{0\}$ forms a semigroup with respect to multiplication of complex numbers.

Proof. Multiplication of complex numbers is an associative binary operation, and the product of two non-zero complex numbers of modulus less than one yields another object of this kind. □

Consequently, one arrives at the intermediate ciphertext block $\widehat{c}_{1,1} \ldots \widehat{c}_{1,n}$. The final step is to apply one last permutation, σ_c, to this intermediate ciphertext block according to

$$c_{1,j} := \widehat{c}_{1,\sigma_c(j)}$$

for all $j = 1, \ldots, n$, yielding the final encrypted first ciphertext block $C_1 := c_{1,1} \cdots c_{1,n} \in \mathcal{C}$. This is the encryption process of the first message block P_1.

Now the remaining $V-1$ message blocks P_v with $v = 2, \ldots, V$ are encrypted in the same manner, with one central alteration: Instead of using the original initialization vector IV_1 in (4.19) in the encryption process, the vector IV_1 is replaced as

$$IV_v := C_{v-1} \tag{4.23}$$

i.e. in the encryption process of message block P_v, the previously encrypted ciphertext block C_{v-1} is utilized as the initialization vector.

Finally, in the encryption process of the last block P_V, it is possibly necessary to add a padding character $t_{\text{pad}} \notin \mathcal{A}$ sufficiently often at the end of P_V in order to arrive at the required block length n (see also the introductory remarks at the beginning of the current Section 4.3). If this situation

[13]This multiplication is simply the common multiplication of complex numbers.

occurs, the encryption process is to be adjusted appropriately in its last step, i.e. for $v = V$, by inserting an encoded wildcard value $z_{\text{pad}} \in \mathbb{C}$ with $|z_{\text{pad}}| > 1$ at the corresponding ciphertext character positions:

$$\text{If } \widehat{p}_{V,j} = Enc(t_{\text{pad}}) \implies \widehat{c}_{V,j} \coloneqq z_{\text{pad}}$$

for the respective index values j. In this regard, the canonical encoding mapping Enc (see Definition 4.3.4) has to assign a reasonable value to the padding character t_{pad} by extending its domain of definition suitably. Consequently, the value of $\widehat{c}_{V,j-1}$ in (4.20) is to be replaced by the neutral element of multiplication $z = 1$ in the subsequent iteration.

The proposed encryption scheme (without the special padding treatment in the last block P_V) is schematically summarized in Algorithm 1 below:

Algorithm 1: Encryption process (without padding)

Data: Plaintext message $P = P_1 \cdots P_V$ with $P_v = p_{v,1} \cdots p_{v,n} \in \mathcal{M}$ for $v = 1, \ldots, V$
Encryption key $k = \big((t_1, \ldots, t_n), (\sigma_f, \sigma_p, \sigma_c)\big) \in \mathcal{K}$
Encryption mappings $\widehat{f}_j = (f_{\sigma_f(j)})^{t_j}$ for $j = 1, \ldots, n$
Initialization vector $IV_1 = (w_{1,1}, \ldots, w_{1,n}) \in (\mathbb{D} \setminus \{0\})^n$

Result: The encrypted ciphertext $C = C_1 \cdots C_V$

1 **for** $v = 1$ **to** V **do**
2   Define $\widehat{c}_{v,0} \coloneqq 1$;
3   **for** $j = 1$ **to** n **do**
4   Compute intermediate plaintext character $\widehat{p}_{v,j} \coloneqq p_{v,\sigma_p(j)}$; `// Permutation`
5   Compute $\chi_{v,j} \coloneqq w_{v,j} \cdot \widehat{c}_{v,j-1} \cdot \widehat{p}_{v,j}$; `// Substitution`
6   Compute $\widehat{f}_j(\chi_{v,j})$; `// Substitution`
7   **if** $\widehat{f}_j(\chi_{v,j}) = 0$ **then**
8   Define $\widehat{c}_{v,j} \coloneqq 1$
9   **else**
10   Define $\widehat{c}_{v,j} \coloneqq \widehat{f}_j(\chi_{v,j})$
11   **end**
12   Compute $c_{v,j} \coloneqq \widehat{c}_{v,\sigma_c(j)}$; `// Permutation`
13   **end**
14   Set $C_v \coloneqq c_{v,1} \cdots c_{v,n}$;
15   Update initialization vector $IV_{v+1} \coloneqq C_v$;
16 **end**

Remark 4.3.8 (Encryption scheme)**.**
As can be seen in (4.20)–(4.22), the j–th intermediate ciphertext character $\widehat{c}_{v,j}$ depends on the previous $(j-1)$–th ciphertext character in general. Moreover, due to the alteration of the initialization vector according to (4.23), each ciphertext block C_v depends on the previous ciphertext block C_{v-1}. These steps ensure to a large part the adherence of the cryptographic primitives confusion and diffusion:

a) ***Confusion*** *is the idea of "...mak[ing] the relationship between the key and the ciphertext as complex as possible" (see [MvOV01, Remark 1.36, p. 20]). More concrete, it is for example intended to obscure the letter frequency of a given plaintext in the corresponding ciphertext: If a plaintext M contains a double letter, e.g. M = KEEP, then it is desirable that the encryption algorithm does not map both (encoded) characters E to the same ciphertext character, but to different ciphertexts. In view of line 12 in Algorithm 1, one sees that the ciphertext character $c_{v,j}$ depends on every part of the chosen encryption key k (except, as the case may be, for the exponents $t_{v+1}, \ldots, t_n$), thus obscuring the relationship between key and ciphertext.*

b) ***Diffusion*** *means to "[rearrange] or [spread] out the [information] in the message so that any redundancy in the plaintext is spread out over the ciphertext" (see [MvOV01, Remark 1.36, p. 20]). More concrete, changing a character of the plaintext is intended to affect as many characters of the ciphertext as possible, thus obscuring the relationship between plaintext and ciphertext. In view of Algorithm 1, this cryptographic paradigm is assured by the usage of the permutations* σ_p *and* σ_c *which operate on the corresponding message blocks.*

The mode of operation of the proposed quasiconformal encryption scheme is related to and at the same time inspired by a widely used design pattern for classical block ciphers, the so–called ***chained Cipher–Block–Chaining Mode (cCBC Mode)****. This particular encryption mode also uses an initialization vector and successively chains the previous ciphertext block with the next plaintext block to be encrypted; see [Buc16, Subsection 3.10.2] and [KL20, pp. 90–91] for further information.*

In addition, the design of the encryption algorithm explained above is – to a certain extent – motivated by an important paradigm of modern cryptography, namely so–called *Substitution–Permutation networks* (SP networks) which form the basis for numerous cryptographic algorithms used in real–world applications such as the *Advanced Encryption Standard* (AES) (see [KL20, p. 220–224] and [MvOV01, Definition 7.79, p. 251]). As the name suggests, a SP network consists of a combination of (one or several) substitutions and permutations. In the proposed quasiconformal encryption scheme, explained in Algorithm 1, these operations are indicated by the comments on the right–hand side in the lines 4, 5, 6 and 12.

A central part in Algorithm 1 is played by the choice of the initialization vector in (4.19). In principle, there are multiple ways for the primary determination of IV_1. Depending on whether the initialization vector is chosen in a deterministic or a probabilistic way, the corresponding encryption scheme is called *context–dependent* or *randomized*, respectively. For example, it is possible to use a (hardware–based) counter as the initialization vector, resulting in a context–dependent scheme. Another possibility is given by choosing an appropriately encoded timestamp, which yields a randomized encryption scheme in general ([Buc16, p. 90]). The randomized approach offers the advantage of allowing for the implementation of so–called *rolling codes* due to the possibility of introducing certain parameter values in the encryption scheme (see also [Dar20, Remark 5, p. 69] for another different potential approach).

Remark 4.3.9 (Extensibility of the encryption scheme)**.**
The encryption scheme proposed above is only one of numerous possible ideas for defining a cryptographic algorithm based on quasiconformal automorphisms of $\mathbb{D}$*. For example, in (4.17), one could easily change the order of the permutation* σ_f *and the exponentiation using the integers* t_j*, resulting in completely different encryption mappings* $\widehat{f_j}$ *for* $j = 1, \ldots, n$ *(see also Remark 4.3.6). Similarly, several other modifications are conceivable and possibly reasonable in order to achieve a secure cryptographic algorithm, for example by choosing the initialization vectors* IV_v *(pseudo)–randomly in (4.23) instead of using the previously encrypted ciphertext blocks (see [KL20, p. 91]). Moreover, Algorithm 1 is constructed in such a manner that it can easily be extended by one or several consecutive encryption steps, e.g. by introducing repeated* ***round functions*** *based on (parts of) the encryption key k, so–called* ***round keys****. This flexibility is permitted by the fact that the proposed encryption scheme is based on/related to the SP network design, which was already mentioned previously.*

A concrete example

The proposed quasiconformal encryption scheme will now be applied to a concrete plaintext message in order to exemplify the underlying algorithm. Suppose the plaintext message is given as

$$M = \text{ATESTMESSAGE}$$

and the corresponding block length is $n = 3$. Thus, according to the assumptions mentioned in the previous subsections, divide the plaintext in blocks of 3 characters:

$$|\text{ATE}|\text{STM}|\text{ESS}|\text{AGE}|$$

with $M_1 = \text{ATE}$, $M_2 = \text{STM}$, $M_3 = \text{ESS}$ and $M_4 = \text{AGE}$, i.e. $V = 4$. Hence, for example in the first block, it is $M_1 = m_{1,1}m_{1,2}m_{1,3} = \text{ATE}$. In order to apply the quasiconformal encryption scheme to the plaintext message M, use the following canonical encoding mapping (see Definition 4.3.4):

$$Enc : \mathcal{A} \longrightarrow \mathbb{D},\ m \longmapsto Enc(m) := \frac{1}{2}\exp\left(\pi i \frac{\eta_2(m)}{13}\right)$$

in which only the subalphabet $\mathcal{A}_2$ is involved due to the particular form of the plaintext message M, and it is $r_2 = \frac{1}{2}$. Encoding M via the mapping Enc yields the following data set written in matrix form, in which the encoded characters $p_{v,j}$ of the v–th plaintext block M_v are contained in the v–th row:

$$\begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix} = \frac{1}{2} \cdot \begin{pmatrix} e^{\pi i \frac{1}{13}} & e^{\pi i \frac{20}{13}} & e^{\pi i \frac{5}{13}} \\ e^{\pi i \frac{19}{13}} & e^{\pi i \frac{20}{13}} & e^{\pi i} \\ e^{\pi i \frac{5}{13}} & e^{\pi i \frac{19}{13}} & e^{\pi i \frac{19}{13}} \\ e^{\pi i \frac{1}{13}} & e^{\pi i \frac{7}{13}} & e^{\pi i \frac{5}{13}} \end{pmatrix} \approx \begin{pmatrix} 0.4855 + 0.1197i & 0.0603 - 0.4964i & 0.1773 + 0.4675i \\ -0.0603 - 0.4964i & 0.0603 - 0.4964i & -0.5 \\ 0.1773 + 0.4675i & -0.0603 - 0.4964i & -0.0603 - 0.4964i \\ 0.4855 + 0.1197i & -0.0603 + 0.4964i & 0.1773 + 0.4675i \end{pmatrix} \tag{4.24}$$

In the second matrix, numerically approximated values are displayed. For example, the first plaintext character $m_{1,1} = \text{A}$ is encoded to $p_{1,1} = Enc(m_{1,1}) = \frac{1}{2}e^{\pi i \frac{1}{13}} \approx 0.4855 + 0.1197i$.

Now the initial data for Algorithm 1 is defined: For the encryption key $k \in \mathcal{K}$, the following data will be used:

(i) The exponents for the encryption mappings are $t_1 = 2$, $t_2 = 1$, $t_3 = 2$;
(ii) The permutation for the encryption functions will be $\sigma_f = (1 \mapsto 3, 2 \mapsto 1, 3 \mapsto 2)$;
(iii) The plaintext permutation is given by $\sigma_p = (1 \mapsto 3, 2 \mapsto 2, 3 \mapsto 1)$;
(iv) The ciphertext permutation is given by $\sigma_c = (1 \mapsto 3, 2 \mapsto 1, 3 \mapsto 2)$;

The primary encryption mappings are defined as

$$f_1(z) = z|z|,\ \ f_2(z) = \frac{z - \frac{1}{3}}{1 - \frac{1}{3}z},\ \ f_3(z) = \sqrt{|z|}\frac{z}{|z|} \tag{4.25}$$

That is, the mapping f_1 is given as a monomial–like radial stretching with coefficient $K = 2$ (see (2.3)). The mapping f_2 represents a conformal unit disk automorphism (see e.g. [Kra06, p. 260]), whereas the third mapping f_3 is a general radial stretching with radial dilation mapping $x \longmapsto \sqrt{x}$ subject to Definition 2.3.1. Consequently, the actual encryption mappings are given by

$$\widehat{f_1}(z) = (f_3)^2(z) = \sqrt[4]{|z|}\frac{z}{|z|},\ \ \widehat{f_2}(z) = (f_1)^1(z) = z|z|,\ \ \widehat{f_3}(z) = (f_2)^2(z) = \frac{z - \frac{3}{5}}{1 - \frac{3}{5}z} \tag{4.26}$$

for $z \in \mathbb{D}$. Finally, for the initialization vector, choose

$$IV_1 = (w_{1,1}, w_{1,2}, w_{1,3}) = \left(-\frac{i}{4}, \frac{i}{2}, -\frac{1}{3}\right)$$

according to the required input data of Algorithm 1. Now the required data is defined in order to begin with the encryption process. First of all, by applying the plaintext permutation σ_p, the original message becomes

|ETA|MTS|SSE|EGA|

e.g. the first block M_1 is transformed into ETA, and for the encoded plaintext, this yields

$$\widehat{P}_1 = (\widehat{p}_{1,1}, \widehat{p}_{1,2}, \widehat{p}_{1,3}) = (p_{1,3}, p_{1,2}, p_{1,1}) = \frac{1}{2}\left(e^{\pi i \frac{5}{13}}, \quad e^{\pi i \frac{20}{13}}, \quad e^{\pi i \frac{1}{13}}\right)$$

Following the algorithm for $\widehat{P}_1$, the first intermediate ciphertext character $\widehat{c}_{1,1}$ computes to

$$\chi_{1,1} = w_{1,1}\widehat{c}_{1,0}\widehat{p}_{1,1} = -\frac{i}{4} \cdot 1 \cdot \frac{1}{2} e^{\pi i \frac{5}{13}} = -\frac{i}{8} e^{\pi i \frac{5}{13}}$$
$$\widehat{c}_{1,1} = \widehat{f}_1(\chi_{1,1}) = f_3^2\left(-\frac{i}{8} e^{\pi i \frac{5}{13}}\right) = -\frac{i}{\sqrt[4]{8}} e^{\pi i \frac{5}{13}} \approx 0.5560 - 0.2108i$$

and consequently

$$\chi_{1,2} = w_{1,2}\widehat{c}_{1,1}\widehat{p}_{1,2} = \frac{i}{2} \cdot \frac{-i}{\sqrt[4]{8}} e^{\pi i \frac{5}{13}} \cdot \frac{1}{2} e^{\pi i \frac{20}{13}} = \frac{1}{\sqrt[4]{2^{11}}} e^{\pi i \frac{25}{13}}$$
$$\widehat{c}_{1,2} = \widehat{f}_2(\chi_{1,2}) = f_1\left(\frac{1}{\sqrt[4]{2^{11}}} e^{\pi i \frac{25}{13}}\right) = \frac{1}{\sqrt{2^{11}}} e^{\pi i \frac{25}{13}} \approx 0.0215 - 0.0053i$$

as well as

$$\chi_{1,3} = w_{1,3}\widehat{c}_{1,2}\widehat{p}_{1,3} = -\frac{1}{3} \cdot \frac{1}{\sqrt{2^{11}}} e^{\pi i \frac{25}{13}} \cdot \frac{1}{2} e^{\pi i \frac{1}{13}} = -\frac{1}{3\sqrt{2^{13}}}$$
$$\widehat{c}_{1,3} = \widehat{f}_3(\chi_{1,3}) = f_2^2\left(-\frac{1}{3\sqrt{2^{13}}}\right) = \frac{-368635 - 1024\sqrt{2}}{614397} \approx -0.60235$$

Finally, permuting the computed intermediate ciphertext characters using the permutation σ_c yields the first ciphertext block

$$C_1 = (c_{1,1}, \ c_{1,2}, \ c_{1,3}) = (\widehat{c}_{1,3}, \ \widehat{c}_{1,1}, \ \widehat{c}_{1,2}) = \left(\frac{-368635 - 1024\sqrt{2}}{614397}, \ -\frac{i}{\sqrt[4]{8}} e^{\pi i \frac{5}{13}}, \ \frac{1}{\sqrt{2^{11}}} e^{\pi i \frac{25}{13}}\right)$$
$$\approx (-0.60235, \ 0.5560 - 0.2108i, \ 0.0215 - 0.0053i)$$

This first encrypted block will be used in the next iteration step ($v = 2$) of Algorithm 1 as the initialization vector in the proposed "cCBC–like" encryption mode (see also Remark 4.3.8), i.e.

$$IV_2 = (w_{2,1}, \ w_{2,2}, \ w_{2,3}) = (c_{1,1}, \ c_{1,2}, \ c_{1,3})$$

The further process of the encryption scheme for computing the remaining ciphertext blocks C_2, C_3 and C_4 is executed analogously as depicted above. The numerically approximated values of the final result is contained in the following matrix–like scheme:

$$\begin{pmatrix} C_2 \\ \\ C_3 \\ \\ C_4 \end{pmatrix} \approx \begin{pmatrix} -0.6003 + 0.0002i & 0.7408 & -0.0116 - 0.0471i \\ \\ -0.5992 - 0.0009i & 0.0895 + 0.7347i & 0.0730 - 0.0180i \\ \\ -0.5994 + 0.0017i & -0.2613 - 0.6921i & 0.0265 + 0.0701i \end{pmatrix} \tag{4.27}$$

Noteworthy in this context is the following interesting observation with regard to cryptographic security: The plaintext blocks M_1 = ATE and M_4 = AGE are very similar, as they differ only in the second character. However, the quasiconformal cryptosystem encrypts these two blocks to the ciphertext messages

$$C_1 \approx (-0.60235,\ \ 0.5560 - 0.2108i,\ \ 0.0215 - 0.0053i)$$

and

$$C_4 \approx (-0.5994 + 0.0017i,\ \ -0.2613 - 0.6921i,\ \ 0.0265 + 0.0701i)$$

as seen above. These ciphertext blocks show a large deviation from each other, especially in the second character. This fact indicates a strong crytographic property of the proposed quasiconformal encryption scheme.

Remark 4.3.10.
The calculations used in the example above were executed using the numerical computation software MATLAB in the version R2018b. In particular, the numerically approximated values, especially in (4.24) and (4.27), were obtained from it. In order to efficiently compute the described algorithm, a MATLAB script "QCryptosystem.m" was developed containing all previously described steps, from the definition of the required data and encoding the message into $\mathbb{D}$ *to the actual encryption algorithm. For computing the values of the iterated encryption mappings* $\widehat{f_j}$ *(see (4.17)), an appropriate MATLAB function "IterateMapping.m" was implemented.*
Moreover and for the sake of completeness, the MATLAB script "QCryptosystem.m" also contains the corresponding decryption part of the proposed quasiconformal cryptosystem, not least in order to verify for the correct implementation of Algorithm 1. Due to the sophisticated computations, the irregular values involved in the corresponding formulas (e.g. the transcendental[14] *number* π*) and the complicated function terms (e.g. the mapping* $\widehat{f_3} = f_2^2$ *in the example above), some minor numerical errors occur during the execution of the MATLAB file. To make this statement more precise, one may consider the (formal) difference matrix between the original (encoded) plaintext values* P *and the decrypted ciphertext characters* D*, i.e.*

$$P - D = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix} - \begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{pmatrix}$$

In this formal matrix, the absolute values of the resulting entries are in the magnitude of 10^{-12} *and below, i.e. the encryption/decryption computations are executed correctly. The main reason for these negligible minor inaccuracies is given by the extremely small absolute values of the involved complex numbers, resulting in binary objects beyond the capable computational accuracy.*

[14] The fact that π is a transcendental number was proved by Lindemann in 1882, see [KM17, p. 277].

Open questions

This final part of the thesis at hand summarizes the most important open questions about the topological and group–theoretic nature of the quasiconformal automorphism groups $Q(G)$ that arose throughout the text:

1. Is the inverse conjugation mapping $\Phi^{-1} : Q(G) \longrightarrow Q(\mathbb{D})$ always continuous, not only for domains $G \in \mathcal{JD}$? (see Remark 2.1.3(ii))
   What can be said about the uniform continuity of Φ^{-1}? (see Remark 2.1.5(ii))
2. Is $Q(\overline{G})$ also incomplete if $G \notin \mathcal{JD}$? If so, what is its completion? (see Question 2.4.20)
3. If $Q(G)$ is separable, what subsets are countable and dense? (see Question 3.1.6)
4. Is $Q(G)$ path–connected only if the domain G has solely prime ends of the first kind? What necessary conditions for the connectedness of $Q(G)$ can be derived? (see Question 3.4.4)
5. Is the necessary compactness criterion $K(M) < +\infty$ with $M < Q(G)$ also fulfilled for general domains G and/or general subsets M (rather than subgroups)? (see Theorem 3.5.3)
6. Are the sufficient compactness criterion and the Arzelà–Ascoli Theorem for $Q(G)$ also valid for general domains G? (see Theorem 3.5.8 and Corollary 3.5.9)
7. Is $Q(G)$ a σ–compact space if the domain G satisfies $\mathcal{P}(G) = \mathcal{P}_1(G)$? (see Question 3.5.19)

Clearly, this list of open questions allows for the addition of further items when studying $Q(G)$ with more refined and elaborated tools, e.g. from Geometric Group Theory, Descriptive Set Theory, Representation Theory, Teichmüller Theory or also (co)homological methods. Among others, the following questions are conceivably in range:

- Is $Q(G)$ a *simple group*, i.e. containing no non–trivial proper normal subgroup?[15]
- For which domains G does $Q(G)$ carry the structure of a (possibly infinite–dimensional) *Lie group*?

[15] In fact, certain results in this direction have been found. For example, it was shown by Fisher that if M is a closed n–manifold with $n = 2, 3$, then the identity component of $\mathcal{H}(M)$ is simple; see [Fis60, Theorems 7 and 9, pp. 201–205].

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